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ON A CLASS OF CERTAIN ANALYTIC FUNCTIONS OF COMPLEX ORDER

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Abstract. We introduce a class, namely, $F_n(b, M)$ of certain analytic functions. For this class we determine coefficient estimate, sufficient condition in terms of coefficients, maximization theorme concerning the coefficients, radius problem and a necessary and sufficient condition in terms of convolution. Our results generalize and correct some results of Nasr and Aouf ([2],[3]).

1. Introduction.

Let A denote the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the unit disc $U = \{z : | z |< 1\}$. We use B to denote the class of analytic functions $\omega(z)$ in U satisfying the conditions $\omega(0) = 0$ and $| \omega(z) |< 1$ for $z \in U$. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ belong to A, then the convolution or Hadamard product of $f(z)$ and $g(z)$, denoted by $f(z) * g(z)$, is defined by

$$f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \ z \in U. \tag{1.1}$$

Let $n \in N_0 = \{0, 1, 2, 3, \ldots\}$. The n^{th} order Ruscheweyh derivative (see, [1]) of $f(z)$, denoted by $D^n f(z)$, is defined by

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}, \ n \in N_0. \tag{1.2}$$

Ruscheweyh [9] determined that

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z). \tag{1.3}$$

We now introduce a class, namely, $F_n(b, M)$ of analytic functions.

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A function $f(z)$ of A belongs to the class $F_n(b,M)$ if and only if $\frac{D^n f(z)}{z} \neq 0$ in U and

$$(1.4) \qquad \left| \frac{(b-1)+z\frac{(D^n f(z))'}{D^n f(z)}}{b} - M \right| < M, \ z \in U,$$

where $M > \frac{1}{2}$ and b is any non-zero complex number.

It follows by [6] that $g(z) \in F_{0,M}$ if and only if for $z \in U$

$$(1.5) \qquad z\frac{g'(z)}{g(z)} = \frac{1+\omega(z)}{1-m\omega(z)},$$

where $m = 1 - \frac{1}{M}$ and $\omega(z) \in B$.

One can easily show that $f(z) \in F_n(b,M)$ if and only if there is a function $g(z) \in F_{0,M}$ such that

$$(1.6) \qquad D^n f(z) = z[\frac{g(z)}{z}]^b.$$

Thus from (1.5) and (1.6) it follows that $f(z) \in F_n(b,M)$ if and only if for $z \in U$

$$(1.7) \qquad z\frac{(D^n f(z))'}{D^n f(z)} = \frac{1+\{b(1+m)-m\}\omega(z)}{1-m\omega(z)},$$

where $\omega(z) \in B$ and $m = 1 - \frac{1}{M}(M > \frac{1}{2})$.

By giving specific values to n, b and M, we obtain the following important subclasses studied by various researchers in earlier works:

(i) Taking $n = 0$, the class $F_n(b,M)$ coincides with the class $F(b,M)$ studied by Nasr and Aouf [2].

(ii) Taking $n = 1$, the class $F_n(b,M)$ coincides with the class $G(b,M)$ investigated by Nasr and Aouf [3].

(iii) Taking $n = 0, b = \cos\lambda e^{-i\lambda}(|\lambda| < \frac{\pi}{2})$ and $M = \infty$, the class $F_n(b,M)$ coincides with the class $S^\lambda, |\lambda| < \frac{\pi}{2}$ introduced by Špaček [14].

(iv) Taking $n = 0$ and $M = \infty$, the class $F_n(b,M)$ coincides with the class $S(1-b)$ investigated by Nasr and Aouf [4].

(v) Taking $n = 0$ and $b = \cos\lambda e^{-i\lambda}$, the class $F_n(b,M)$ coincides with the class $F_{\lambda,M'}$ $|\lambda| < \frac{\pi}{2}$ studied by Kulshrestha [6].

(vi) Taking $n = 1$ and $b = \cos\lambda e^{-i\lambda}$, the class $F_n(b,M)$ coincides with the class $G_{\lambda,M'}$ $|\lambda| < \frac{\pi}{2}$ investigated by Kulshrestha [7].

