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ON INTEGRAL INEQUALITIES RELATED TO OPIAL'S INEQUALITY

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1. Introduction

In 1960, Z. Opial [4] proved the following integral inequality:

Theorem A. *Let u be of class c^1 on $[0,b]$, and satisfy $u(0)=u(b)=0, u>0$ on $(0,b)$. Then*

$$\int_0^b | u(x)u'(x) | dx \le \frac{b}{4}\int_0^b | u'(x) |^2 dx \tag{1}$$

where the constant $b/4$ is the best possible.

We note that if we replaced $u(0)=u(b)=0$ by $u(0)=0$ (or $u(b)=0$), then (1) becomes

$$\int_0^b | u(x)u'(x) | dx \le \frac{b}{2}\int_0^b | u'(x) |^2 dx \tag{1'}$$

C. Olech [5] showed that (1) is valid for any function u which is absolutely continuous on $[0,b]$, and satisfies the boundary conditions $u(0)=u(b)=0$, and Olech's proof of (1) was simplier than that of Opial.

In 1967, E.K.Godunova, and V.I.Levin [1] generalized (1′) and (1) in the following forms:

Theorem B. *Let u be absolutely continuous on $[a,b]$ with $u(a)=0$. If f is convex increasing on $[0,\infty)$ and $f(0)=0$. Then*

$$\int_a^b f'(| u(x) |) | u'(x) | dx \le f(\int_a^b | u'(x) | dx) \tag{2}$$

Theorem C. *Let u be absolutely continuous on $[a,b]$ with $u(a)=u(b)=0$. Let p be positive on $[a,b]$ and $\int_a^b p(x)dx=1$, and let f,h be convex increasing on $(0,\infty)$, and $f(0)=0$. Then*

$$\int_a^b f'(| u(x) |) | u'(x) | dx \le 2f(h^{-1}(\int_a^b p(x)h(\frac{| u'(x) |}{2p(x)})dx)) \tag{3}$$

In 1966, G.S.Yang [6] generalized (1′) and (1) in the following forms:

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Theorem D. *If u is absolutely continuous on $[a,b]$ with $u(a)=0$, then*

$$\int_a^b |u(x)|^m |u'(x)|^n \, dx \le \frac{n}{m+n}(b-a)^m \int_a^b |u'(x)|^{m+n} \, dx \tag{4}$$

where $m \ge 0, n \ge 1$.

Theorem E. *If u is absolutely continuous on $[a,b]$ with $u(a)=u(b)=0$, then*

$$\int_a^b |u(x)|^m |u'(x)|^n \, dx \le \frac{n}{m+n}\left(\frac{b-a}{2}\right)^m \int_a^b |u'(x)|^{m+n} \, dx \tag{5}$$

where $m \ge 0, n \ge 1$.

Theorem F. *Let p be positive on $[a,b]$ with $\int_a^b \frac{1}{p(x)}dx < \infty$, and let q be positive, bounded and nonincreasing on $[a,b]$. If u is absolutely continuous on $[a,b]$ with $u(a)=0$, then*

$$2\int_a^b q(x) |u(x)||u'(x)| \, dx \le \left(\int_a^b \frac{1}{p(x)}dx\right)\left(\int_a^b p(x)q(x) |u'(x)|^2 \, dx\right) \tag{6}$$

The aim of this paper is to establish some new integral inequalities which generalize (2), (3), (4), (5), and (6).

2. Main results

Throughout, we assume that n is a real number such that $n \ge 1$, and $k=(b-a)^{n-1}$.

