TAMKANG JOURNAL OF MATHEMATICS
Volume 40, Number 2, 165-172, Summer 2009

COMPLEMENTARY NIL DOMINATION NUMBER OF A GRAPH

T. TAMIZH CHELVAM AND S. ROBINSON CHELLATHURAI

Abstract. A set $S \subseteq V$ is said to be a complementary nil dominating set of a graph G if it is a dominating set and its complement $V - S$ is not a dominating set for G. The minimum cardinality of a *cnd*-set is called the complementary nil domination number of G and is denoted by $\gamma_{\rm cnd}(G)$. In this paper some results on the complementary nil domination number are obtained.

1. Introduction

A set $S \subseteq V$ of vertices of a simple graph $G = (V, E)$ is a dominating set if for every vertex v in $V - S$, there exists a vertex u in S such that v is adjacent to u. The minimum cardinality of a dominating set in G is called the domination number of G and is denoted by $\gamma(G)$ [2]. Similarly the maximum cardinality of a minimal dominating set of a graph G is called the upper domination number $\Gamma(G)$[2]. A dominating set D of a graph G is a split dominating set if the induced subgraph $<V - D>$ is disconnected. The split domination number $\gamma_s(G)$ is the minimum cardinality of a split dominating set [4]. The uniform domination number $\gamma_u(G)$ of a graph G is the least positive integer k such that any set with k vertices is a dominating set of G [5]. Let $S \subseteq V$. Then a vertex $v \in S$ is said to be an enclave of S if $N[v] \subseteq S$. The graphs considered here are finite, undirected, without loops or multiple edges and connected with p vertices and q edges. The corona of two graphs G and H is the graph GoH formed from one copy of G and $|V(G)|$ copies of H where the i^{th} vertex of G is adjacent to every vertex in the i^{th} copy of H. Any undefined terms in this paper may be found in Harary [1].

Definition 1.1. A set $S \subseteq V$ is said to be a *cnd*-set of a graph G if it is a dominating set and its complement $V - S$ is not a dominating set. The minimum cardinality of a *cnd*-set is called the complementary nil domination number of G and is denoted by $\gamma_{\rm cnd}(G)$.

Corresponding author: T. Tamizh Chelvam.
Received April 12, 2007; revised September 23, 2008.
2000 *Mathematics Subject Classification*. 05C.
Key words and phrases. Domination number, split domination number, independence number, uniform domination number and enclave.

Hereafter by a *cnd*-set we mean a complementary nil dominating set. We note that γ_{cnd} sets exist if and only if the graph is not complete. Here after, we assume that G is a non-complete connected graph.

2. Characterization of complementary nil dominating sets

First let us prove some basic results on the newly introduced parameter.

Lemma 2.1. *Let S be a cnd-set of a graph G. Then S contains at least one enclave of S.*

Proof. Let S be a *cnd*-set of a graph G. By Definition 1.1, $V-S$ is not a dominating set, which implies that there exists a vertex $v \in S$ such that v is not adjacent to any vertex in $V-S$ and so $N[v] \subseteq S$.

Remark 2.2. The enclave of a γ_{cnd}-set need not be unique. For example, in the graph $C_5 o C_3$, every γ_{cnd}-set contains three enclaves.

Proposition 2.3. *Let G be a graph and S be a γ_{cnd}-set. If u and v are two enclaves of S, then $N[u] \cap N[v] \neq \phi$ and u and v are adjacent.*

Proof. Let u and v be two enclaves of S. Suppose $N[u] \cap N[v] = \phi$. Then u is an enclave of $S - N$(v). Clearly $S - N(v)$ is a *cnd*-set of G and $|S - N(v)| < |S| = \gamma_{\text{cnd}}(G)$, which is a contradiction to the minimality of S. Hence $N[u] \cap N[v] \neq \phi$. Suppose u and v are non-adjacent, $u \notin N(v)$ and so $S - \{v\}$ contains an enclave u of $S - \{v\}$. Hence $S - \{v\}$ is a *cnd*-set, which is a contradiction to the minimality of S.