(vii) Taking $n = 1, b = \cos\lambda e^{-i\lambda}(|\lambda| < \frac{\pi}{2})$ and $M = \infty$, the class $F_n(b,M)$ coincides with the class $C^\lambda, |\lambda| < \frac{\pi}{2}$ introduced by Robertson [11].

(viii) Taking $n = 1$ and $M = \infty$, the class $F_n(b,M)$ coincides with the class $C(b)$ introduced by Wiatrowski [15] and studied by Nasr and Aouf [5].

(ix) Taking $b = 1$ and $M = \infty$, the class $F_n(b, M)$ coincides with the class R_n introduced and studied by Singh and Singh [12].

(x) Taking $b = 1 - \beta(0 \leq \beta < 1), n = 1 - 2\alpha(0 \leq \alpha < 1)$ and $M = \infty$, the class $F_n(b, M)$ coincides with the class $R(\alpha, \beta)$ introduced and studied by -Sheil-Small, Silverman and Silvia [13].

(xi) Taking $b = 1 - \alpha, n = 1 - 2\alpha(0 \leq \alpha < 1)$ and $M = \infty$, the class $F_n(b, M)$ coincides with the class R_α introduced and studied by Ruscheweyh [10].

From the definitions of the classes $F(b, M)$ and $F_n(b, M)$, we observe that

$$f(z) \in F_n(b, M) \Longleftrightarrow D^n f(z) \in F(b, M). \tag{1.8}$$

The purpose of the present paper is to determine coefficient estimate, sufficient condition in terms of coefficients for a function to belong to $F_n(b, M)$ and maximization of $| a_3 - \mu a_2^2 |$ over the class $F_n(b, M)$ for complex value of μ. Further we obtain the radius of disc in which $\mathrm{Re}\{z\frac{(D^n f(z))'}{D^n f(z)}\} > 0$, whenever $f(z)$ belongs to $F_n(b, M)$ and a necessary and sufficient condition in terms of convolution for a function to be in $F_n(b, M)$. We also obtain the correct form of Maximization theorems concerning the coefficients for the classes $F(b, M)$ and $G(b, M)$, by taking $n = 0$ and $n = 1$, in our maximization theorem for the class $F_n(b, M)$ respectively.

Our results generalize the some results of Nasr and Aouf ([2], [3], [4], [5]), Kulshrestha ([6], [7]), Ruscheweyh [10], Robertson [11], Singh and Singh [12], Sheil-Small, Silverman and Silvia [13], Špaček [14] and Wiatrowski [15].

2. Preliminary lemma

In our investigation we require the following Lemma due to Keogh and Merkes [8]:

Lemma 2.1. *Let $\omega(z) = \sum_{k=1}^{\infty} c_k z^k$ belongs to B. If μ is any complex number then*

$$| c_2 - \mu c_1^2 | \leq \max\{1, | \mu |\}. \tag{2.1}$$

Equality may be attained with the functions $\omega(z) = z^2$ and $\omega(z) = z$.

3. Coefficient estimate

Theorem 3.1. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belongs to $F(b, M)$ and*

$$L = | b |^2 (1 + m) + 2m(j - 1)Re(b) + (m - 1)(j - 1)^2,$$

then

$$| a_k | \leq \frac{1}{\underline{| k - 1}\delta(n - 1, k)} \prod_{\ell=0}^{k-2} | b(1 + m) + m\ell |, \quad when\ L > 0 \tag{3.1}$$

and

$$| a_k | \leq \frac{1}{(k - 1)\delta(n - 1, k)} | b(1 + m) |, \quad when\ L \leq 0, \tag{3.2}$$

where $m = 1 - \frac{1}{M}(M > \frac{1}{2})$ and $\delta(n, k) = \binom{n+k}{n+1}$.

The inequalities (3.1) and (3.2) are sharp.

