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CR-submanifolds of SQ-Sasakian manifold

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Abstract. In this paper we discussed the geometry of CR-submanifolds of a SQ-Sasakian manifold. Next, we considered Chaki pseudo parallel as well as Deszcz pseudo parallel CR-submanifolds of SQ-Sasakian manifolds. Further we studied almost Ricci soliton and almost Yamabe soliton with torse forming vector field on a CR-submanifold of a SQ-Sasakian manifold using semi-symmetric metric connection.

Keywords. Pseudo parallel, Ricci soliton, Yamabe soliton

1 Introduction

In 1978 A. Bejancu [8, 9] gave the notion of CR-submanifolds as a generalization of invariant and anti-invariant submanifolds of Kähler manifold. After that a lot of investigation were done on CR-submanifolds in both complex and almost contact manifolds. Moreover, CR-submanifolds of Sasakian manifold were studied by many authors [16, 24, 16]. M. Shahid [25, 26] studied CR-submanifolds of trans Sasakian manifold (a generalization of α-Sasakian and β-Kenmotsu manifolds). The same author also studied CR-submanifolds of Sasakian manifold with vanishing contact Bochner curvature tensor [27]. In [6, 30] CR-submanifolds were studied in cosymplectic and Kenmotsu manifold. K. Yano [32] introduced the notion of f-structure on a $(2n+s)$-dimensional manifold as a tensor field f of type (1,1) and rank $2n$ satisfying $f^3+f=0$. Almost complex $(s=0)$ and almost contact $(s=1)$ structures are well-known examples of f-structures. Further the study of CR-submanifolds was extended to f-structures by I. Mihai, L. Ornea and L. M. Fernandez [1, 2, 3, 12, 18, 20, 22, 23].

In 1993, J. H. Kwon and B. H. Kim [17] introduced a new class of almost contact metric manifolds known as a special quasi-Sasakian manifold or briefly as SQ-Sasakian manifold. In [29] Shaikh and Ahmad obtained some interesting results on SQ-Sasakian manifolds. Further S. K. Hui and J. Roy [15] studied invariant and anti-invariant submanifolds of SQ-Sasakian manifolds with respect to Levi-Civita connection as well as semi-symmetric metric connection. They also deal with the Chaki-pseudo parallel as well as Deszcz-pseudo parallel invariant submanifolds of SQ-Sasakian manifolds with respect to Levi-Civita connection as well as semi-symmetric metric connection.

In the present paper, we discussed integrability and totally geodesic condition of CR-submanifold of SQ-Sasakian manifold. Next we studied almost Ricci soliton and almost Yamabe soliton with torse forming vector field on a CR-submanifold of a SQ-Sasakian manifold using semi-symmetric metric connection.

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2 Preliminaries

Let $\bar{\mathcal{M}}^{2n+1}$ with the structure (ϕ, ξ, η, g) be an almost contact metric manifold such that

$$\eta(\xi) = 1, \quad \phi^2(X) = -X + \eta(X)\xi,$$
$$\phi\xi = 0, \quad g(X, \xi) = \eta(X),$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad g(\phi X, Y) = -g(X, \phi Y)$$

for any vector fields X, Y in $T\bar{\mathcal{M}}$, where ϕ, ξ and η are the (1,1) tensor field, characteristic vector field and one form respectively.
The fundamental two form Φ is defined on $\bar{\mathcal{M}}$ by $\Phi(X, Y) = g(X, \phi Y)$. If $d\eta(X, Y) = g(X, \phi Y)$ for all vector fields X, Y on $\bar{\mathcal{M}}^{2n+1}(\phi, \xi, \eta, g)$, then the almost contact metric manifold $\bar{\mathcal{M}}^{2n+1}(\phi, \xi, \eta, g)$ is called a contact metric manifold. A normal contact metric manifold is called a Sasakian manifold [10]. A normal almost contact metric manifold is called quasi-Sasakian if Φ is closed. A three dimensional almost contact metric manifold is called quasi-Sasakian if and only if [19]

