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CR-SUBMANIFOLDS OF TWO DIMENSIONAL COMPLEX PROJECTIVE SPACE

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1.

Among all submanifolds of a Kaehler manifold there are three typical classes: the complex submanifolds, the totally real submanifolds and the CR-submanifolds. The notion of a CR-submanifold of a Kaehler manifold was introduced by Bejancu [1] and it includes the other two classes as special cases. If $\overline{M}$ is a Kaehler manifold with complex structure J and M is a submanifold of $\overline{M}$, M is called a *CR-submanifold* of $\overline{M}$ if there exists a pair of orthogonal complementary distributions D and $D^{\perp}$ on M satisfying $JD = D$ and $JD^{\perp} \subset \nu$, where ν is the normal bundle of M.

It is known that every compact and orientable 3-manifold M possesses a contact structure, that is, M carries a globally defined 1-form η with $\eta \wedge d\eta \neq 0$ everywhere on M [4]. One can associate with η a vector field t determined by $\eta(t) = 1$ and $d\eta(t, X) = 0$ for all vector fields X on M. If, in addition, M is a Riemannian manifold with metric g and η satisfies $\eta(t) = g(X, t)$, then η is called the contact metric stucture on M.

The object of the present paper is to study the 3-dimensional CR-submanifold of the 2-dimensional complex projective space CP^2. It is shown that a simply connected 3-dimensional CR-submanifold M of CP^2 is either a contact manifold or a certain 2-dimensional distribution on M is integrable. We next consider those compact and simply connected CR-submanifolds which admit contact metric structure with respect to the induced metric, and prove that they are either diffeomorphic to S^3 or minimal submanifolds.

2.

Let J be the almost complex structure and g be the metric of constant holomorphic sectional curvature 4 on CP^2. If $\overline{\nabla}$ is the Riemannian connection on CP^2, then we have

$$(\overline{\nabla}_X J)(Y) = 0. \tag{2.1}$$

Let M be a 3-dimentional CR-submanifold of CP^2. Then on M there are two orthogonal complementary distributions, D and $D^{\perp}$, satisfying $JD = D$ and $JD^{\perp} = T^{\perp}M$,

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where $T^{\perp}M$ is the normal line boundle of M (cf. [1]). It is clear that dim $D = 2$ and dim $D^{\perp} = 1$. Suppose N is the unit normal vector field to M and put $\xi = -JN$. Then ξ is a globally defined unit vector field on M which lies in $D^{\perp}$. We shall denote by g both the metric on CP^2 and the induced metric on M. The Riemannian connection $\overline{\nabla}$ of CP^2 induces a Riemannian connection ∇ on M and they are related by the formulae

$$(2.2) \qquad \overline{\nabla}_X Y = \nabla_X Y + h(X,Y),\ \overline{\nabla}_X J\xi = -AX,\ X, Y \in \mathcal{X}(M),$$

where $h(X,Y)$ is the second fundamental form, A is the Weingarten map and $\mathcal{X}(M)$ is the lie-algebra of vector fields on M. We also have the following relations for the hypersurface M :

$$(2.3)\ g(h(X,Y), J\xi) = g(AX,Y)$$

$$(2.4) \qquad \overline{R}(X,Y)J\xi = (\nabla_Y A)(X) - (\nabla_X A)(Y)$$

$$(2.5)\ R(X,Y;Z,W) = \overline{R}(X,Y;Z,W) + g(h(Y,Z), h(X,W) - g(h(X,Z), h(Y,W)),$$
$$X, Y, Z, W \in \mathcal{X}(M),$$

where R is the curvature tensor of M and $\overline{R}$ is the curvature tensor of CP^2 given by

$$(2.6) \qquad \overline{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY$$
$$+ 2g(X,JY)JZ, \qquad X, Y, Z \in \mathcal{X}(CP^2).$$

If the trace of the second fundamental form $h(X,Y)$ is zero, then M is called a minimal submanifold of CP^2.