Proof. Since $f(z) \in F_n(b, M)$, so from (1.7) we have that

$$z\frac{(D^n f(z))'}{D^n f(z)} = \frac{1 + \{b(1+m) - m\}\omega(z)}{1 - m\omega(z)}$$

or

$$\sum_{j=2}^{\infty}(j-1)\delta(n-1,j)a_j z^j = [b(1+m)z + \sum_{j=2}^{\infty}\{b(1+m) + m(j-1)\}\delta(n-1,j)a_j z^j]\omega(z)$$

or

$$\sum_{j=2}^{k}(j-1)\delta(n-1,j)a_j z^j + \sum_{j=k+1}^{\infty} d_j z^j = [b(1+m)z + \sum_{j=2}^{k-1}\{b(1+m) + m(j-1)\}\delta(n-1,j)a_j z^j]\omega(z),$$

where d_j's are some complex numbers. Then since $|\omega(z)| < 1$, we have

$$|b(1+m)z + \sum_{j=2}^{k-1}\{b(1+m) + m(j-1)\}\delta(n-1,j)a_j z^j| \geq |\sum_{j=2}^{k}(j-1)\delta(n-1,j)a_j z^j + \sum_{j=k+1}^{\infty} d_j z^j|. \tag{3.3}$$

Squaring both sides of (3.3) and integrating round $|z| = r < 1$, we get, after taking the limit when $r \to 1$

$$(k-1)^2\{\delta(n-1,k)\}^2 |a_k|^2 \leq |b(1+m)|^2 + \sum_{j=2}^{k-1}[|b(1+m) + m(j-1)|^2 - (j-1)^2]\{\delta(n-1,j)\}^2 |a_j|^2. \tag{3.4}$$

On solving, we get

$$(k-1)^2\{\delta(n-1,k)\}^2 |a_k|^2 \leq |b(1+m)|^2 + \sum_{j=2}^{k-1}[(1+m)\{|b|^2(1+m) + 2m(j-1)\mathrm{Re}(b) + (m-1)(j-1)^2\}]\{\delta(n-1,j)\}^2 |a_j|^2$$

or

$$(k-1)^2\{\delta(n-1,k)\}^2 |a_k|^2 \leq |b(1+m)|^2 + \sum_{j=2}^{k-1}(1+m)L\{\delta(n-1,j)\}^2 |a_j|^2,$$

where

$$L = \{| b |^2 (1+m) + 2m(j-1)\mathrm{Re}(b) + (m-1)(j-1)^2\}.$$

Now we consider the following two cases:

Cases I: When $L > 0$. In this case (3.4) gives successively for $k = 2, 3, 4, \ldots$

$$| a_2 | \leq \frac{| b(1+m) |}{\delta(n-1,2)}, \quad | a_3 | \leq \frac{| b(1+m) || b(1+m)+m |}{2\delta(n-1,3)},$$

and hence by induction

$$| a_k | \leq \frac{1}{| \underline{k-1}\delta(n-1,k)} \prod_{l=0}^{k-2} | b(1+m)+ml | .$$

The above inequality is sharp for the function $f(z)$ defined by

$$D^n f(z) = \begin{cases} \frac{z}{(1-mz)^{b(1+m)/m}}, & \text{for } m \neq 0 \\ z\exp(bz), & \text{for } m = 0. \end{cases}$$

Case II: When $L \leq 0$. In this case (3.4) gives

$$| a_k | \leq \frac{1}{(k-1)\delta(n-1,k)} | b(1+m) | .$$

This inequality is sharp.

Theorem 3.2. *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. If for some $n \geq 0$ and A, B with $-1 \leq B < A \leq 1$,*

$$\sum_{k=2}^{\infty} [(k-1) + | b(1+m) + m(k-1) |]\delta(n-1,k) | a_k | \leq | b(1+m) |, \tag{3.5}$$

holds, then $f(z)$ belongs to $F_n(b,m)$, where $\delta(n,k) = \binom{n+k}{n+1}$ and $m = 1 - \frac{1}{M} (M > \frac{1}{2})$.

Proof. Suppose that the inequality (3.5) holds. Since $f(z) = z + \sum_{k=2}^{\infty} \delta(n-1,k) a_k z^k$, we have for $z \in U$

$$\begin{aligned} &| z(D^n f(z))' - D^n f(z) | - | b(1+m)D^n f(z) + m\{z(D^n f(z))' - D^n f(z)\} | \\ &= | \sum_{k=2}^{\infty} (k-1)\delta(n-1,k) a_k z^k | - | b(1+m)\{z + \sum_{k=2}^{\infty} \delta(n-1,k) a_k z^k\} \\ &\quad + m \sum_{k=2}^{\infty} (k-1)\delta(n-1,k) a_k z^k | \\ &\leq \sum_{k=2}^{\infty} (k-1)\delta(n-1,k) | a_k | r^k \end{aligned}$$