Theorem 1. *Let u_1, u_2 be absolutely continuous on $[a,b]$ with $u_1(a)=u_2(a)=0$. Let f_1, f_2 be nonnegative, continuous on $[0,\infty)$ with $f_1(0)=0$ such that f_1', f_2' exist, nonnegative, continuous, and nondecreasing on $[0,\infty)$. Then*

$$\begin{aligned}\int_a^b [f_1(|u_1(x)|^n) f_2'(|u_2'(x)|^n) |u_2'(x)|^n + f_2(|u_2(x)|^n) f_1'(|u_1(x)|^n) |u_1'(x)|^n] dx \\ \le \frac{1}{k} f_1\left(k\int_a^b |u_1'(x)|^n \, dx\right) f_2\left(k\int_a^b |u_2'(x)|^n \, dx\right)\end{aligned} \tag{7}$$

Proof. For $x \in [a,b]$, and $i=1,2$, define

$$y_i(x) = \int_a^x |u_i'(x)|^n \, dx.$$

Then $y_i'(x) = |u_i'(x)|^n$, and using Holder inequality, we have

$$|u_i(x)|^n \le \left(\int_a^x |u_i'(x)| \, dx\right)^n \le \{b-a\}^{(n-1)}\left(\int_a^x |u_i'(x)|^n \, dx\right) \le k\, y_i(x).$$

Since f_i, f_i' are nondecreasing, nonnegative, and continuous on $[0,\infty)$, we have

$$\begin{aligned}
&\int_a^b [f_1(| u_1(x) |^n)f_2'(| u_2(x) |^n) | u_2'(x) |^n + f_2(| u_2(x) |^n)f_1'(| u_1(x) |^n) | u_1'(x) |^n]dx \\
&\leq \int_a^b [f_1(ky_1)f_2'(ky_2)y_2' + f_2(ky_2)f_1'(ky_1)y_1']dx \\
&= \frac{1}{k}\int_a^b \frac{d}{dx}[f_1(ky_1)f_2(ky_2)]dx \\
&= \frac{1}{k} f_1(k\int_a^b | u_1'(x) |^n \, dx) f_2(k\int_a^b | u_2'(x) |^n \, dx)
\end{aligned}$$

This completes the proof of Theorem 1.

Remark.

1. Let $n = 1$, and $f_1(x) = f(x)$, $f_2(x) = 1$, $u_1(x) = u_2(x) = u(x)$ in Theorem 1. Then, it follows from (7) that

$$\int_a^b f'(| u(x) |) | u'(x) | \, dx \leq f(\int_a^b | u'(x) | \, dx),$$

which is the inequality (2). [See also [2], p159, Theorem 13]

2. Let $f_1(x) = x^{(m+n)/n}$, where $m \geq 0$, and let $f_2(x) = 1$, $u_1(x) = u_2(x) = u(x)$ in Theorem 1. Then it follows from (7), and Holder inequality that

$$\begin{aligned}
\int_a^b | u(x) |^m | u'(x) |^n \, dx &\leq \frac{n}{m+n}\frac{1}{k}\{k\int_a^b | u'(x) |^n \, dx\}^{(m+n)/n} \\
&\leq \frac{n}{m+n} k^{\frac{m}{n}} (\int_a^b dx)^{\frac{m}{n}} (\int_a^b | u'(x) |^{m+n} \, dx) \\
&= \frac{n}{m+n}(b-a)^m \int_a^b | u'(x) |^{m+n} \, dx,
\end{aligned}$$

which is the inequality (4). [See [6], Lemma 7]

3. If q is a positive, bounded, and nonincreasing function on $[a,b]$, and p is a positive function with $\int_a^b \frac{1}{p(x)}dx < \infty$, let $f_1(x) = x^2$, $f_2(x) = 1$, $n = 1$, and $u_1(x) = u_2(x) = \int_a^x \sqrt{q(t)} \, | u'(t) | \, dt$ in Theorem 1. We have

$$2\int_a^b (\int_a^x \sqrt{q(t)} \, | u'(t) | \, dt)\sqrt{q(x)} \, | u'(x) | \, dx \leq (\int_a^b \sqrt{q(x)} \, | u'(x) | \, dx)^2$$

Since q is nonincreasing, we have

$$\int_a^x \sqrt{q(t)} \, | u'(t) | \, dt \geq \sqrt{q(x)}\int_a^x | u'(t) | \, dt \geq | \sqrt{q(x)}\int_a^x u'(t)dt | = \sqrt{q(x)} \, | u(x) |$$

