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INTEGRALS OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

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Abstract. The moments of the weight functions $w(x) = e^{-x}x^{\mu}(\ln x)^{\rho}$, $\rho = 0, 1, 2,$ on $[0, \infty)$ with respect to the Confluent Hypergeometric function $\phi(a - n, c; x), n = 0, 1, 2, \ldots$, are explicitly evaluated.

1. Introduction

In a previous paper Blue [1] gave a simply expression for the integral $\int_0^1 p_n(2x-1)\ln(1/x)dx$, where $P_n(2x-1)$ is the shifted Legendre polynomial. Gautschi [5] treated $\int_0^1 x^{\alpha}\ln(1/x)P_n(2x-1)dx, \alpha > -1$.

The evaluation of these integrals is related to the construction of the modified moments with respect to certain classes of polynomials [4].

Afterwards Gatteschi [2] generalized these results by considering and explicitly evaluating the modified moments of the weight functions $w(x) = x^{\rho}(1-x)^{\alpha}\ln(1/x), \alpha, \rho > -1$, on $[0,1]$, with respect to the shifted Jacobi polinomials $P{*}_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$, and $w_p(x) = x^{\rho}e^{-x}(\ln x)^p, \rho > -1, p = 1, 2$ on $[0,\infty)$, with respect to the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$.

In a more recent paper [6], Kalla, Conde and Luke have considered a different problem, not necessarily related to the modified moments. More precisely, they have examined the integral $\int_{-1}^1(1-x)^a(1+x)^bP_{\nu}^{(\alpha,\beta)}(x)dx, \mathrm{Re}(a), \mathrm{Re}(b) > -1$, and its partial derivatives with respect to a and b, where $P_{\nu}^{(\alpha,\beta)}(x)$ is the Jacobi function which reduces to the Jacobi polynomial if ν is a positive integer. Gatteschi's result about Laguerre polynomial integrals has been considered again by Kalla and Conde in a more recent paper [7].

The purpose of this paper is to evaluate the moments of the weight functions $w(x) = e^{-x}x^{\mu}(\ln x)^{\rho}, \rho = 0, 1, 2, \ldots$ on $[0,\infty)$ with respect to the Confluent Hypergeometric function $\phi(a, c; x)$, that is the integrals

$$(1.1)\quad I_{\rho}(a,\mu) = \int_0^{\infty} e^{-x}x^{\mu}(\ln x)^{\rho}\phi(a,c;x)dx, c \neq 0, -1, -2, \ldots, \mu > -1, \rho = 1, 2, \ldots$$

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Since, by putting

$$J(a,\mu)=\int_0^\infty e^{-x}x^\mu\phi(a,c;x)dx, \qquad c\neq 0,-1,-2,\dots, \qquad \mu>-1, \tag{1.2}$$

it is

$$I_\rho(a,\mu)=\frac{\partial^\rho}{\partial\mu^\rho}J(a,\mu), \qquad \rho=1,2,\dots,$$

the evaluation of integrals (1.1) follows from that of $J(a,\mu)$ and its partial derivatives with respect to μ.

Here and throughout this paper, we use the notation of Gatteschi [3].

2. Integral $J(a,\mu)$ and related derivatives

For the computation of the integral $J(a,\mu)$ we make use of the known expansion [3, p.61]

$$\phi(a,c;x)=\sum_{k=0}^\infty\frac{(a)_k}{(c)_k}\frac{x^k}{k!}. \tag{2.1}$$

Indeed, by using (2.1) and termwise integration, we can write

$$\begin{aligned} J(a,\mu)&=\Gamma(\mu+1)\sum_{k=0}^\infty\frac{(a)_k(\mu+1)_k}{(c)_k}\frac{1}{k!}\\ &=\Gamma(\mu+1)F(a,\mu+1;c;1), \end{aligned} \tag{2.2}$$

where $F(a,\mu+1;c;1)$ is the gaussian hypergeometric function.

To insure the absolute convergence of the series (2.2), we must require $\mathrm{Re}(c-a-\mu-1)>0$.

Consequently, taking into account that [3, p.50]

$$F(a,\mu+1;c;1)=\frac{\Gamma(c)\Gamma(c-a-\mu-1)}{\Gamma(c-a)\Gamma(c-\mu-1)},$$

we obtain

$$\begin{aligned} J(a,\mu)&=\Gamma(\mu+1)\frac{\Gamma(c)\Gamma(c-a-\mu-1)}{\Gamma(c-a)\Gamma(c-\mu-1)},\\ &c\neq 0,-1,-2,\dots,\ \mu>-1,\ \mathrm{Re}(c-a-\mu-1)>0. \end{aligned} \tag{2.3}$$

We now consider the problem of evaluating the integrals (1.1).