Since the complement of an independent set is a dominating set, we get the following.

Proposition 2.4. *A cnd-set of a graph G is not an independent dominating set.*

The following result is used in the sequel.

Theorem 2.5.([2]) *A dominating set S is a minimal dominating set if and only if for each vertex $u \in S$, one of the following two conditions holds:*

(a) *u is an isolate of S,*

(b) *there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$.*

Theorem 2.6. *Let S be a cnd-set of a graph G. Then S is minimal if and only if for each vertex $u \in S$ one of the following conditions is satisfied.*

(i) *u has a private neighbour*

(ii) *$V - (S - \{u\})$ is a dominating set of G.*

Proof. Suppose S is minimal. On the contrary if there exists a vertex $u \in S$ such that u does not satisfy any of the given conditions (i) and (ii), then by Theorem 2.5,

$S_1 = S - \{u\}$ is a dominating set of G. Also by (ii), $V - (S - \{u\})$ is not a dominating set. This implies that S_1 is a *cnd*-set of G, which is a contradiction.

Conversely, suppose that S is a *cnd*-set and for each vertex $u \in S$, one of the two stated conditions holds. We show that S is a minimal *cnd*-set of G. On the contrary, we assume that S is not a minimal *cnd*-set. That is, there exists a vertex $u \in S$ such that $S - \{u\}$ is a *cnd* of G. Hence u is adjacent to atleast one vertex in $S - \{u\}$. Also $S - \{u\}$ is a dominating set, every vertex in $V - S$ is adjacent to atleast one vertex in $S - \{u\}$. That is, condition (i) does not hold. Since $S - \{u\}$ is a *cnd*-set, $V - (S - \{u\})$is not a dominating set. That is condition (ii) does not hold. Therefore there exists a vertex $u \in S$ such that which does not satisfy conditions (i) and (ii), a contradiction to the assumption.

Theorem 2.7. *For any graph G, every γ_{cnd}-set intersects with every γ-set of G.*

Proof. Let S_1 be a γ_{cnd}- set and S be a γ-set of G. Suppose that $S_1 \cap S = \phi$, then S $\subseteq V - S_1, V - S_1$ contains a dominating set S. Therefore $V - S_1$ itself is a dominating set, which is a contradiction.

Corollary 2.8. *In any graph G, any two γ_{cnd}-sets intersect.*

Proposition 2.9. *Let S be a cnd-set of a graph G. Then there exists a vertex v in S such that v has no private neighbour.*

Proof. Since complement of a minimal dominating set is a dominating set, S cannot be a minimal dominating set of G. So there exists a vertex v in S such that v has no private neighbour.

3. Bounds for complementary nil domination number

In this section, we obtain some bounds for the *cnd*-number of graphs.

Theorem 3.1. *For any graph G, $\delta + 1 \leq \gamma_{\text{cnd}}(G) \leq \gamma(G) + \delta$.*

Proof. Let S be a γ_{cnd}-set of G. Since $V - S$ is not a dominating set, there exists a vertex $v \in S$ which is not adjacent to any of the vertices in $V - S$. Therefore $N[v] \subseteq S$ which implies that $|N[v]| \leq |S|$, that is $d(v)+1 \leq |S|$ and so $\delta + 1 \leq \gamma_{\text{cnd}}(G)$. Let S_1 be a γ i set of G. Let $u \in V$ such that $d(u) = \delta$. Then atleast one vertex $u_1 \in N[u]$ such that $u_1 \in S_1$. Now $S_1 \cup (N[u] - \{u_1\})$ is a *cnd*-set of G, which implies that $\gamma_{\text{cnd}}(G) \leq |S_1 \cup (N[u] - \{u_1\})| \leq |S_1| + |N[u] - \{u_1\}| = \gamma + \delta$. Therefore $\delta + 1 \leq \gamma_{\text{cnd}}(G) \leq \gamma + \delta$.

In view of the above Theorem 3.1, we have the following corollaries.