$$\bar{\nabla}_X \xi = -\beta\phi X, \tag{2.1}$$

for some smooth function β on $\bar{\mathcal{M}}$ such that $(\xi\beta) = 0$. The equation (2.1) does not hold for a quasi-Sasakian manifold of dimension greater than three. An almost contact metric manifold is called SQ-Sasakian manifold [17] if the following conditions are satisfied

$$\bar{\nabla}_X \xi = -\beta\phi X, \quad d\Phi = 0 \tag{2.2}$$

and (ϕ, ξ, η) is normal for some smooth function β on $\bar{\mathcal{M}}$ such that $(\xi\beta) = 0$. In 1993, Kwon and Kim [17] have constructed a non trivial example of SQ-Sasakian manifold. It is to be noted that a SQ-Sasakian manifold is a cosymplectic manifold if and only if $\beta = 0$ and a Sasakian manifold if and only if $\beta = 1$.

In an SQ-Sasakian manifold $\bar{\mathcal{M}}$, the following relations hold [29]

$$(\bar{\nabla}_X \phi)(Y) = \beta[g(X, Y)\xi - \eta(Y)X] \tag{2.3}$$
$$(\bar{\nabla}_X \eta)(Y) = \beta g(X, \phi Y) \tag{2.4}$$
$$\bar{R}(X, Y)\xi = (Y\beta)\phi X - (X\beta)\phi Y + \beta^2[\eta(Y)X - \eta(X)Y] \tag{2.5}$$
$$\eta(\bar{R}(X, Y)Z) = (X\beta)g(\phi Y, Z) - (Y\beta)g(\phi X, Z) + \beta^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \tag{2.6}$$
$$\bar{R}(\xi, X)Y = g(X, \phi Y)g\,\mathrm{rad}\,\beta + (Y\beta)\phi X + \beta^2[g(X, Y)\xi - \eta(Y)X] \tag{2.7}$$
$$\bar{S}(Y, \xi) = 2\eta\beta^2\eta(Y) - ((\phi Y)\beta) \tag{2.8}$$

for all vector fields X, Y, Z on $\bar{\mathcal{M}}$ and $\bar{R}$ and $\bar{S}$ are the curvature tensor and Ricci tensor of $\bar{\mathcal{M}}$ respectively.
Let $\mathcal{M}$ be an n-dimensional submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$, g be the induced metric tensor on $\mathcal{M}$ and $\bar{\nabla}$ (*resp.* ∇) be the covariant derivatives in $\bar{\mathcal{M}}$ (resp. $\mathcal{M}$). For vector fields X and Y tangent to the submanifold and normal vector field N, the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.9}$$

where $\nabla^{\perp}$ is the connection in the normal bundle $T^{\perp}\mathcal{M}$, h is the second fundamental form of $\mathcal{M}$ and A_N is the shape operator associated with N and satisfy

$$g(A_N X, Y) = g(h(X, Y), N). \tag{2.10}$$

If R and $\bar{R}$ are the curvature tensors of $\mathcal{M}$ and $\bar{\mathcal{M}}$ respectively, then the Gauss and Codazzi's equations are given by

$$\begin{aligned}\bar{R}(X,Y,Z,W) &= R(X,Y,Z,W)+g(h(X,Z),h(Y,W)) \\ &- g(h(X,W),h(Y,Z)), \end{aligned}\tag{2.11}$$

$$(\bar{R}(X,Y,Z,W))^{\perp} = (\bar{\nabla}_X h)(Y,Z)-(\bar{\nabla}_Y h)(X,Z). \tag{2.12}$$

In [13], Friedmann and Schouten introduced the notion of semi-symmetric linear connection on a smooth manifold. A linear connection on a SQ-Sasakian manifold $\bar{\mathcal{M}}$ is said to be a semi-symmetric connection if its torsion tensor τ of the connection $\tilde{\bar{\nabla}}$ given by