We first examine the case $\rho=1$. Differentiating (2.3) with respect to μ gives

$$\begin{aligned} \frac{\partial}{\partial\mu}J(a,\mu)&=I_1(a,\mu)\\ &=\Gamma(\mu+1)\frac{\Gamma(c)\Gamma(c-a-\mu-1)}{\Gamma(c-a)\Gamma(c-\mu-1)}\\ &\quad\cdot\{\psi(\mu+1)+\psi(c-\mu-1)-\psi(c-a-\mu-1)\},\\ &c\neq 0,-1,-2,\dots,\ \mu>-1,\ \mathrm{Re}(c-a-\mu-1)>0, \end{aligned} \tag{2.4}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function.

The evaluation of (1.1) when $\rho = 2, 3, \ldots$ can be obtained by repeatedly differentiating (2.4) with respect to μ. We shall only examine, with some details, the case $\rho = 2$.

By a partial differentiation of (2.4) with respect to μ, we have

$$(2.5) \qquad \begin{aligned} \frac{\partial^2}{\partial \mu^2} J(a,\mu) &= I_2(a,\mu) \\ &= \Gamma(\mu+1)\frac{\Gamma(c)\Gamma(c-a-\mu-1)}{\Gamma(c-a)\Gamma(c-\mu-1)} \\ &\quad \cdot \{[\psi(\mu+1)+\psi(c-\mu-1)-\psi(c-a-\mu-1)]^2 \\ &\quad + [\psi'(\mu+1)+\psi'(c-a-\mu-1)-\psi'(c-\mu-1)]\}, \\ &c \neq 0,-1,-2,\ldots,\ \mu > -1,\ \mathrm{Re}(c-a-\mu-1) > 0. \end{aligned}$$

3. Some sequences

In this section we will evaluate the sequences of integrals (1.1) and (1.2) obtained from (2.3), (2.4), (2.5) when we change a into $a-n$, where $n = 0, 1, 2, \ldots$ and $a \neq 0$.

By remembering that $\Gamma(x+1) = x\Gamma(x)$, we may derive a useful algorithm for the computation of the moments $J(a-n,\mu)$.

Indeed, it is easily seen that, by putting

$$J(a-n,\mu) = \alpha_n,$$

that is,

$$\alpha_n = \Gamma(\mu+1)\frac{\Gamma(c)\Gamma(c-a-\mu+n-1)}{\Gamma(c-a+n)\Gamma(c-\mu-1)},$$

we have

$$(3.1) \qquad \begin{aligned} \alpha_{n+1} &= \alpha_n \frac{c-a-\mu+n-1}{c-a+n}, \qquad n = 0,1,2,\ldots, \\ \alpha_0 &= J(a,\mu). \end{aligned}$$

Moreover, for the computation of (2.4) when a is substituted by $a-n$, $n = 0, 1, 2, \ldots$, if we use the previously recalled property of function $\Gamma(x)$ and the recurrent relation $\psi(x+1) = \psi(x) + 1/x$, we may construct, in addition to (3.1), the following sequence

$$(3.2) \qquad \begin{aligned} \beta_{n+1} &= \beta_n + \frac{1}{c-a-\mu+n-1},\ n = 0,1,2,\ldots, \\ \beta_0 &= \psi(\mu+1)+\psi(c-\mu-1)-\psi(c-a-\mu-1), \end{aligned}$$

hence

$$I_1(a-n,\mu) = \alpha_n \cdot \beta_n,\ n = 0,1,2,\ldots.$$

Similarly, by replacing in (2.5) a with $a-n$, $n = 0, 1, 2, \ldots$, and recalling that $\psi'(x+1) = \psi'(x) - 1/x^2$, we obtain, together with the recurrent relationships (3.1) and

(3.2), the recursion formula

$$\gamma_{n+1} = \gamma_n - \frac{1}{(c-a-\mu+n-1)^2}, \quad n = 0,1,2,\ldots, \tag{3.3}$$
$$\gamma_0 = \psi'(\mu+1) + \psi'(c-a-\mu-1) - \psi'(c-\mu-1),$$

and, finally, the useful algorithm

$$I_2(a-n,\mu) = \alpha_n\{\beta_n^2 + \gamma_n\}, \quad n = 0,1,2,\ldots. \tag{3.4}$$

4. Particular cases

The results of the previous Sections reduce to the Gatteschi ones [2] when we assume $a = -n$ in (2.3), (2.4) and (2.5) or $a = 0$ in Section 3 and, after setting $c = \alpha + 1$, we change μ in $\alpha + \mu$.