3.

If we take $X \in D$, add the expansions of the equations $(\overline{\nabla}_{JX}J)(X) = 0$ and $J(\overline{\nabla}_X J)(X) = 0$, and use (2.2) we get

$$(3.1) \qquad \nabla_X X + \nabla_{JX} JX + h(X,X) + h(JX,JX) + J[X,JX] = 0, X \in D.$$

Theorem 3.1 *Let M be a simply connected 3-dimensional CR-submanifold of CP^2. Then there exists a 1-form α on M which either defines a contact structure on M or the distribution given by the kernel of α is integrable.*

Proof. We note that for each unit vector field $e \in D, \{e, Je\}$ is an orthonormal local frame for D. Let $\{\omega^1, \omega^2\}$ be the dual frame of $\{e, Je\}$. We shall now show that the 2-form $\omega^1 \wedge \omega^2$ is independent of the frame $\{e, Je\}$. If $\{X, JX\}$ is another orthonormal frame for D with dual frame $\{p, q\}$, then there exist smooth functions f and g such that, locally, we have $X = fe + gJe$, $JX = -ge + fJe$ and $f^2 + g^2 = 1$. Consequently $p = f\omega^1 + g\omega^2$, $q = -g\omega^1 + f\omega^2$ and $p \wedge q = \omega^1 \wedge \omega^2$. Taking the inner product with ξ in (3.1), since $J\xi$ is normal, we get $g(\nabla_X X + \nabla_{JX} JX, \xi) = 0, X \in D$. And since ξ is a unit vector field we have $\xi \cdot g(\xi,\xi) = 0$, that is, $g(\nabla_\xi \xi, \xi) = 0$. For the orthonormal frame $\{e, Je, \xi\}$ on M, we get $d(\omega^1 \wedge \omega^2)(e, Je, \xi) = 0$, which means that the form $\omega^1 \wedge \omega^2$ is closed. Since M is simply connected the cohomology group $H^2(M,R)$ is trivial. Thus the closed 2-form $\omega^1 \wedge \omega^2$ is exact and there exists a 1-form α on M such that $d\alpha = \omega^1 \wedge \omega^2$. Now consider the set $U = \{p \in M : (\alpha \wedge d\alpha)_p \neq 0\}$, which is an open subset of M. U is orientable as it has a non-vanishing 3-form $\alpha \wedge d\alpha$. The 1-form α defines a contact structure on U, so there exists a vector field t on U determined by $\alpha(t) = 1, d\alpha(t, X) = 0$ for all vector fields X on U (cf. [2]). Since $d\alpha = \omega^1 \wedge \omega^2$, it follows from $d\alpha(t, e) = 0$ and $d\alpha(t, Je) = 0$ that $\omega^1(t) = 0$ and $\omega^2(t) = 0$, that is, the vector field t is parallel to ξ. On the other hand it is not difficult to see that $(\alpha \wedge d\alpha)_p \neq 0$ if and only if $\alpha_p(t) = 1, \omega_p^1(t) = 0$ and $\omega_p^2(t) = 0$. For $\alpha_p(t) = 1, \omega_p^1(t) = 0$ and $\omega_p^2(t) = 0$ imply that $\{t, e, Je\}$ are mutually orthogonal tangent vectors at p and therefore $(\alpha \wedge d\alpha)_p(t, e, Je) = (\alpha \wedge \omega^1 \wedge \omega^2)_p(t, e, Je) = 1 \neq 0$. Hence $U = \{p \in M : \alpha_p(t) = 1, \omega_p^1(t) = 0, \omega_p^2(t) = 0\}$ is a closed subset of M. Thus the set U is both an open and a closed subset of M. M, being simply connected, is connected, and therefore either $U = M$ or U is empty. When $U = M$, α defines a contact structure on M and when U is empty we have $\alpha \wedge d\alpha = 0$ everywhere on M, which is the condition for integrability of the distribution $\{X \in \mathcal{X}(M) : \alpha(X) = 0\}$.