$$\tau(X,Y)=\tilde{\bar{\nabla}}_X Y-\tilde{\bar{\nabla}}_Y X-[X,Y] \tag{2.13}$$

satisfies $\tau(X,Y)=\eta(Y)X-\eta(X)Y$, where η is an 1-form. Further, if the semi-symmetric connection $\tilde{\bar{\nabla}}$ satisfies the condition $(\tilde{\bar{\nabla}}_X g)(Y,Z)=0$ for all $X,Y,Z\in\chi(\bar{\mathcal{M}})$ then $\tilde{\bar{\nabla}}$ is said to be semi-symmetric metric connection. The relation between semi-symmetric metric connection $\tilde{\bar{\nabla}}$ and Levi-Civita connection on SQ-Sasakian manifold $\bar{\mathcal{M}}$ is [15]

$$\tilde{\bar{\nabla}}_X Y=\bar{\nabla}_X Y+\eta(Y)X-g(X,Y)\xi. \tag{2.14}$$

Let ∇ and $\tilde{\nabla}$ be the induced connection on $\mathcal{M}$ from the connection $\bar{\nabla}$ and $\tilde{\bar{\nabla}}$ respectively. Then we have

$$\tilde{\bar{\nabla}}_X Y=\tilde{\nabla}_X Y+\tilde{h}(X,Y). \tag{2.15}$$

By virtue of (2.9) and (2.14), (2.15) yields

$$\tilde{\nabla}_X Y+\tilde{h}(X,Y)=\nabla_X Y+h(X,Y)+\eta(Y)X-g(X,Y)\xi, \tag{2.16}$$

where h and $\tilde{h}$ are the second fundamental forms with respect to Levi-Civita connection and semi-symmetric metric connections respectively. The covariant derivative of the second fundamental form h is defined as

$$\left(\bar{\nabla}_X h\right)(Y,Z)=\nabla^{\perp}_X(h(Y,Z))-h\left(\nabla_X Y,Z\right)-h\left(Y,\nabla_X Z\right). \tag{2.17}$$

If $\bar{R}$ and $\tilde{\bar{R}}$ are respectively the curvature tensor with respect the Levi-Civita connection $\bar{\nabla}$ and semi-symmetric metric connection $\tilde{\bar{\nabla}}$ in a SQ-Sasakian manifold then the following relations hold

$$\begin{aligned}\tilde{\bar{R}}(X,Y)Z &= \bar{R}(X,Y)Z+g(X,Z)Y-g(Y,Z)X+\eta(Z)\{\eta(Y)X \\ &- \eta(X)Y\}+\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi+\beta\{g(Y,Z)\phi X \\ &- g(X,Z)\phi Y+\Phi(X,Z)Y-\Phi(Y,Z)X\},\end{aligned}\tag{2.18}$$

$$\tilde{\bar{S}}(Y,Z) = \bar{S}(Y,Z)-(2n-1)\{g(Y,Z)-\eta(Y)\eta(Z)+\Phi(Y,Z)\}, \tag{2.19}$$

$$\begin{aligned}\tilde{\bar{R}}(X,Y)\xi &= \beta^2\{\eta(Y)X-\eta(X)Y\}+\{\beta\eta(Y)+(Y\beta)\}\phi X \\ &- \{\beta\eta(X)+(X\beta)\}\phi Y,\end{aligned}\tag{2.20}$$

$$\begin{aligned}\tilde{\bar{R}}(\xi,Y)Z &= \beta^2\{g(Y,Z)\xi-\eta(Z)Y\}+\{Z\beta-\beta\eta(Z)\}\phi Y \\ &+ \Phi(Y,Z)\{\operatorname{grad}\beta-\beta\xi\},\end{aligned}\tag{2.21}$$

for arbitrary vector fields X,Y and Z on $\bar{\mathcal{M}}$.

A SQ-Sasakian manifold $\bar{\mathcal{M}}$ is said to be pseudo quasi-Einstein (or pseudo η -Einstein) manifold if its Ricci tensor $\bar{S}$ of type (0,2) is not identically zero and satisfies the following [28]:

$$\bar{S}(X,Y)=pg(X,Y)+q\eta(X)\eta(Y)+sD(X,Y), \tag{2.22}$$

where p, q, s are non-zero scalars.
Now for any X $\in T\mathcal{M}$, we have

$$\phi X = TX + FX, \tag{2.23}$$

where TX is the tangential component and FX is the normal component of ϕX. Then T is an endomorphism of $T\mathcal{M}$ and F is normal bundle valued 1-form on $T\mathcal{M}$. T (resp. F) is parallel if $\nabla T = 0$ (resp. $\nabla F = 0$).
Similarly for $N \in T^{\perp}M$, we can write

$$\phi N = tN + fN, \tag{2.24}$$

where tN (resp. fN) denotes the tangential and normal components of ϕN . Then f is an endomorphism of the normal bundle and t is tangent bundle valued 1-form on $T^{\perp}\mathcal{M}$.

