TAMKANG JOURNAL OF MATHEMATICS
Volume 21, Number 2, Summer 1990

BOUNDED SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS

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1. In this note we are concerned with the question of boundedness of solutions of the following difference equation

$$\Delta(p_{k-1}\Delta x_{k-1}) = q_k f(x_k), \qquad k = 1,2,3,\ldots \tag{1}$$

where $p_k > 0$ for $k = 0,1,2,\ldots$, f is a real function definded on R such that sign $f(x) =$ sign x and $q_k \geq 0$ for $k = 1,2,3,\ldots$. In several recent papers [1,2,3,4], oscillatory, periodic and asymptotic behavior of solutions of second order difference equations have been investigated. In particular, the case when $f(x) = x$ and the case $f(x) = x^r$ where r is a quotient of odd positive integers, have been studied [1,3].

A solution of (1) is a real sequence $x = \{x_k\}_0^\infty$ satisfying (1). It is clear that the standard existence and uniqueness theorem holds for (1). It is also not difficult to see that a nontrivial solution x of (1) is eventually nonnegative or eventually nonpositive. Indeed, suppose without loss of generality that $x_0 \leq 0$. If $x_k \leq 0$ for all $k = 1,2,\ldots$, our assertion holds. Otherwise, $x_1 \leq 0,\ldots,x_{N-1} \leq 0$ but $x_N > 0$ for some $N > 0$. Since x_{N-1} and x_N cannot be zero simultaneously, we have from (1) that

$$p_N\Delta x_N = p_{N-1}\Delta x_{N-1} + q_N f(x_N) > 0,$$

which implies that $x_{N+1} > x_N \geq 0$, and by induction that $x_k > x_N \geq 0$ for all $k > N$.

Theorem 1. *A nontrivial solution x of (1) is either eventually positive increasing, eventually positive nonincreasing, eventually negative decreasing or eventually negative nondecreasing.*

Indeed, if $x_k \geq 0$ for $k \geq N$, then

$$\Delta(x_k p_{k-1}\Delta x_{k-1}) = p_k(\Delta x_k)^2 + q_k x_k f(x_k) \geq 0, \qquad k > N.$$

If $\Delta x_k \leq 0$ for all $k \geq N$, then x is nonincreasing for $k \geq N$. Otherwise there is some $m \geq N$ such that $\Delta x_m > 0$. In this case, $x_{m+1} > x_m \geq 0$ which implies $x_{m+1}p_m\Delta x_m >$

Received March 7, 1989; revised June 3, 1989.

0. But then $\Delta(x_{m+1}p_m\Delta x_m) \geq 0$ implies $x_{m+2}\Delta x_{m+1} > 0$. Since $x_{m+2} \geq 0$, we have $\Delta x_{m+1} > 0$. By similar arguments, we show by induction that $\Delta x_k > 0$ for $k > m$. It is clear that if x is an eventually nonnegative nonincreasing then it is an eventually positive nonincreasing. The case that $x_k < 0$ for $k \geq N$ is similarly proved.

2. It is not clear whether (1) has a bounded solution. However, if f is continuous, then it can be shown that (1) has a positive nonincreasing solution so that a bounded solution exists. The proof of this statement is similar to that of Theorem 3.2 in [3] and is thus omitted. It is also not known what conditions are sufficient or necessary for the boundedness of all solutions of (1). Some of these conditions will be given below.

Theorem 2. *Suppose f is nondecreasing, $f(x)/x$ is nonincreasing for $x > 0$, $q_k \neq 0$ for infinitely many k, and*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{q_k}{p_n} < \infty, \tag{2}$$

then every eventually positive solution of (1) is bounded.

Proof. Suppose x is an eventually positive and unbounded solution of (1). By Theorem 1, we may assume without loss of generality that $x_k > 0$ and $\Delta x_k > 0$ for $k \geq N$. From (1), we have

$$\begin{aligned} q_k &= \frac{\Delta(p_{k-1}\Delta x_{k-1})}{f(x_k)} = \frac{p_k\Delta x_k}{f(x_k)} - \frac{p_{k-1}\Delta x_{k-1}}{f(x_k)} \\ &\geq \frac{p_k\Delta x_k}{f(x_k)} - \frac{p_{k-1}\Delta x_{k-1}}{f(x_{k-1})} = \Delta\left\{\frac{p_{k-1}\Delta x_{k-1}}{f(x_{k-1})}\right\}, \quad k > N. \end{aligned} \tag{3}$$

Summing both sides of (3) from $k = N+1$ to $k = m$, we obtain

$$\sum_{k=N+1}^{m} q_k + \frac{p_N\Delta x_N}{f(x_N)} \geq \frac{p_m\Delta x_m}{f(x_m)}. \tag{4}$$