4.

In this section we study the simply connected 3-dimensional CR-submanifolds of CP^2 on which α defines a contact structure. Our main result is

Theorem 4.1 *Let M be a compact and simply connected 3-dimensional CR-submanifold of CP^2. If the 1-form α defines a contact structure on M which is also a contact metric structure with respect to the induced metric on M, then either M is a minimal submanifold or M is diffeomorphic to S^3.*

Proof. Since α defines a contact metric structure on M with respect to the induced metric structure on M, we have $\alpha(X) = g(X, t), X \in \mathcal{X}(M)$. From this it follows that $g(t, t) = 1$, that is, t is a unit vector field which is parallel to ξ, and thus $t = \xi$. If η is a 1-form dual to ξ, then we get $\alpha = \eta$. Since $d\eta = \omega^1 \wedge \omega^2$, from $d\eta(\xi, e) = 0$ and $d\eta(\xi, Je) = 0$, we get $g(\nabla_\xi \xi, e) = 0$ and $g(\nabla_\xi \xi, Je) = 0$. As we already have $g(\nabla_\xi \xi, \xi) = 0$ we conclude that $\nabla_\xi \xi = 0$. Now using (2.2) in $(\overline{\nabla}_\xi J)(\xi) = 0$, we get $A\xi + Jh(\xi, \xi) = 0$. If we take $h(\xi, \xi) = \nu J\xi$, where ν is a smooth function, we obtain $A\xi = \nu\xi$, that is, ξ is an eigenvector of A. The other two eigenvectors of A will be from D. Suppose the two eigenvectors from D are e and Je with $Ae = \lambda e$ and $AJe = \mu Je$. Since the frame $\{e, Je, \xi\}$ diagonalizes A we have $h(e, Je) = 0$, $h(e, \xi) = 0$ and $h(Je, \xi) = 0$. Using (2.2) in $(\overline{\nabla} eJ)(e) = 0$ and taking the inner product with $J\xi$ we get $g(\nabla_e e, \xi) = 0$. Similarly we get $g(\nabla_{Je} Je, \xi) = 0$. Furthermore $\nabla_\xi \xi = 0$ implies that $g(\nabla_\xi e, \xi) = 0$ and $g(\nabla_\xi Je, \xi) = 0$. Then, in view of the equations $(\overline{\nabla} eJ)(\xi) = 0$, $(\overline{\nabla}_{Je} J)(\xi) = 0$ and the structure equations of M, we have the following local equations

$$
(4.1)\qquad
\begin{array}{lll}
\nabla_e e = aJe, & \nabla_{Je} Je = be, & \nabla_\xi \xi = 0 \\
\nabla_e \xi = \lambda Je, & \nabla_{Je} \xi = -\mu e, & \nabla_\xi e = fJe \\
\nabla_\xi Je = -fe, & \nabla_e Je = -ae - \lambda \xi, & \nabla_{Je} e = -bJe + \mu \xi,
\end{array}
$$

where a, b, λ, μ, ν and f are smooth functions.

Now consider the set $V = \{p \in M : \lambda\mu\nu \neq 0\}$. Since V is an open subset of M every point in V has a neighbourhood where the equations (4.1) hold.