$$-\{| b(1+m) | r - \sum_{k=2}^{\infty} | b(1+m) + m(k-1) | \delta(n-1,k) | a_k | r^k\}$$

$$= \sum_{k=2}^{\infty} [(k-1) + | b(1+m) + m(k-1) |] \delta(n-1,k) | a_k | r^k - | b(1+m) | r$$

$$\leq \sum_{k=2}^{\infty} [(k-1) + | b(1+m) + m(k-1) |] \delta(n-1,k) | a_k | - | b(1+m) |$$

$$\leq 0, \text{ by } (3.5)$$

Hence it follows that

$$\left| \frac{\{z\frac{(D^n f(z))'}{D^n f(z)} - 1\}}{b(1+m) + m\{z\frac{(D^n f(z))'}{D^n f(z)} - 1\}} \right| < 1, \ z \in U.$$

Letting

$$\omega(z) = \frac{\{z\frac{(D^n f(z))'}{D^n f(z)} - 1\}}{b(1+m) + m\{\frac{z(D^n f(z))'}{D^n f(z)} - 1\}},$$

then $\omega(0) = 0, \omega(z)$ is analytic in $| z |< 1$ and $| \omega(z) |< 1$. Hence we have

$$\frac{z(D^n f(z))'}{D^n f(z)} = \frac{1 + \{b(1+m) - m\}\omega(z)}{1 - m\omega(z)}$$

which shows that $f(z)$ belongs to $F_n(b, M)$.

4. Maximization of $| a_3 - \mu a_2^2 |$

Theorem 4.1. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belongs to $F_n(b, M)$ and μ is any complex number then*

(4.1) $$| a_3 - \mu a_2^2 | \leq \frac{| b(1+m) |}{2\delta(n-1,3)} \max\{1, | d |\},$$

where

$$d = \frac{2\mu b(1+m)\delta(n-1,3) - \{b(1+m) + m\}\{\delta(n-1,2)\}^2}{\{\delta(n-1,2)\}^2}.$$

This inequality is sharp for each μ.

Proof. Since $f(z)$ belongs to $F_n(b, M)$, so that from (1.7) we have

$$z\frac{(D^n f(z))'}{D^n f(z)} = \frac{1 + \{b(1+m) - m\}\omega(z)}{1 - m\omega(z)}$$

or

$$\begin{aligned}\omega(z) &= \frac{\{z(D^n f(z))' - D^n f(z)\}}{[\{b(1+m)-m\}D^n f(z) + mz(D^n f(z))']} \\ &= \frac{\sum_{k=2}^{\infty}(k-1)\delta(n-1,k)a_k z^{k-1}}{[b(1+m)+\sum_{k=2}^{\infty}\{b(1+m)+m(k-1)\}\delta(n-1,k)a_k z^{k-1}]} \\ &= \frac{\sum_{k=2}^{\infty}(k-1)\delta(n-1,k)a_k z^{k-1}}{b(1+m)} \\ &\quad [1+\frac{\sum_{k=2}^{\infty}\{b(1+m)+m(k-1)\}\delta(n-1,k)a_k z^{k-1}}{b(1+m)}]^{-1}.\end{aligned}$$

Equating the coefficients of z and z^2 on both sides, we get

$$a_2 = \frac{b(1+m)c_1}{\delta(n-1,2)} \tag{4.2}$$

and

$$a_3 = \frac{b(1+m)}{2\delta(n-1,3)}[c_2 + \{b(1+m)+m\}c_1^2]. \tag{4.3}$$

Hence

$$a_3 - \mu a_2^2 = \frac{b(1+m)}{2\delta(n-1,3)}[c_2 - dc_1^2], \tag{4.4}$$

where

$$d = \frac{2\mu b(1+m)\delta(n-1,3) - \{b(1+m)+m\}\{\delta(n-1,2)\}^2}{\{\delta(n-1,2)\}^2}.$$

Taking modulus both sides in (4.4), we have

$$| a_3 - \mu a_2^2 | = \frac{| b(1+m) |}{2\delta(n-1,3)} | c_2 - dc_1^2 | . \tag{4.5}$$

Using Lemma 2.1 in (4.5), we have

$$| a_3 - \mu a_2^2 | \le \frac{| b(1+m) |}{2\delta(n-1,3)} \max\{1, | d |\}.$$

Since the inequality (2.1) is sharp, so that the inequality (4.1) must also be sharp.