Since $f(x)/x$ is nonincreasing for $x > 0$, we have from (4) that

$$\frac{\Delta x_m}{x_m} \leq \frac{f(x_N)\Delta x_m}{x_N f(x_m)} \leq \frac{f(x_N)}{x_N}\sum_{k=N+1}^{m}\frac{q_k}{p_m} + \frac{p_N\Delta x_N}{p_m x_N}. \tag{5}$$

Let $g(t) = x_m + (t-m)\Delta x_m$ for $m \leq t \leq m+1$. Then $g'(t) = \Delta x_m$ and $g(t) \geq x_m$ for $m \leq t \leq m+1$. Hence

$$\frac{\Delta x_m}{x_m} = \int_m^{m+1}\frac{g'(t)}{x_m}dt \geq \int_m^{m+1}\frac{g'(t)}{g(t)}dt = \log(x_{m+1}) - \log(x_m). \tag{6}$$

If we now sum $\Delta x_m/x_m$ from $m = N+1$ to $m = n$, then (5) and (6) imply

$$\log(x_{n+1}) - \log(x_{N+1}) \le \sum_{m=N+1}^{n} \frac{\Delta x_m}{x_m}$$
$$\le \frac{f(x_N)}{x_N} \sum_{m=N+1}^{n} \sum_{k=N+1}^{m} \frac{q_k}{p_m} + \frac{p_N \Delta x_N}{x_N} \sum_{m=N+1}^{n} \frac{1}{p_m}.$$

Next choose $T \ge N+1$ such that $q_T \ne 0$, then by (2)

$$q_T \sum_{n=T}^{\infty} \frac{1}{p_n} \le \sum_{n=1}^{\infty} \frac{1}{p_n} \sum_{k=1}^{\infty} q_k < \infty,$$

so that $\{\log(x_n)\}$ is bounded. This contradiction completes our proof.

We remark that if the assumption, that $f(x)/x$ is nonincreasing for $x > 0$, is replaced by the assumption that $f(x)/x$ is nondecreasing for $x < 0$, then we may conclude that every eventually negative solution of (1) is bounded. Analogous remarks also hold for the following theorem.

Theorem 3. *Suppose $f(x)/x$ is nonincreasing for $x > 0$, $q_k \ne 0$ for infinitely many k and that*

$$\sum_{n=1}^{\infty} q_n \sum_{k=1}^{\infty} \frac{1}{p_k} < \infty, \tag{7}$$

then every eventually positive solution of (1) is bounded.

Proof. Assume that x is an eventually positive solution of (1). We first show that the sequence $\{p_k \Delta x_k\}$ is bounded. Indeed, if x is eventually nonincreasing, then the sequence $\{p_k \Delta x_k\}$ is eventually nonpositive and nondecreasing since $\Delta(p_k \Delta x_k) = q_{k+1} f(x_{k+1}) \ge 0$ for all large k. If x is eventually increasing such that $x_k > 0$ and $\Delta x_k > 0$ for $k \ge N$. Then

$$\Delta\left\{\frac{x_{k+1}}{p_k \Delta x_k}\right\} = \frac{p_k \Delta x_k \Delta x_{k+1} - q_{k+1} x_{k+1} f(x_{k+1})}{p_k p_{k+1} \Delta x_k \Delta x_{k+1}} \le \frac{1}{p_{k+1}}$$

for $k \ge N$. Summing both sides from $k = N$ to $k = m-1$, we have

$$\frac{x_{m+1}}{p_m \Delta x_m} \le \frac{x_{N+1}}{p_N \Delta x_N} + \sum_{k=N+1}^{m} \frac{1}{p_k}.$$

Since $f(x)/x$ is nonincreasing for $x \ne 0$, we have from (1) and the above inequality that

$$\frac{\Delta(p_m \Delta x_m)}{p_m \Delta x_m} = \frac{q_{m+1}}{p_m \Delta x_m} f(x_{m+1})$$
$$\le \frac{q_{m+1}}{p_m \Delta x_m} \frac{f(x_{N+1}) x_{m+1}}{x_{N+1}}$$
$$\le \frac{q_{m+1} f(x_{N+1})}{p_N \Delta x_N} + \frac{q_{m+1} f(x_{N+1})}{x_{N+1}} \sum_{k=N+1}^{m} \frac{1}{p_k}.$$

As in the proof of Theorem 2, we then obtain

$$\log(p_{n+1}\Delta x_{n+1}) - \log(p_{N+1}\Delta x_{N+1}) \le \sum_{m=N+1}^{n} \frac{\Delta(p_m \Delta x_m)}{p_m \Delta x_m}$$
$$\le \frac{f(x_{N+1})}{p_N \Delta x_N} \sum_{m=N+1}^{n} q_{m+1} + \frac{f(x_{N+1})}{x_{N+1}} \sum_{m=N+1}^{n} \sum_{k=N+1}^{m} \frac{q_{m+1}}{p_k}.$$

Since (7) implies that

$$\sum_{m=1}^{\infty} q_m < \infty,$$

thus $\{\log(p_k \Delta x_k)\}$ and $\{p_k \Delta x_k\}$ are bounded.