3 CR-submanifold of a SQ-Sasakian manifold.

In this section we are going to discuss some basic results for the CR-submanifolds of a SQ-Sasakian manifold.

Definition 1. A submanifold $\mathcal{M}$ of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ is said to be a CR-submanifold if ξ is tangent to M and there exist two orthogonal differentiable distributions $\mathfrak{D}$ and $\mathfrak{D}^{\perp}$ such that

- $T\mathcal{M} = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$.
- $\mathfrak{D}$ is invariant under ϕ for each point on $\mathcal{M}$, i.e., $\phi\mathfrak{D}_x \subset \mathfrak{D}_x$ for each $x \in \mathcal{M}$.
- $\mathfrak{D}^{\perp}$ is anti-invariant under ϕ for each point on $\mathcal{M}$, i.e., $\phi\mathfrak{D}^{\perp}_x \subset T^{\perp}_x\mathcal{M}$.

Also, $\mathcal{M}$ is called horizontal (resp. vertical) if $\xi \in \Gamma(\mathfrak{D})$ (resp. $\Gamma(\mathfrak{D}^{\perp})$).

Theorem 3.1. *Let $\mathcal{M}$ be a submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$. Then we have*

$$\nabla_X TY - T\nabla_X Y - A_{FY}X - th(X,Y) = \beta\{g(X,Y)\xi - \eta(Y)X\} \tag{3.1}$$
$$\nabla^{\perp}_X FY - F\nabla_X Y = fh(X,Y) - h(X,TY). \tag{3.2}$$

for all $X,\ Y \in T\mathcal{M}$.

Proof. For any X, $Y \in T\mathcal{M}$, from (2.3) we have

$$\bar{\nabla}_X \phi Y - \phi\bar{\nabla}_X Y = \beta[g(\phi X, Y)\xi - \eta(Y)X]. \tag{3.3}$$

Now, by using Gauss and Weingartan formulas together with (2.23) and (2.24), we obtain

$$\begin{gathered}\nabla_X TY - T\nabla_X Y - A_{FY}X - th(X,Y) + \nabla^{\perp}_X FY - F\nabla_X Y \\ +h(X,TY) - fh(X,Y) = \beta\{g(X,Y)\xi - \eta(Y)X\}.\end{gathered} \tag{3.4}$$

Comparing tangential and normal components of (3.4), we obtain the required result. □

Lemma 3.1. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$, then the distribution $\mathfrak{D}$ is integrable if and only if*

$$h(X, TY) = h(Y, TX), \tag{3.5}$$

for all $X,\ Y \in \Gamma(\mathfrak{D})$.

Proof. Let $X,\ Y \in \Gamma(\mathfrak{D})$, then $FX = FY = 0$. Therefore (3.2) becomes

$$F\nabla_X Y = -fh(X,Y) + h(X,TY). \tag{3.6}$$

Now interchanging X and Y, we get

$$F\nabla_Y X = -fh(Y,X) + h(Y,TX). \tag{3.7}$$

Subtracting (3.6) from (3.7), we obtain

$$h(X,TY) - h(Y,TX) = F[X,Y].$$

Now $[X,Y] \in \Gamma(\mathfrak{D})$ if and only if $F[X,Y] = 0$, which proves our result. □

Lemma 3.2. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$, then the distribution $\mathfrak{D}^\perp$ is integrable if and only if*

$$A_{FZ}W - A_{FW}Z = \beta[\eta(Z)W - \eta(W)Z]. \tag{3.8}$$

for all $W, Z \in \Gamma(\mathfrak{D}^\perp)$.