Corollary 3.2. *For any graph G with $\delta = 1, \gamma_{\text{cnd}}(G) = \gamma(G) + 1$.*

Since, for any graph G, $\gamma(G) \leq p - \Delta[2]$, we have the following corollary.

Corollary 3.3. *For any graph G, $\gamma_{\text{cnd}}(G) \leq p - \Delta + \delta$.*

Observation 3.4.
(i) *For any graph $G, \gamma(G) < \gamma_{\text{cnd}}(G)$.*
(ii) *For any tree $T, \gamma_{\text{cnd}}(T) = \gamma(T) + 1$. In particular $\gamma_{\text{cnd}}(P_p) = \gamma(P_p) + 1$.*
(iii) $\gamma_{\text{cnd}}(C_p) = \begin{cases} \gamma(C_p) + 1 \text{ if p is not a multiple of 3.} \\ \gamma(C_p) + 2 \text{ if p is a multiple of 3.} \end{cases}$
(iv) $\gamma_{\text{cnd}}(W_p) = 4, p \geq 5$, *where W_p is a wheel with p vertices.*
(v) $\gamma_{\text{cnd}}(K_{m,n}) = min\{m, n\} + 1$.
(vi) $\gamma_{\text{cnd}}(K_p - \{e\}) = p - 1$, *where e is an edge in* $\mathrm{K_p}$.
(vii) $\gamma_{\text{cnd}}(\overline{mK_2}) = 2m - 1$, *for $m > 1$.*
(viii) $2 \leq \gamma_{\text{cnd}}(G) \leq p - 1$, *for $p \geq 3$.*

One can easily prove the following propositions.

Proposition 3.5. *For* $\mathrm{k} > 1$
(i) $\gamma_{\text{cnd}}(P_2 \times P_k) = \begin{cases} \gamma(P_2 \times P_k) + 1 \text{ if k is even} \\ \gamma(P_2 \times P_k) + 2 \textit{ if k is odd} \end{cases}$
(ii) $\gamma_{\text{cnd}}(C_3 \times P_k) = \gamma(C_3 \times P_k) + 2$.

Proposition 3.6.
(i) $\gamma_{\text{cnd}}(\overline{K_{m,n} - e}) = m + 1$ *for $m \leq n$, where e is an edge in $K_{m,n}$.*
(ii) $\gamma_{\text{cnd}}(\bar{P}_p) = p - 2$ *for $p > 4$ and* $\gamma_{\text{cnd}}(\bar{P}_4) = 3$.
(iii) $\gamma_{\text{cnd}}(\bar{C}_p) = p - 2$ *for $p > 4$.*

Theorem 3.7. *For any graph G with $p > 1, \lceil \frac{p}{\Delta+1} \rceil < \gamma_{\text{cnd}}(G) \leq 2q - p + 1$. Also if $\gamma_{\text{cnd}}(G) = 2q - p + 1$ then G is a tree.*

Proof. Since $\lceil \frac{p}{\Delta+1} \rceil \leq \gamma(G) < \gamma_{\text{cnd}}(G)$, the first inequality follows. For any graph G, $\gamma_{\text{cnd}}(G) \leq p - 1 = 2(p - 1) - p + 1 \leq 2q - p + 1$.
Also if $\gamma_{\text{cnd}}(G) = 2q - p + 1$. Then $2q - p + 1 \leq p - 1$ and so $q \leq p - 1$. Hence $q = p - 1$. Therefore G must be a tree.

Theorem 3.8. *Let G be a graph such that both* G *and its complement $\bar{G}$ are connected. Then $\gamma_{\text{cnd}}(G)+\gamma_{\text{cnd}}(\bar{G}) \leq (p - 1)(p - 2)$. Equality holds for $G = P_4$.*

Proof. By Theorem 3.7, $\gamma_{\text{cnd}}(G) \leq 2q - p + 1$ and $\gamma_{\text{cnd}}(\bar{G}) \leq 2q - p + 1$, then $\gamma_{\text{cnd}}(G) + \gamma_{\text{cnd}}(\bar{G}) \leq 2(q + q) - 2(p - 1) = p(p - 1) - 2(p - 1) = (p - 1)(p - 2)$.