Thus

$$\begin{aligned}2\int_a^b q(x)\mid u(x)\mid\mid u'(x)\mid dx &\leq 2\int_a^b(\int_a^x \sqrt{q(t)}\mid u'(t)\mid dt)\sqrt{q(x)}\mid u'(x)\mid dx\\ &\leq (\int_a^b \sqrt{q(x)}\mid u'(x)\mid dx)^2\\ &= (\int_a^b \frac{1}{\sqrt{p(x)}}\sqrt{p(x)}\sqrt{q(x)}\mid u'(x)\mid dx)^2\\ &\leq (\int_a^b \frac{1}{p(x)}dx)(\int_a^b p(x)q(x)\mid u'(x)\mid^2 dx),\end{aligned}$$

which is the inequality (6). ⟦ See [6], Theorem 3 ⟧

Theorem 2. *For $i = 1,2$, let u_i, f_i, f_i' be as in Theorem 1. Let p_i be positive on $[a,b]$, and $\int_a^b p_i(x)dx = 1$. If h is a positive, convex, and increasing function on $[0,\infty)$, then*

$$\int_a^b [f_1(\mid u_1(x)\mid^n)f_2'(\mid u_2(x)\mid^n)\mid u_2'(x)\mid^n + f_2(\mid u_2(x)\mid^n)f_1'(\mid u_1(x)\mid^n)\mid u_1'(x)\mid^n]dx$$
$$\leq \frac{1}{k}f_1(2kh^{-1}(\int_a^b p_1(x)h(\frac{\mid u_1'(x)\mid^n}{2p_1(x)})dx))f_2(2kh^{-1}(\int_a^b p_2(x)h(\frac{\mid u_2'(x)\mid^n}{2p_2(x)})dx))$$

Proof. For $i = 1,2$, it follows from Jensen's inequality, that

$$h(\frac{1}{2}\int_a^b \mid u_i'(x)\mid^n dx) \leq \int_a^b p_i(x)h(\frac{\mid u_i'(x)\mid^n}{2p_i(x)})dx.$$

Since h is increasing, so that

$$\int_a^b \mid u_i'(x)\mid^n dx \leq 2h^{-1}(\int_a^b p_i(x)h(\frac{\mid u_i'(x)\mid^n}{2p_i(x)})dx),$$

which together with Theorem 1 imply that

$$\int_a^b [f_1(\mid u_1(x)\mid^n)f_2'(\mid u_2(x)\mid^n)\mid u_2'(x)\mid^n + f_2(\mid u_2(x)\mid^n)f_1'(\mid u_1(x)\mid^n)\mid u_1'(x)\mid^n]dx$$
$$\leq \frac{1}{k}f_1(2kh^{-1}(\int_a^b p_1(x)h(\frac{\mid u_1'(x)\mid^n}{2p_1(x)})dx))f_2(2kh^{-1}(\int_a^b p_2(x)h(\frac{\mid u_2'(x)\mid^n}{2p_2(x)})dx))$$

This is the desired inequality.

Theorem 3. *For $i = 1,2$, let u_i be absolutely continuous functions on $[a,b]$ with $u_i(a) = u_i(b) = 0$. Let f_i be nonnegative, continuous, nondecreasing functions on $[0,\infty)$*

with $f_1(0) = 0$ *and that* f_i' *exist, nonnegative, continuous, and nondecreasing on* $[0, \infty)$. *Then*

$$\int_a^b [f_1(|u_1(x)|^n)f_2'(|u_2(x)|^n)|u_2'(x)|^n + f_2(|u_2(x)|^n)f_1'(|u_1(x)|^n)|u_1'(x)|^n]dx$$
$$\leq \frac{2^{n-1}}{k} f_2(2^{1-n}k\int_a^b |u_2'(x)|^n\, dx)[f_1(2^{1-n}k\int_a^{(a+b)/2} |u_1'(x)|^n\, dx)$$
$$+ f_1(2^{1-n}k\int_{(a+b)/2}^b |u_1'(x)|^n\, dx)] \tag{8}$$