Proof. Let $Z,\ W \in \Gamma(\mathfrak{D}^\perp)$, then from (3.1), we obtain

$$-A_{FW}Z - T\nabla_Z W - th(Z,W) = \beta[g(Z,W)\xi - \eta(W)Z]. \tag{3.9}$$

Interchanging Z and W in (3.9), we get

$$-A_{FZ}W - T\nabla_W Z - th(Z,W) = \beta[g(Z,W)\xi - \eta(W)Z]. \tag{3.10}$$

Subtracting (3.10) from (3.9), we obtain

$$A_{FZ}W - A_{FW}Z - T[Z,W] = \beta[\eta(X)Y - \eta(W)Z].$$

Now $[Z,W] \in \Gamma(\mathfrak{D}^\perp)$ if and only if $T[Z,W] = 0$, which proves the result. □

Theorem 3.2. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$, then we have*

(i) $\nabla_X \xi = -\beta TX$,

(ii) $h(X,\xi) = -\beta FX$,

(iii) $h(X,\xi) = 0, \quad \forall \quad X \in \Gamma(\mathfrak{D})$,

(iv) $\nabla_X \xi = 0, \quad \forall \quad X \in \Gamma(\mathfrak{D}^\perp)$,

(v) $A_N \xi \in \Gamma(\mathfrak{D}^\perp), \quad \forall \quad N \in T^\perp \mathcal{M}$,

(vi) $\eta(A_N X) = 0, \forall \quad X \in \Gamma(\mathfrak{D})$.

Proof. By using Gauss formula in (2.1) we easily obtain

$$\nabla_X \xi + h(X,\xi) = -\beta\phi X.$$

Now comparing tangential and normal components, we have

$$\nabla_X \xi = -\beta TX, \tag{3.11}$$

$$h(X,\xi) = -\beta FX, \tag{3.12}$$

for all $X \in T\mathcal{M}$. These proves (i) and (ii). Now for $X \in \Gamma(\mathfrak{D})$, we get

$$h(X,\xi) = 0. \tag{3.13}$$

For $X \in \Gamma(\mathfrak{D}^{\perp})$, we have $\nabla_X \xi = 0$.
Again for $X \in \Gamma(\mathfrak{D})$, and $N \in T^{\perp}\mathcal{M}$

$$g(A_N\xi, X) = g(h(X,\xi), N) = 0,$$

which means $A_N\xi \in \Gamma(\mathfrak{D}^{\perp})$. Also $0 = g(A_N\xi, X) = g(A_N X, \xi) = \eta(A_N X)$. □

The covariant derivatives of T and F are, respectively, defined by

$$\left(\bar{\nabla}_X T\right) Y = \nabla_X TY - T\nabla_X Y \tag{3.14}$$

and

$$\left(\bar{\nabla}_X F\right) Y = \nabla^{\perp}_X FY - F\nabla_X Y, \tag{3.15}$$

also the covariant derivatives of t and f are, respectively, defined by

$$\left(\bar{\nabla}_X t\right) N = \nabla_X tN - t\nabla^{\perp}_X N \tag{3.16}$$

and

$$(\bar{\nabla}_X f)(N) = \nabla^{\perp}_X fN - f\nabla^{\perp}_X N \tag{3.17}$$

for any vector field X, Y tangent to $\mathcal{M}$ and any vector field N normal to $\mathcal{M}$.

Proposition 3.1. *Let $\mathcal{M}$ be a submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$. Then*

$$\begin{aligned}
(\bar{\nabla}_X T)(Y) &= A_{FY}X + th(X,Y) + \beta[g(X,Y)\xi - \eta(Y)X], &(3.18)\\
(\bar{\nabla}_X F)(Y) &= fh(X,Y) - h(X,TY), &(3.19)\\
(\bar{\nabla}_X t)(N) &= A_{fN}X - TA_N X, &(3.20)\\
(\bar{\nabla}_X f)(N) &= -h(X,tN) - FA_N X, &(3.21)
\end{aligned}$$

for any vector field X and Y tangent to $\mathcal{M}$ and N normal to $\mathcal{M}$.

