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A NOTE ON COPSON'S INEQUALITY INVOLVING SERIES OF POSITIVE TERMS

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Abstract. In the present note we establish some generalizations of Copson's inequality concerning series of positive terms. The method used in the proofs is elementary and our results provide new estimates on inequalities of this type.

1. Introduction.

In 1927, Copson [2] found an interesting inequality concerning series of positive terms of the following form.

Let $p>1$, $\lambda_n>0$, $a_n>0$, $\Sigma\lambda_n a_n^p$ *converge, and further let* $\Lambda_n=\sum_{i=1}^n \lambda_i$, $A_n=\sum_{i=1}^n \lambda_i a_i$. *Then*

$$\sum \lambda_n\left(\frac{A_n}{\Lambda_n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum \lambda_n a_n^p. \tag{1}$$

The special case of (1) when $\lambda_n=1$ is first proved by Hardy [4] in an attempt to give a simple proof of Hilbert's double series inequality [7, p.226]. Hardy deduced his result from the corresponding theorem for integrals. However, the best possible constant in Hardy's series inequality [7, p239] is determined by Landu in [11]. In [2], Copson proved inequality (1) by using the method employed by Elliott [3] in presenting a simplified proof of Hardy's inequality [7, p.239]. The purpose of this note is to give some generalizations of Copson's inequality (1) concerning series of positive terms. The analysis used in the proofs is simple and based on the direct adaptation of Elliott's argument given in [3] and some elementary inequalities.

2. Statement of results

In this section we state our main results concerning the series of positive terms to be proved in this paper.

The following result is central to the concerns of this paper.

Theorem 1. *Let* $f(u)$ *be a real-valued positive convex function defined for* $u>0$. *Let* $p>1$ *be a constant,* $\lambda_n>0$, $a_n>0$, $\Sigma\lambda_n f^p(a_n)$ *converge, and further let* $\Lambda_n=\sum_{i=1}^n \lambda_i$, $A_n=\sum_{i=1}^n \lambda_i a_i$. *Then*

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$$\sum \lambda_n f^p\left(\frac{A_n}{\Lambda_n}\right) \le \left(\frac{p}{p-1}\right)^p \sum \lambda_n f^p(a_n). \tag{2}$$

Remark 1. We note that in the special case when $f(u) = u$, the inequality (2) reduces to the Copson's inequality given in (1) which in itself contains as a special case the series inequality given by Hardy in [4] involving best constant.

We next establish the following inequalities which in the special cases reduce to the variants of inequality given in Theorem 1.

Theorem 2. *Let $f_j(u)$, $j = 1, 2$ be real-valued positive convex functions defined for $u > 0$. Let $p_j \ge 1$, $j = 1, 2$ be constants, $\lambda_n > 0$, $a_n^{(j)} > 0$, $j = 1, 2$, $\Sigma \lambda_n f_j^{p_1+p_2}(a_n^{(j)})$, $j = 1, 2$ converge, and further let $\Lambda_n = \sum_{i=1}^n \lambda_i$, $A_n^{(j)} = \sum_{i=1}^n \lambda_i a_i^{(j)}$, $j = 1, 2$. Then*

$$\sum \lambda_n f_1^{p_1}\left(\frac{A_n^{(1)}}{\Lambda_n}\right) f_2^{p_2}\left(\frac{A_n^{(2)}}{\Lambda_n}\right) \le \left(\frac{p_1+p_2}{p_1+p_2-1}\right)^{p_1+p_2}$$
$$\cdot \left[\left(\frac{p_1}{p_1+p_2}\right) \sum \lambda_n f_1^{p_1+p_2}(a_n^{(1)}) + \left(\frac{p_2}{p_1+p_2}\right) \sum \lambda_n f_1^{p_1+p_2}(a_n^{(2)})\right] \text{ (3).}$$

Remark 2. If we take $p_1 = p_2 = p$, $a_n^{(1)} = a_n^{(2)} = a_n$, $A_n^{(1)} = A_n^{(2)} = A_n$, $f_1 = f_2 = f$, then the inequality (3) reduces to the following inequality

$$\sum \lambda_n f^{2p}\left(\frac{A_n}{\Lambda_n}\right) \le \left(\frac{2p}{2p-1}\right)^{2p} \sum \lambda_n f^{2p}(a_n). \tag{4}$$

We note that the inequality obtained in (4) is a slight variant of the inequality given in Theorem 1.