Now that

$$\Delta x_k \le \frac{M}{p_k}, \quad k \ge N$$

where M is an upper bound for the sequence $\{p_k \Delta x_k\}$, we have by summing from $k = N+1$ to $k = n$ that

$$x_{n+1} \le x_{N+1} + \sum_{k=N+1}^{n} \frac{M}{p_k}.$$

Again since (7) implies

$$\sum_{k=N+1}^{\infty} \frac{1}{p_k} < \infty,$$

the solution x must be bounded. Q.E.D.

Theorem 4. *Suppose $f(x)$ is nonincreasing for $x > 0$, $q_k \neq 0$ for infinitely many k and (2) holds. Then every eventually positive solution of (1) is bounded.*

Proof. Assume that x is an eventually positive solution of (1) and that $x_k > 0$ and $\Delta x_k > 0$ for $k \ge N$. Since $f(x_{k+1}) \le f(x_k)$ for $k \ge N$, we have

$$\Delta(p_{k-1}\Delta x_{k-1}) = q_k f(x_k) \le q_k f(x_N), \quad k \ge N$$

so that

$$\Delta x_n \le p_n^{-1} p_N \Delta x_N + f(x_N) p_n^{-1} \sum_{k=N+1}^{n} q_k, \quad n > N$$

and

$$x_{m+1} \le x_{N+1} + p_N \Delta x_N \sum_{n=N+1}^{m} p_n^{-1} + f(x_N) \sum_{n=N+1}^{m} p_n^{-1} \sum_{k=N+1}^{n} q_k.$$

Since (2) implies

$$\sum_{n=N+1}^{m} p_k^{-1} < \infty,$$

thus x is bounded.

We remark that if the assumption, that $f(x)$ is nonincreasing for $x > 0$, is replaced by the assumption that $f(x)$ is nondecreasing for $x < 0$, then we may conclude that every eventually negative solution of (1) is bounded.

3. We now turn our attention to unbounded solutions. Note first that, in view of Theorem 1, an unbounded solution is either an eventually positive increasing or negative decreasing solution.

Theorem 5. *Suppose f is nondecreasing, $q_k \neq 0$ for infinitely many k, and*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{q_k}{p_n} = \infty \tag{8}$$

then every eventually positive increasing or negative decreasing solution x of (1) is unbounded.

Proof. Suppose $x_k > 0$ and $\Delta x_k > 0$ for $k \geq N$. Since

$$\Delta(p_{k-1}\Delta x_{k-1}) = q_k f(x_k) \geq q_k f(x_N), \quad k \geq N \tag{9}$$

by summing both sides from $k = N+1$ to $k = n$, we obtain

$$\Delta x_n \geq p_n^{-1} p_N \Delta x_N + p_n^{-1} f(x_N) \sum_{k=N+1}^{n} q_k, \quad n \geq N+1. \tag{10}$$

Again, by summing both sides of (10) from $n = N+1$ to $n = m$, we obtain

$$x_{m+1} \geq x_N + p_N \Delta x_N \sum_{n=N+1}^{m} p_n^{-1} + f(x_N) \sum_{n=N+1}^{m} p_n^{-1} \sum_{k=N+1}^{n} q_k, \tag{11}$$

which, in view of (8), implies $x_m \to \infty$ as $m \to \infty$. Q.E.D.

The assumption that $q_k \neq 0$ for infinitely many k can be removed if a stronger condition than (8) is assumed.

Theorem 6. *Suppose*

$$\sum_{k=1}^{\infty}\frac{1}{p_k} = \infty, \tag{12}$$

then every eventually positive increasing or negative decreasing solution x of (1) is unbounded.

Proof. Suppose $x_k > 0$ and $\Delta x_k > 0$ for $k \geq N$, then similar to (11), we may derive the inequality

$$x_{m+1} \geq x_N + p_N \Delta x_N \sum_{n=N}^{m} p_n^{-1}.$$

our assertion now follows from (12).

When $f(x) = x$ Theorems 2 and 5 imply the following consequence which has been derived previously [1, Theorem 4].

Corollary. *Suppose $f(x) = x$ and $q_k \neq 0$ for infinitely many k, then every solution of (1) is bounded if and only if (2) holds.*

The above Corollary, since it contains a necessary and sufficient condition, indicates that Theorems 2 and 6 are sharp.

References

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