Proof. From (3.1), (3.2) and making use of (3.14), (3.15) we easily obtained (3.18) and (3.19). Now for $X \in T\mathcal{M}$ and $N \in T^{\perp}\mathcal{M}$, using Gauss and Weingarten formulas together with (2.23) and (2.24), we have

$$\begin{aligned}
(\bar{\nabla}_X \phi)N &= \bar{\nabla}_X \phi N - \phi\bar{\nabla}_X N &(3.22)\\
&= \bar{\nabla}_X (tN + fN) - \phi(-A_N X + \nabla^{\perp}_X N)\\
&= \nabla_X tN + h(X,tN) - A_{fN}X + \nabla^{\perp}_X fN + \phi A_N X - \phi\nabla^{\perp}_X N\\
&= \nabla_X tN + h(X,tN) - A_{fN}X + \nabla^{\perp}_X fN + TA_N X + FA_N X\\
&- t\nabla^{\perp}_X N - f\nabla^{\perp}_X N.
\end{aligned}$$

Also from (2.3) we get

$$(\bar{\nabla}_X \phi)N = 0. \tag{3.23}$$

From (3.22) and (3.23), we have

$$\begin{aligned}\nabla_X tN + h(X, tN) - A_{fN}X + \nabla^{\perp}_X fN + TA_N X \\ + FA_N X - t\nabla^{\perp}_X N - f\nabla^{\perp}_X N = 0.\end{aligned} \tag{3.24}$$

Comparing tangential and normal components, we get (3.20) and (3.21). □

Proposition 3.2. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$. Then the leaf $\mathcal{M}^{\perp}$ of $\mathfrak{D}^{\perp}$ is totally geodesic in $\mathcal{M}$ if and only if*

$$g(h(Y, W), FZ) + \beta g(W, Z)\eta(Y) = 0, \tag{3.25}$$

for any $Y \in \Gamma(\mathfrak{D})$ and $W, Z \in \Gamma(\mathfrak{D}^{\perp})$.

Proof. Putting $X = W \in \Gamma(\mathfrak{D}^{\perp})$ and $Y = Z \in \Gamma(\mathfrak{D}^{\perp})$ in (3.1), we have

$$T\nabla_W Z = -A_{FZ}W - th(W, Z) - \beta[g(W, Z)\xi - \eta(Z)W]. \tag{3.26}$$

Now taking inner product of (3.26) with $Y \in \Gamma(\mathfrak{D})$, we get

$$g(T\nabla_W Z, Y) = -g(A_{FZ}W, Y) - \beta[g(W, Z)\eta(Y)], \tag{3.27}$$

which implies

$$g(\nabla_W Z, TY) = g(h(Y, W), FZ) + \beta[g(W, Z)\eta(Y)]. \tag{3.28}$$

We know that $\mathfrak{D}^{\perp}$ is totally geodesic in $\mathcal{M}$ if and only if $\nabla_W Z \in \Gamma(\mathfrak{D}^{\perp})$, for all Z, $W \in \Gamma(\mathfrak{D}^{\perp})$. If (3.25) holds then from (3.28) we get $\nabla_W Z \in \Gamma(\mathfrak{D}^{\perp})$, which proves that $\mathfrak{D}^{\perp}$ is totally geodesic. Conversely if $\mathfrak{D}^{\perp}$ is totally geodesic then $\nabla_W Z \in \Gamma(\mathfrak{D}^{\perp})$. From (3.28) we get (3.25). □

Proposition 3.3. *Let $\mathcal{M}$ be a CR-submanifold of a special quasi Sasakian $\bar{\mathcal{M}}$. Then the distribution $\mathfrak{D}$ is integrable if and only if*

$$\begin{aligned} g(h(X, TY), FZ) - g(h(Y, TX), FZ) &= \eta(\nabla_X Z)\eta(Y) - \eta(\nabla_Y Z)\eta(X) \\ &- 2\beta\eta(Z)g(X, TY), \end{aligned} \tag{3.29}$$

for all X, $Y \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(\mathfrak{D}^{\perp})$.