and

$$\int_a^b [f_1(|u_1(x)|^n)f_2'(|u_2(x)|^n)|u_2'(x)|^n + f_2(|u_2(x)|^n)f_1'(|u_1(x)|^n)|u_1'(x)|^n]dx$$
$$\leq \frac{2}{k} f_1(\frac{k}{2}\int_a^b |u_1'(x)|^n\, dx) f_2(k\int_a^b |u_2'(x)|^n\, dx) \tag{9}$$

Proof. For $i = 1, 2$, we note first that, by defining $y_i(x) = \int_x^b |u_i'(x)|^n\, dx$, and using a similar argument as in the proof of Theorem 1. We see that the inequality (1) still holds if we replaced $u_1(a) = u_2(a) = 0$ by $u_1(b) = u_2(b) = 0$. Now for any $c \in (a, b)$, we have

$$\int_a^c [f_1(|u_1(x)|^n)f_2'(|u_2(x)|^n)|u_2'(x)|^n + f_2(|u_2(x)|^n)f_1'(|u_1(x)|^n)|u_1'(x)|^n]dx$$
$$\leq \frac{1}{k_1} f_1(k_1\int_a^c |u_1'(x)|^n\, dx) f_2(k_1\int_a^c |u_2'(x)|^n\, dx)$$

where $k_1 = (c-a)^{n-1}$, and

$$\int_c^b [f_1(|u_1(x)|^n)f_2'(|u_2(x)|^n)|u_2'(x)|^n + f_2(|u_2(x)|^n)f_1'(|u_1(x)|^n)|u_1'(x)|^n]dx$$
$$\leq \frac{1}{k_2} f_1(k_2\int_c^b |u_1'(x)|^n\, dx) f_2(k_2\int_c^b |u_2'(x)|^n\, dx)$$

where $k_2 = (b-c)^{n-1}$.

Hence

$$\int_a^b [f_1(|u_1(x)|^n)f_2'(|u_2(x)|^n)|u_2'(x)|^n + f_2(|u_2(x)|^n)f_1'(|u_1(x)|^n)|u_1'(x)|^n]dx$$
$$\leq \frac{1}{k_1} f_1(k_1\int_a^c |u_1'(x)|^n\, dx) f_2(k_1\int_a^c |u_2'(x)|^n\, dx)$$
$$+ \frac{1}{k_2} f_1(k_2\int_c^b |u_1'(x)|^n\, dx) f_2(k_2\int_c^b |u_2'(x)|^n\, dx) \tag{10}$$

By taking $c = \frac{(a+b)}{2}$ in (10), we have $k_1 = k_2 = 2^{1-n}k$, and since f_2 is nondecreasing, so that

$$\int_a^b [f_1(|u_1(x)|^n)f_2'(|u_2(x)|^n)|u_2'(x)|^n + f_2(|u_2(x)|^n)f_1'(|u_1(x)|^n)|u_1'(x)|^n]dx$$
$$\le \frac{2^{n-1}}{k} f_2(2^{1-n}k\int_a^b |u_2'(x)|^n\,dx)[f_1(2^{1-n}k\int_a^{(a+b)/2} |u_1'(x)|^n\,dx)$$
$$+ f_1(2^{1-n}k\int_{(a+b)/2}^b |u_1'(x)|^n\,dx)]$$

This is the inequality (8).