This can be shown remembering that the Confluent Hpergeometric functions reduce to Laguerre polynomials when $a = -n, n = 0,1,2,\ldots$, and, more precisely,

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n}\phi(-n,\alpha+1;x).$$

For instance from (2.3) we derive

$$\int_0^\infty e^{-x}x^{\alpha+\mu}L_n^{(\alpha)}(x)dx = \frac{(-1)^n}{n!}\frac{\Gamma(\mu+1)\Gamma(\mu+\alpha+1)}{\Gamma(\mu-n+1)},$$

which is the formula given by Gatteschi [2, p.1295].

We consider now the case $\mu + 1 - c = m, m = 0,1,2,\ldots.$ Recalling that, for any integer $r \geq 0$,

$$\lim_{\epsilon\to 0}\frac{\psi(-r+\epsilon)}{\Gamma(-r+\epsilon)} = (-1)^{r-1}r!,$$

from (2.4) we obtain

$$\begin{aligned} I_1(a,\mu) &= \lim_{\epsilon\to 0} I_1(a,\mu-\epsilon) \\ &= \Gamma(\mu+1)\Gamma(c)\frac{\Gamma(c-a-\mu-1)}{\Gamma(c-a)}\lim_{\epsilon\to 0}\frac{\psi(c-\mu-1+\epsilon)}{\Gamma(c-\mu-1+\epsilon)} \\ &= (-1)^{m-1}\frac{\Gamma(c)}{\Gamma(c-a)}\Gamma(\mu+1)\Gamma(-a-m)\Gamma(m+1), \\ &\mu+1-c = m,\ m = 0,1,2,\ldots,\ \mathrm{Re}(a) < -m,\ \mu > -1. \end{aligned}$$

Analogously, for the evaluation of $I_2(a,\mu)$, we have from (2.5)

$$\begin{aligned} I_2(a,\mu) &= \lim_{\epsilon\to 0} I_2(a,\mu-\epsilon) \\ &= \frac{\Gamma(\mu+1)\Gamma(c)\Gamma(c-a-\mu-1)}{\Gamma(c-a)}\{A(\mu)+2B(\mu)\}, \end{aligned} \tag{4.1}$$

where

$$A(\mu) = \lim_{\epsilon \to 0} \frac{\psi^2(c-\mu-1+\epsilon) - \psi'(c-\mu-1+\epsilon)}{\Gamma(c-\mu-1+\epsilon)},$$
$$B(\mu) = \lim_{\epsilon \to 0} \{[\psi(\mu+1-\epsilon) - \psi(c-a-\mu-1+\epsilon)] \frac{\psi(c-\mu-1+\epsilon)}{\Gamma(c-\mu-1+\epsilon)}\}.$$

By means of the two series expansions

$$\Gamma(x) = \frac{(-1)^r}{r!}\frac{1}{x+r} + \sum_{k=0}^{\infty} a_k (x+r)^k, \qquad r = 0,1,2\ldots,$$
$$\psi(x) = \frac{-1}{x+r} + \psi(1+r) + \sum_{k=1}^{\infty} b_k (x+r)^k,$$

which hold for $| x+r |< 1$, it is easily seen that

$$A(\mu) = (-1)^{\mu-c} 2\psi(2+\mu-c)\Gamma(2+\mu-c),$$
$$B(\mu) = (-1)^{\mu-c}\Gamma(2+\mu-c)\{\psi(\mu+1) - \psi(c-a-\mu-1)\}.$$

Hence, substitution into (4.1), yields the final result

$$I_2(a,\mu) = (-1)^{m-1} 2 \frac{\Gamma(\mu+1)\Gamma(c)\Gamma(-m-a)}{\Gamma(c-a)} \Gamma(m+1)$$
$$\cdot \{\psi(m+1) + \psi(\mu+1) - \psi(-m-a)\},$$
$$\mu+1-c = m, \; m = 0,1,2,\ldots, \; \mathrm{Re}(a) < -m, \; \mu > -1.$$

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