Corollary. *If $f(z)$ belongs to $F_n(b, M)$, then*

$$| a_2 | \le \frac{| b | (1+m)}{\delta(n-1,2)} \tag{4.6}$$

and

$$| a_3 | \le \frac{| b | (1+m)}{2\delta(n-1,3)} \max\{1, | b(1+m)+m |\}. \tag{4,7}$$

The inequalities (4.6) and (4.7) follow directly from (4.2) and (4.3) respectively.

Remarks.

(i) We obtain the maximization of $| a_3 - \mu a_2^2 |$ over the class $F(b, M)$ which is studied by Nasr and Aouf [2], in correct form, by taking $n = 0$ in our maximization of $| a_3 - \mu a_2^2 |$ over the class $F_n(b, M)$.

(ii) We also obtain the maximization of $| a_3 - \mu a_2^2 |$ over the class $G(b, M)$ which is studied by Nasr and Aouf [3], in correct form, by taking $n = 1$ in our maximization of $| a_3 - \mu a_2^2 |$ over the class $F_n(b, M)$.

5. Radius theorem

The following theorem may by obtained with the help of (1.8) and Lemma 3 of Nasr and Aouf [2].

Theorem 5.1. *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belongs to $F_n(b, M)$. Then*

$$Re\{\frac{z(D^n f(z))'}{D^n f(z)}\} > 0 \ for \ | z |< r_n,$$

where

$$r_n = 2\{| b | (1+m) + [| b |^2 (1+m)^2 - 4\{Re(b)(\frac{1+m}{m}) - 1\}]^{\frac{1}{2}}\}^{-1}.$$

Remarks. In the above mentioned result

(i) Putting $n = 0$, we get the sharp radius of starlikeness of the class $F(b, M)$ which is studied by Nasr and Aouf [2].

(ii) Putting $n = 1$, we get the sharp radius of convexity of the class $G(b, M)$ which is investigated by Nasr and Aouf [3].

6. Necessary and sufficient condition

Theorem 6.1. *A function $f(z)$ belongs to the class $F_n(b, M)$ if and only if*

$$f(z) * [\frac{z + [\frac{(n+1)+x\{b(1+m)-m(n+1)\}}{-b(1+m)x}]z^2}{(1-z)^{n+2}}] \neq 0 \tag{6.1}$$

in $0 <| z |< 1$, where $x = 1$ and $x \neq 1$.

Proof. Let $f(z)$ belongs to the class $F_n(b, M)$, then

$$z\frac{(D^n f(z))'}{D^n f(z)} \neq \frac{1 + \{b(1+m) - m\}x}{1 - mx},$$

$| x |= 1$ and $x \neq 1$ in $0 <| z |< 1$. Equivalently

$$(1 - mx)z(D^n f(z))' - [1 + \{b(1+m) - m\}x]D^n f(z) \neq 0 \text{ in } 0 <| z |< 1. \tag{6.2}$$

We know that, by the definition of $D^n f(z)$

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z). \tag{6.3}$$

Using (6.3) in (6.2), we have

$$(1-mx)[(n+1)D^{n+1}f(z) - nD^n f(z)] - [1+\{b(1+m)-m\}x]D^n f(z) \neq 0 \text{ in } 0<|z|<1$$

or

$$f(z) * [\frac{-b(1+m)xz + [(n+1)+x\{b(1+m)-m(n+1)\}]z^2}{(1-z)^{n+2}}] \neq 0.$$

Since $x \neq 1$ and b is any non-zero complex number, we have

$$f(z) * [\frac{z + [\frac{(n+1)+x\{b(1+m)-m(n+1)\}}{-b(1+m)x}]z^2}{(1-z)^{n+2}}] \neq 0$$

which is the required condition.

The converse part follows easily since all the steps can be retraced back.

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