Theorem 3.9. *Let G_1 and G_2 be two connected graphs. Then $\gamma_{\text{cnd}}(G_1 \circ G_2) = |V(G_1)| + \delta(G_1 \circ G_2)$.*

Proof. Clearly $V(G_1)$ is a $\gamma(G_1 \circ G_2)$-set. We choose a vertex v in $G_1 \circ G_2$ such that $d(v) = \delta(G_1 \circ G_2)$. Then v must be one of the vertices in $V(G_2)$. Let u be the vertex in $V(G_1)$ adjacent to v. Now $(V(G_1) - \{u\}) \cup N[v]$ is a γ_{cnd}-set of $G_1 \circ G_2$ and $\gamma_{\text{cnd}}(G_1 \circ G_2) = |(V(G_1) - \{u\}) \cup N[v]| = |V(G_1)| + \delta(G_1 \circ G_2)$.

In view of Theorem 3.9, we have the following corollaries.

Corollary 3.10. *Let G_1 be any connected graph and G_2 be a complete graph. Then* $\gamma_{\text{cnd}}(G_1 \circ G_2) = |V(G_1)| + |V(G_2)|$.

Corollary 3.11. *For any connected graph H, $\gamma_{\text{cnd}}(HoK_1) = |V(H)| + 1$.*

Theorem 3.12. *Let T be a tree with diam $(T) > 2$. Then $\gamma_{\text{cnd}}(\bar{T}) \leq p + 1 - \Delta(T)$. Equality holds for $G = P_4$.*

Proof. Since $\gamma(\bar{T}) = 2$, then $\gamma_{\text{cnd}}(\bar{T}) \leq \gamma(\bar{T}) + \delta(\bar{T}) = 2 + p - 1 - \Delta(T)$. Hence $\gamma_{\text{cnd}}(\bar{T}) \leq p + 1 - \Delta(T)$.

Theorem 3.13. *Let $\bar{G}$ be the connected complement of a graph G. Then $\gamma_{\text{cnd}}(G) + \gamma_{\text{cnd}}(\bar{G}) \leq \frac{3p}{2} + 1 + \delta(G) - \Delta(G)$. Equality holds for $G = P_4$.*

Proof. From Theorem 3.1, $\gamma_{\text{cnd}}(G) \leq \gamma(G) + \delta(G)$ and $\gamma_{\text{cnd}}(\bar{\text{G}}) \leq \gamma(\bar{\text{G}}) + \delta(\bar{\text{G}}) = \gamma(\bar{\text{G}}) + \text{p} - 1 - \Delta(\text{G})$. Thus $\gamma_{\text{cnd}}(\text{G}) + \gamma_{\text{cnd}}(\bar{\text{G}}) \leq \gamma(\text{G}) + \gamma(\bar{\text{G}}) + \text{p} - 1 + \delta(\text{G}) - \Delta(\text{G})$. Since $\gamma(G) + \gamma(\bar{G}) \leq \frac{p}{2} + 2$ (by Theorem 9.5 [2]), from the above, we have $\gamma_{\text{cnd}}(G) + \gamma_{\text{cnd}}(\bar{G}) \leq \frac{p}{2} + 2 + p - 1 + \delta(G) - \Delta(G)$. Hence $\gamma_{\text{cnd}}(G) + \gamma_{\text{cnd}}(\bar{G}) \leq \frac{3p}{2} + 1 + \delta(G) - \Delta(G)$.

Proposition 3.14. *For any tree T, $\gamma_{\text{cnd}}(T) + \in (T) \leq p + 1$, where $\in (T)$ is the number of pendant vertices in T. Equality holds for $G = K_{1,p-1}$, where $p > 2$.*

Proof. All the non pendant vertices together a pendant vertex form a complementary nil dominating set. Therefore $\gamma_{\text{cnd}}(T) \leq p - \in (T) + 1$ and so $\gamma_{\text{cnd}}(T) + \in (T) \leq p + 1$.