Equation (2.4) with different combinations of the frame vectors $\{e, Je, \xi\}$, in view of (4.1) and (2.6), gives

$$
(4.2)\qquad f(\mu-\lambda)+\lambda(\nu-\mu) = -1,\ f(\mu-\lambda)+\mu(\lambda-\nu) = 1,\ \nu(\lambda+\mu) = 2(\lambda\mu-1)
$$

and

$$
(4.3)\qquad \xi.\lambda = 0,\ \xi.\mu = 0,\ e.\nu = 0,\ Je.\nu = 0,\ e.\mu = b(\mu-\lambda),\ Je.\lambda = a(\lambda-\mu).
$$

Adding the first two equations in (4.2), we get $(\lambda-\mu)(\nu-2f) = 0$. Thus either $\lambda = \mu$ or $\nu = 2f$. Also from $d\eta = \omega^1 \wedge \omega^2$, we get $-\eta([e, Je]) = 2$, and hence, using (4.1), we have $\lambda + \mu = 2$. In case $\lambda = \mu$, we get $\lambda = \mu = 1$ and, from (4.2), $\nu = 0$, which does not occur on V. Thus on V we have $\lambda \neq \mu$, and therefore $\nu = 2f$. To compute the values of λ, μ, ν we note from (4.2) that $\nu = \lambda\mu - 1$. Using the first two equations in (4.3), we get $\xi.\nu = 0$, and , in view of (4.3), this implies that ν is constant.

Now solving the equations $\lambda + \mu = 2$ and $\lambda - \mu = \sqrt{4-4\lambda\mu} = 2\sqrt{-\nu}$, we get $\lambda = 1 + \sqrt{-\nu}$ and $\mu = 1 - \sqrt{-\nu}$. This shows that λ and μ are both constants and, since $\lambda \neq \mu$, from (4.3) we get $a = 0$ $b = 0$. Using (2.5) and (2.6) together with (4.1) to compute $R(e, Je, Je, e)$, we get

$$
e.b + Je.a = a^2 + b^2 + f(\lambda+\mu) + 4 + 2\lambda\mu.
$$

From the above equation we have $f + \nu + 3 = 0$. Solving it with $\nu = 2f$, we get $\nu = -2$ and consequently $\lambda = 1 + \sqrt{2}$, $\mu = 1 - \sqrt{2}$. With the set $V = \{p \in M : \lambda\mu\nu = 2\}$, and which is therefore closed, using the connectedness of M we get either $V = M$ or V is empty. When $V = M$, we have $\lambda + \mu + \nu = 0$, that is M is a minimal submanifold of CP^2, and when V is empty we have $\lambda = 1$, $\mu = 1$ and $\nu = 0$. If Ric denotes the Ricci tensor of M, then from (2.5) and (2.6) the Ricci tensor of M is given by $Ric(X,Y) = 5g(X,Y) + tr.Ag(AX,Y) - g(AX,AY)$. For $\lambda = 1$, $\mu = 1$ and $\nu = 0$, we get $Ric(e,e) = 6$, $Ric(Je,Je) = 6$, $Ric(\xi,\xi) = 5$. $Ric(e,Je) = 0$, $Ric(Je,\xi) = 0$. Thus for any non-zero vector field $X \in \mathcal{X}(M)$, we have locally $X = \omega^1(X)e + \omega^2(X)Je + \eta(X)\xi$, and

$$
Ric(X,X) = [\omega^1(x)]^2 Ric(e,e) + [\omega^2(x)]^2 Ric(Je,Je) + [\eta(x)]^2 Ric(\xi,\xi) > 0.
$$

M is therefore of strictly positive Rici curvature and, being compact and simply connected, by Hamiltons Theorem M is diffeomorphie to S^3 (cf. [3]).

Remark. We note that if M is any real hypersurface of CP^2, then the unit normal vector field N gives rise to the unit vector field $\xi = -JN$ on M. Then the kernel of the 1-form η dual to ξ gives rise to a smooth 2-dimensional distribution D which satisfies $JD = D$, and ξ spans the 1-dimensional distribution $D^{\perp}$ which satisfies $JD^{\perp} = T^{\perp}M$. Thus a real hypersurfaces of CP^2 is a 3-dimensional CR-submanifold of CP^2 and our theorems hold for the real hypersurface of CP^2. In a forthcoming paper we shall study 3-dimensional simply connected CR-submanifolds of CP^2 on which α does not define the contact structure given by theorem 3.1.

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