If we choose c in (4) so that

$$\int_a^c |u_1'(x)|^n\,dx = \int_c^b |u_1'(x)|^n\,dx = \frac{1}{2}\int_a^b |u_1'(x)|^n\,dx,$$

then,

$$\int_a^b [f_1(|u_1(x)|^n)f_2'(|u_2(x)|^n)|u_2'(x)|^n + f_2(|u_2(x)|^n)f_1'(|u_1(x)|^n)|u_1'(x)|^n]dx$$
$$\le \frac{1}{k_1}f_1(\frac{k_1}{2}\int_a^b |u_1'(x)|^n\,dx)f_2(k_1\int_a^c |u_2'(x)|^n\,dx)$$
$$+ \frac{1}{k_2}f_1(\frac{k_2}{2}\int_a^b |u_1'(x)|^n\,dx)f_2(k_2\int_c^b |u_2'(x)|^n\,dx)$$
$$\le f_2(k\int_a^b |u_2'(x)|^n\,dx)[\frac{1}{k_1}f_1(\frac{k_1}{2}\int_a^b |u_1'(x)|^n\,dx)$$
$$+ \frac{1}{k_2}f_1(\frac{k_2}{2}\int_a^b |u_1'(x)|^n\,dx)]$$
$$= f_2(k\int_a^b |u_2'(x)|^n\,dx)[\frac{1}{k_1}f_1(\frac{k_1}{k}\frac{k}{2}\int_a^b |u_1'(x)|^n\,dx)$$
$$+ \frac{1}{k_2}f_1(\frac{k_2}{k}\frac{k}{2}\int_a^b |u_1'(x)|^n\,dx)]$$
$$\le \frac{2}{k}f_1(\frac{k}{2}\int_a^b |u_1'(x)|^n\,dx)f_2(k\int_a^b |u_2'(x)|^n\,dx)$$

This completes the proof of Theorem 3.

Remark. For $m \ge 0$, let $f_1(x) = x^{(m+n)/n}$, $f_2(x) = 1$, $u_1(x) = u_2(x) = u(x)$ in Theorem 3, it follows from (8), and Holder inequality that

$$\int_a^b |u(x)|^m |u'(x)|^n\,dx \le \frac{n}{m+n}(\frac{b-a}{2})^m \int_a^b |u'(x)|^{m+n}\,dx,$$

which is the inequality (5). [See [6], Theorem 6]

Theorem 4. *For $i = 1, 2$, let u_i, f_i, f_i' be as in Theorem 3. Let p_i be positive on $[a, b]$, and $\int_a^b p_i(x)dx = 1$. If h is a positive, convex, and increasing function on $[0, \infty)$, then*

$$\int_a^b [f_1(| u_1(x) |^n) f_2'(| u_2(x) |^n) | u_2'(x) |^n + f_2(| u_2(x) |^n) f_1'(| u_1(x) |^n) | u_1'(x) |^n] dx$$
$$\leq \frac{2}{k} f_1(kh^{-1}(\int_a^b p_1(x) h(\frac{| u_1'(x) |^n}{2p_1(x)})dx)) f_2(2kh^{-1}(\int_a^b p_2(x) h(\frac{| u_2'(x) |^n}{2p_2(x)})dx)) \qquad (11)$$

Proof. For $i = 1, 2$, it follows from Jensen's inequality, that

$$h(\frac{1}{2}\int_a^b | u_i'(x) |^n dx) \leq \int_a^b p_i(x) h(\frac{| u_i'(x) |^n}{2p_i(x)})dx.$$

Since h is increasing, we have

$$\int_a^b | u_i'(x) |^n dx \leq 2h^{-1}(\int_a^b p_i(x) h(\frac{| u_i'(x) |^n}{2p_i(x)})dx). \qquad (12)$$

The desired inequality (11) is then follows from (9), and (12).

Remark. For $i = 1, 2$, let $p_i(x) = p(x)$, $u_i(x) = u(x)$, and let $n = 1$, $f_1(x) = f(x)$, $f_2(x) = 1$ in Theorem 4. Then it follows from the inequality (11) that

$$\int_a^b f'(| u(x) |) | u'(x) | dx \leq 2f(h^{-1}(\int_a^b p(x) h(\frac{| u'(x) |}{2p(x)})dx)),$$

which is the inequality (3). ⟦ See also [2], p159, Theorem 12 ⟧

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