Proposition 3.15. *For any graph G, $\gamma_s(G) < \gamma_{\text{cnd}}(G)$.*

Proof. Let S be a γ_{cnd}-set of G. Then there exists a vertex $v \in S$ such that $N[v] \subseteq S$. Clearly $S - \{v\}$ is a split dominating set and so $|S - \{v\}| \geq \gamma_s(G)$. Therefore $\gamma_{\text{cnd}}(G) - 1 \geq \gamma_s(G)$. Hence $\gamma_s(G) \leq \gamma_{\text{cnd}}(G) - 1 < \gamma_{\text{cnd}}(G)$.

Theorem 3.16. *For any graph G, $\Gamma(G) + \gamma_{\text{cnd}}(G) \leq p + 1$.*

Proof. Let S be Γ-set of G. Then there exists a vertex $v \in S$ such that $S - \{v\}$ is not a dominating set of G. But $V - (S - \{v\})$ is a dominating set and so $V - (S - \{v\})$ is a *cnd*-set. Therefore $|(V - S) \cup \{v\}| \geq \gamma_{\text{cnd}}(G)$. Hence $\Gamma(G) + \gamma_{\text{cnd}}(G) \leq p + 1$.

Remark 3.17. In the above Theorem 3.16, the bound $\Gamma(\text{G}) + \gamma_{\text{cnd}}(\text{G}) = \text{p} + 1$ is reachable for the following graphs.

(i) $\text{K}_\text{p} - \{\text{e}_1, \text{e}_2, \ldots\ldots, \text{e}_\text{k}\}$ where $1 \leq k \leq \lfloor p/2 \rfloor$, e_i's are independent edges, $p > 2$.

(ii) All the trees with every non-end vertex is adjacent to atleast one end vertex.

4. Particular values of complementary nil domination number

Theorem 4.1. *For any graph G,* $\gamma_{\rm cnd}({\rm G}) = 2$ *if and only if* $\gamma({\rm G}) = 1$ *and* $\delta = 1$.

Proof. Suppose $\gamma_{\rm cnd}({\rm G}) = 2$. Then $\gamma({\rm G}) < \gamma_{\rm cnd}({\rm G}) = 2$, implies $\gamma({\rm G}) = 1$. Therefore there exits a vertex u in G such that ${\rm d(u)} = {\rm p} - 1$. If $\delta > 1$, since $\delta + 1 \ \leq \gamma_{\rm cnd}(G)$, which implies that $\gamma_{\rm cnd}(G) > \ 2$, which is a contradiction to $\gamma_{\rm cnd}(G) = 2$. Converse follows from Theorem 3.1.

Theorem 4.2. *Let G be a graph with* $\delta > 1$ *and* $p > 3$. *Then* $\gamma_{\rm cnd}({\rm G}) = {\rm p} - 1$ *if and only if* $\delta = {\rm p} - 2$.

Proof. Suppose that $\gamma_{\rm cnd}(G) = p - 1$. On the contrary, let us assume that $\delta \leq p - 3$. Then there exists a vertex $v \in V(G)$ such that v is not adjacent to atleast two vertices say u, w in $V(G)$. Then $V - \{u, w\}$ is a *cnd*-set. So $|V - \{u, w\}| \geq \gamma_{\rm cnd}(G)$. Hence $\gamma_{\rm cnd}(G) \leq p-2$, which is a contradiction to the hypothesis. Conversely if $\delta = {\rm p} - 2$, there exists a vertex v in G with ${\rm d(v)} = {\rm p} - 2$ and N[v] is a *cnd*-set. Therefore $|{\rm N[v]}| \geq \gamma_{\rm cnd}({\rm G})$, so $\delta + 1 \geq \gamma_{\rm cnd}({\rm G})$. By Theorem 3.1, $\gamma_{\rm cnd}({\rm G}) = \delta + 1$. Hence $\gamma_{\rm cnd}({\rm G}) = {\rm p} - 1$.