Theorem 3. *Let $f_j(u), j = 1, 2, 3$ be real-valued positive convex functions defined for $u > 0$. Let $p_j \ge 1$, $j = 1, 2, 3$ be constants, $\lambda_n > 0$, $a_n^{(j)} > 0$, $j = 1, 2, 3$, $\Sigma \lambda_n f_j^{2p_j}(a_n^{(j)})$, $j = 1, 2, 3$ converge, and further let $\Lambda_n = \Sigma_{i=1}^n \lambda_i$, $A_n^{(j)} = \Sigma_{i=1}^n \lambda_i a_i^{(j)}$, $j = 1, 2, 3$. Then*

$$\sum \lambda_n \left[f_1^{p_1}\left(\frac{A_n^{(1)}}{\Lambda_n}\right) f_2^{p_2}\left(\frac{A_n^{(2)}}{\Lambda_n}\right) + f_2^{p_2}\left(\frac{A_n^{(2)}}{\Lambda_n}\right) f_3^{p_3}\left(\frac{A_n^{(3)}}{\Lambda_n}\right) + f_3^{p_3}\left(\frac{A_n^{(3)}}{\Lambda_n}\right) f_1^{p_1}\left(\frac{A_n^{(1)}}{\Lambda_n}\right)\right]$$
$$\le \sum_{j=1}^{3} \left(\frac{2p_j}{2p_j-1}\right)^{2p_j} \left\{\sum \lambda_n f_j^{2p_j}(a_n^{(j)})\right\}. \tag{5}$$

Theorem 4. *Let f_j, p_j, λ_n, $a_n^{(j)}$, $\Lambda_n, A_n^{(j)}$ be as defined in Theorem 3 and*

$\Sigma\lambda_n f_j^{4p_j}(a_n^{(j)})$, $j = 1, 2, 3$ *converge. Then*

$$\sum \lambda_n f_1^{p_1}(\frac{A_n^{(1)}}{\Lambda_n}) f_2^{p_2}(\frac{A_n^{(2)}}{\Lambda_n}) f_3^{p_3}(\frac{A_n^{(3)}}{\Lambda_n}) \Big[f_1^{p_1}(\frac{A_n^{(1)}}{\Lambda_n}) + f_2^{p_2}(\frac{A_n^{(2)}}{\Lambda_n}) + f_3^{p_3}(\frac{A_n^{(3)}}{\Lambda_n}) \Big]$$
$$\leq \sum_{j=1}^{3} (\frac{4p_j}{4p_j - 1})^{4p_j} \Big\{ \sum \lambda_n f_j^{4p_j}(a_n^{(j)}) \Big\}. \tag{6}$$

Remark 3. In the special case when $p_1 = p_2 = p_3 = p$, $a_n^{(1)} = a_n^{(2)} = a_n^{(3)} = a_n$ $A_n^{(1)} = A_n^{(2)} = A_n^{(3)} = A_n$, $f_1 = f_2 = f_3 = f$, inequalities (5) and (6) reduce respectively to the inequality (4) and

$$\sum \lambda_n f^{4p}(\frac{A_n}{\Lambda_n}) \leq (\frac{4p}{4p-1})^{4p} \sum \lambda_n f^{4p}(a_n), \tag{7}$$

which are the variants of the inequality (2) given in Theorem 1.

To this end we establish the following interesting generalization of Copson's inequality (1).

Theorem 5. *Let $f_j(u)$, $j = 1, \cdots, m$ be real-valued positive convex functions defined for $u > 0$. Let $p_j > 1$, $j = 1, \cdots, m$ be constants, $\lambda_n > 0$, $a_n^{(j)} > 0$, $j = 1, \cdots, m$, $\Sigma\lambda_n f_j^{mp_j}(a_n^{(j)})$, $j = 1, \cdots, m$ converge, and further let $\Lambda_n = \Sigma_{i=1}^n \lambda_i$, $A_n^{(j)} = \Sigma_{i=1}^n \lambda_i a_i^{(j)}$, $j = 1, \cdots, m$. Then*

$$\sum \lambda_n \prod_{j=1}^{n} f_j^{p_j}(\frac{A_n^{(j)}}{\Lambda_n}) \leq \frac{1}{m} \sum_{j=1}^{m} (\frac{mp_j}{mp_j - 1})^{mp_j} \{ \sum \lambda_n f_j^{mp_j}(a_n^{(j)}) \}. \tag{8}$$

Remark 4. We note that in the special case where $p_j = p$, $a_n^{(j)} = a_n$, $A_n^{(j)} = A_n$, $f_j = f$ for $j = 1, \cdots, m$, inequality (8) reduces to the following inequality

$$\sum \lambda_n f^{mp}(\frac{A_n}{\Lambda_n}) \leq (\frac{mp}{mp-1})^{mp} \sum \lambda_n f^{mp}(a_n), \tag{9}$$

which in turn when $m = 1$ redures to the inequality (2) given in Theorem 1 and in addition if we take $f(u) = u$ it contains the Hardy's inequality given in [7, p.239].