Corollary 4.3.
(i) *Let G be a graph with* $\delta > 1$ *and* ${\rm p} > 3$. *Then* $\gamma_{\rm cnd}(G) = p - 1$ *if and only if* $G = K_p - \{e_1, e_2, \ldots, e_k\}$ *where* $1 \leq {\rm k} \leq \lfloor {\rm p}/2 \rfloor$, ${\rm e_i}$*'s are independent edges.*
(ii) *Let G be a graph with* $\delta = 1$ *and* ${\rm p} \geq 5$. *Then* $\gamma_{\rm cnd}({\rm G}) \leq {\rm p} - 2$.

Remark 4.4.
(i) When $\delta = 1$, Theorem 4.2 need not be true, as can be seen from the graph P_4.
(ii) In view of Corollary 4.3, for $\delta = 1$, only graphs satisfying $\gamma_{\rm cnd}({\rm G}) = {\rm p} - 1$ are P_3 and P_4.

One can easily prove the following propositions.

Proposition 4.5. *Let G be graph with* $\delta = 1$. *Then there exists a* $\gamma_{\rm cnd}$*-set which contains atleast one pendant vertex.*

Proposition 4.6. *Let T be a tree. Then there exists a* $\gamma_{\rm cnd}$*-set which contain all the supports and exactly one pendant vertex. Hence* ${\rm s} + 1 \ \leq \gamma_{\rm cnd}({\rm G})$, *where* s *is the number of supports in T.*

In view of Proposition 4.6, we have the following corollary.

Corollary 4.7. *Let T be a tree such that every non-end vertex is adjacent to atleast one end vertex. Then* $\gamma_{\rm cnd}(G)$.=s+1

Theorem 4.8. *Let G be a graph with* $diam(G) \geq 3$. *Then* $\gamma_{\rm cnd}(G) \leq p - \delta$.

Proof. Let $v \in V$ with $d(v) = \delta$. Since $diam(G) \geq 3$, there exists a vertex $u \in V - N[v]$ but u is not adjacent to any vertex in $N[v]$. Now, $V - N(v)$ is a dominating set but N(v) is not a dominating set. Therefore $|V - N(v)| \geq \gamma_{\rm cnd}(G)$ and $\gamma_{\rm cnd}(G) \leq p - \delta$.

For any graph G, $\gamma_u(G) = p - \delta$ [5] and hence we have the following corollary.

Corollary 4.9. *Let G be a graph with diam* (G) ≥ 3. *Then* $\gamma_{\rm cnd}(\rm G) \leq \gamma_{\rm u}(\rm G)$.

Remark 4.10. Corollary 4.9 fails if $diam(G) = 2$. For consider $\rm K_{2n} - X$ where X is a 1-factor in $\rm K_{2n}$. Here $\gamma_{\rm cnd} = 2n - 1$ for $n > 1$ but $\gamma_{\rm u} = 2$.

Theorem 4.11. *Let G be a graph. Then diam* (G) $= 2$ *if and only if* $\gamma_{\rm cnd}(\rm G) = \delta + 1$.

Proof. Let v be a vertex in G such that $\rm d(v) = \delta$, since G is not complete $\delta \leq p - 2$. Suppose diam (G) $= 2$, every vertex $\rm u \notin N(v)$ must be adjacent to some vertex in N(v). Therefore N[v] is a complement nil dominating set of G. So $|\rm N[v]| \geq \gamma_{\rm cnd}, \delta + 1 \geq \gamma_{\rm cnd}(G)$. Also by Theorem 3.11, $\delta + 1 \leq \gamma_{\rm cnd}(\rm G)$. Hence $\gamma_{\rm cnd}(\rm G) = \delta + 1$. Convesely, suppose that $\gamma_{\rm cnd}(\rm G) = \delta + 1$. Then there exists a vertex $\rm v \in G$ with $\rm d(v) = \delta$ such that N[v] is a $\gamma_{\rm cnd}$-set. Therefore every vertex not in N(v) must be adjacent to some vertex in N(v) and so diam (G) $= 2$.

In view of the above Theorem 4.11 and $\gamma_u(G) = p - \delta$ [5], we have the following corollary.