3. Proofs. of theorems 1-5

From hypotheses of Theorem 1, since f is convex, by Jensen's inequality (see, [10, p.133]), we have

$$f(\frac{A_n}{\Lambda_n}) \leq \frac{F_n}{\Lambda_n}, \tag{10}$$

where $F_n = \Sigma_{i=1}^n \lambda_i f(a_i)$. We write $\alpha_n = F_n \Lambda_n^{-1}$ and agree that any number with suffix 0 is 0. Now, by making use of the elementary inequality (see [1,7])

$$x^{n+1} + ny^{n+1} \geq (n+1)xy^n, \qquad x, y \geq 0 \text{ reals},$$

we observe that

$$\begin{aligned} &\lambda_n \alpha_n^p - \frac{p}{p-1}\lambda_n f(a_n)\alpha_n^{p-1} \\ &= \lambda_n \alpha_n^p - \frac{p}{p-1}\alpha_n^{p-1}\Big[\alpha_n \Lambda_n - \alpha_{n-1}\Lambda_{n-1}\Big] \\ &= (\lambda_n - \frac{p}{p-1}\Lambda_n)\alpha_n^p + \frac{p}{p-1}\Lambda_{n-1}\alpha_{n-1}\alpha_n^{p-1} \\ &\leq (\lambda_n - \frac{p}{p-1}\Lambda_n)\alpha_n^p + \frac{\Lambda_{n-1}}{p-1}\Big[\alpha_{n-1}^p + (p-1)\alpha_n^p\Big] \\ &= \frac{1}{p-1}\Big[\Lambda_{n-1}\alpha_{n-1}^p - \Lambda_n \alpha_n^p\Big]. \end{aligned} \tag{11}$$

By substituting $n = 1, \cdots, N$ in (11) and adding the inequalities we see that

$$\sum_{n=1}^{N}\Big[\lambda_n \alpha_n^p - \frac{p}{p-1}\lambda_n f(a_n)\alpha_n^{p-1}\Big] \leq -\frac{1}{p-1}\Lambda_N \alpha_N^p \leq 0. \tag{12}$$

From (12) we observe that

$$\sum_{n=1}^{N}\lambda_n \alpha_n^p \leq \frac{p}{p-1}\sum_{n=1}^{N}\Big\{\lambda_n^{\frac{1}{p}} f(a_n)\Big\}\Big\{\lambda_n^{\frac{p-1}{p}}\alpha_n^{p-1}\Big\}. \tag{13}$$

Using Hölder's inequality with indices $p, \frac{p}{p-1}$ on the right side of (13) we have

$$\sum_{n=1}^{N}\lambda_n \alpha_n^p \leq \frac{p}{p-1}\Big\{\sum_{n=1}^{N}\lambda_n f^p(a_n)\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{N}\lambda_n \alpha_n^p\Big\}^{\frac{p-1}{p}}.$$

Dividing the above ineqality by the last factor on the right and raising the result to the pth power, we obtain

$$\sum_{n=1}^{N}\lambda_n(\frac{F_n}{\Lambda_n})^p \leq (\frac{p}{p-1})^p\sum_{n=1}^{N}\lambda_n f^p(a_n). \tag{14}$$

Now from (10) and (14), we have

$$\sum_{n=1}^{N}\lambda_n f^p(\frac{A_n}{\Lambda_n}) \leq \sum_{n=1}^{N}\lambda_n(\frac{F_n}{\Lambda_n})^p \leq (\frac{p}{p-1})^p\sum_{n=1}^{N}\lambda_n f^p(a_n). \tag{15}$$

By letting N tend to infinity in (15), we obtain the desired inequality in (2). This completes the proof of Theorem 1.

By using the elementary inequality (see [1,7,12])

$$p_1 x^{p_1+p_2} + p_2 y^{p_1+p_2} - (p_1+p_2)x^{p_1}y^{p_2} \geq 0,$$

where $x, y \geq 0$ and $p_1 > 0$, $p_2 > 0$ real, we observe that

$$\begin{aligned}&\sum \lambda_n f_1^{p_1}(\frac{A_n^{(1)}}{\Lambda_n}) f_2^{p_2}(\frac{A_n^{(2)}}{\Lambda_n}) \\ &\leq (\frac{p_1}{p_1+p_2}) \sum \lambda_n f_1^{p_1+p_2}(\frac{A_n^{(1)}}{\Lambda_n}) + (\frac{p_2}{p_1+p_2}) \sum \lambda_n f_2^{p_1+p_2}(\frac{A_n^{(2)}}{\Lambda_n}).\end{aligned}$$

Now a suitable application of Theorem 1 on the right side of the above inequality yields the required inequality in (3) and the proof of Theorem 2 is complete.