Corollary 4.12. *Let G be a graph. Then diam* (G) $= 2$ *if and only if* $\gamma_{\rm cnd}(\rm G) + \gamma_{\rm u}(\rm G) = p + 1$.

Proposition 4.13. *Let G be a bipartite graph with its complement* $\bar{\rm G}$ *connected. Then* $\gamma_{\rm cnd}(\bar{\rm G}) = \rm p - \Delta(G)$ *or* $\rm p - \Delta(G) + 1$.

Proof. Let (X, Y) be a partition of G. In $\bar{G}$, $<X>$ and $<Y>$ are complete. Let u be a vertex in $\bar{\rm G}$ such that $\delta(\bar{\rm G}) = \rm d(u)$. Without loss of generality we may assume that u is in X. If $\rm N[u] \cap Y \neq \phi$, then $\gamma_{\rm cnd}(\bar{\rm G}) = |\rm N[u]| = \delta(\bar{\rm G}) + 1$, since $\delta(\bar{\rm G}) = \rm p - 1 - \Delta(G)$, $\gamma_{\rm cnd}(\bar{\rm G}) = p - \Delta(G)$. If $\rm N[u] \cap Y = \phi$, then $N[u]$ together with a vertex from Y is a $\gamma_{\rm cnd}$-set of $\bar{\rm G}$. In this case $\gamma_{\rm cnd}(\bar{\rm G}) = \delta(\bar{\rm G}) + 2, \gamma_{\rm cnd}(\bar{\rm G}) = \rm p - \Delta(G) + 1$.

Theorem 4.14. *For any graph G, if* $\gamma(\rm G) = \frac{p}{2}$, *then* $\gamma_{\rm cnd}(\rm G) = \frac{p}{2} + 1$.

Proof. Suppose $\gamma(\rm G) = \frac{p}{2}$. Let S be a γ-set of G, which implies $V - S$ is a dominating set with $|V - S| = \frac{p}{2}$. For any vertex $x \in V - S, (V - S) - \{x\}$ is not a dominating set. But $S \cup \{x\}$ is a dominating set and so $S \cup \{x\}$ is a cnd-set. Therefore $|S \cup \{x\}| \geq \gamma_{\rm cnd}(G)$. So $\gamma(\rm G) + 1 \geq \gamma_{\rm cnd}(\rm G)$. By Theorem 3.1, $\gamma_{\rm cnd}(\rm G) = \gamma(\rm G) + 1$.

Remark 4.15. Converse of the above Theorem 4.14 is not true. For consider C_6. Here $\gamma_{\rm cnd}(C_6) = 4 = \frac{p}{2} + 1$, but $\gamma(C_6) = 2 \neq \frac{p}{2}$. Assume that $\delta = 1$ and $\gamma_{\rm cnd}(G) = \frac{p}{2} + 1$. By Corollary 3.2, $\gamma(G) + 1 = \gamma_{\rm cnd}(G) = \frac{p}{2} + 1$ and so $\gamma(G) = \frac{p}{2}$.

Acknowledgement

This work is supported by the University Grants Commission, New Delhi, under the Faculty Improvement Programme for the second author.

References

[1] F. Harary, Graph Theory, Addision Wesely, Reading M.A., 1969.

[2] Teresa W. Haynes, Stephen T. Hedetnimi, Peter. J. Slater, Fundamentals of Domination in Graphs, Marcel Decker, 1998.

[3] J. Paulraj Joseph and S. Arumugam, *Domination in graphs*, International J. Management Systems **11**(1995), 177–182.

[4] V.R. Kulli and B.Janakiram, *The split domination number of a graph*, Graph Theory Notes of New York, New York Academy of Sciences **XXXII**(1997), 16–19.

[5] S. Arumugam and J. Paulraj Joseph, *The uniform domination number of a graph*, International J. Management. Systems **11**(1995), 111–116.

Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli 627 012, Tamil Nadu, India.

E-mail: tamche_59@yahoo.co.in

Department of Mathematics, Scott Christian College, Nagercoil 629 003, Tamil Nadu, India.

E-mail: robinchel@rediffmail.com