In order to prove the inequalities in Theorems 3-5 we first observe as follows:

Using the elementary inequality $c_1c_2 + c_2c_3 + c_3c_1 \leq c_1^2 + c_2^2 + c_3^2$ (for c_1, c_2, c_3 real) we observe that

$$\begin{aligned}&\sum \lambda_n \Big[f_1^{p_1}(\frac{A_n^{(1)}}{\Lambda_n}) f_2^{p_2}(\frac{A_n^{(2)}}{\Lambda_n}) + f_2^{p_2}(\frac{A_n^{(2)}}{\Lambda_n}) f_3^{p_3}(\frac{A_n^{(3)}}{\Lambda_n}) + f_3^{p_3}(\frac{A_n^{(3)}}{\Lambda_n}) f_1^{p_1}(\frac{A_n^{(1)}}{\Lambda_n}) \Big] \\ &\qquad \leq \sum_{j=1}^{3} \Big\{ \sum \lambda_n f_j^{2p_j}(\frac{A_n^{(j)}}{\Lambda_n}) \Big\}. \qquad (16)\end{aligned}$$

By making use of the elementary inequalities

$$c_1c_2c_3(c_1+c_2+c_3) \leq \frac{1}{3}(c_1c_2 + c_2c_3 + c_3c_1)^2,$$

$$(c_1c_2 + c_2c_3 + c_3c_1) \leq c_1^2 + c_2^2 + c_3^2 \quad \textit{and} \quad (c_1+c_2+c_3)^2 \leq 3(c_1^2 + c_2^2 + c_3^2),$$

(for c_1, c_2, c_3 real) (see [14,pp.201,203]) we observe that

$$\begin{aligned}&\sum \lambda_n f_1^{p_1}(\frac{A_n^{(1)}}{\Lambda_n}) f_2^{p_2}(\frac{A_n^{(2)}}{\Lambda_n}) f_3^{p_3}(\frac{A_n^{(3)}}{\Lambda_n}) \Big[f_1^{p_1}(\frac{A_n^{(1)}}{\Lambda_n}) + f_2^{p_2}(\frac{A_n^{(2)}}{\Lambda_n}) + f_3^{p_3}(\frac{A_n^{(3)}}{\Lambda_n}) \Big] \\ &\qquad \leq \sum_{j=1}^{3} \Big\{ \sum \lambda_n f_j^{4p_j}(\frac{A_n^{(j)}}{\Lambda_n}) \Big\}. \qquad (17)\end{aligned}$$

Using the elementary inequalities $\{\prod_{j=1}^{m} c_j\}^{\frac{1}{m}} \leq \frac{1}{m}\Sigma_{j=1}^{m} c_j$ and $\{\Sigma_{j=1}^{m} c_j\}^m \leq m^{m-1} \sum_{j=1}^{m} c_j^m$ (for $c_1, \cdots, c_m \geq 0$ real and $m \geq 1$) (see [15, p.272]) we observe that

$$\sum \lambda_n \prod_{j=1}^{m} f_j^{p_j}\left(\frac{A_n^{(j)}}{\Lambda_n}\right) = \sum \lambda_n \left[\left\{\prod_{j=1}^{m} f_j^{p_j}\left(\frac{A_n^{(j)}}{\Lambda_n}\right)\right\}^{\frac{1}{m}}\right]^m$$
$$\leq \frac{1}{m}\sum \lambda_n \sum_{j=1}^{m} f_j^{mp_j}\left(\frac{A_n^{(j)}}{\Lambda_n}\right). \qquad (18)$$

Now suitable applications of inequality (2) on the right sides of (16), (17) and (18) yields the required inequalities in (5),(6) and (8) respectively. This completes the proofs of Theorems 3-5.

Remark 5. We note that in [2] Copson also proved a companion inequality under the same hypotheses as in his first inequality (1) except that $A_n = \frac{\lambda_n a_n}{\Lambda_n} + \frac{\lambda_{n+1} a_{n+1}}{\Lambda_{n+1}} + \cdots$, of the following form:

$$\sum \lambda_n A_n^p \leq p^p \sum \lambda_n a_n^p. \qquad (19)$$

In 1928, Hardy in his paper [5] noted that the inequality (19) does not require a separte proof, but can be derived from Copson's first inequality (1) (see, also [9, p.133]). In view of this remark, we do not discuss the generalizations or variants of Copson's inequality (19). For other interesting generalizations and extensions of Hardy's and Copson's series inequalities in different directions, we refer the interested readers to [8,9,13,16] and the references given therein.

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