Proof. For $X, Y \in \Gamma(\mathfrak{D})$, $Z \in \Gamma(\mathfrak{D}^{\perp})$, (3.1) we infer

$$-T\nabla_X Z = A_{FZ}X + th(X, Z) - \beta\{\eta(Z)X\}. \tag{3.30}$$

Taking inner product with TY

$$g(T\nabla_X Z, TY) = -g(A_{FZ}X, TY) - g(th(X, Z), TY) + \beta\eta(Z)g(X, TY),$$

which implies

$$g(\nabla_X Z, Y) - \eta(\nabla_X Z)\eta(Y) = -g(h(X, TY), FZ) + \beta\eta(Z)g(X, TY)$$

or

$$g(\nabla_X Y, Z) = g(h(X, TY), FZ) - \beta\eta(Z)g(X, TY) - \eta(\nabla_X Z)\eta(Y). \tag{3.31}$$

Interchanging X and Y in (3.31) we get

$$g(\nabla_Y X, Z) = g(h(Y, TX), FZ) - \beta\eta(Z)g(Y, TX) - \eta(\nabla_Y Z)\eta(X). \tag{3.32}$$

From (3.31) and (3.32), we obtain

$$\begin{aligned} g([X,Y], Z) &= g(h(X, TY), FZ) - g(h(Y, TX), FZ) - 2\beta\eta(Z)g(X, TY) \\ &+ \eta(\nabla_Y Z)\eta(X) - \eta(\nabla_X Z)\eta(Y). \end{aligned} \tag{3.33}$$

Hence $\mathfrak{D}$ is integrable if $[X, Y] \in \Gamma(\mathfrak{D})$, i.e., $g([X, Y], Z) = 0$, which proves our result. □

Proposition 3.4. *Let $\mathcal{M}$ be a CR -submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$, and the leaf $\mathcal{M}^\perp$ of $\mathfrak{D}^\perp$ is totally geodesic in $\mathcal{M}$. If the endomorphism T satisfies*

$$(\bar{\nabla}_X T)Y = \beta[g(X, Y)\xi - \eta(Y)X], \tag{3.34}$$

for any X, Y tangent to $\mathcal{M}$. Then $dim\Gamma(\mathfrak{D}^\perp) = 0$.

Proof. From (3.25) we get

$$g(A_{FZ}Y + \beta\eta(Y)Z, W) = 0, \tag{3.35}$$

for any $Y \in \Gamma(\mathfrak{D})$, W, $Z \in \Gamma(\mathfrak{D}^\perp)$. For any X, Y tangent to $\mathcal{M}$ from (3.1) and (3.34) we get

$$A_{FY}X + th(X, Y) = 0. \tag{3.36}$$

From above equation, for any $Y \in \Gamma(\mathfrak{D})$, $th(X, Y) = 0$. Thus we have

$$g(h(X, Y), \phi W) = g(A_{\phi W}Y, X) = -g(\phi h(X, Y), W) = 0, \tag{3.37}$$

which implies $A_{\phi W}Y = 0$. Using this in (3.35) gives $\beta\eta(Y)g(Z, W) = 0$, i.e. $g(Z, W) = 0$, for all Z, $W \in \Gamma(\mathfrak{D}^\perp)$. Hence $dim\Gamma\ (\mathfrak{D}^\perp) = 0$. □

4 Contact CR-product submanifold of a SQ-Sasakian manifold

Definition 2. [12] A submanifold $\mathcal{M}$ of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ is called contact CR-product if it is locally a Rieamnnian product of $\mathcal{M}^T$ and $\mathcal{M}^\perp$, where $\mathcal{M}^T$ and $\mathcal{M}^\perp$ denote leaves of the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$ respectively.

Now we prove the following:

Theorem 4.1. *Let $\mathcal{M}$ be a ξ-horizontal CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$. Then $\mathcal{M}$ is a contact CR-product if and only if*

$$A_{FZ}Y + \beta\eta(Y)Z = 0, \tag{4.1}$$

for any $Y \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(\mathfrak{D}^\perp)$.

Proof. If a CR-submanifold $\mathcal{M}$ of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ is contact CR-product then from (3.25), we have

$$g(A_{FZ}W, Y) + \beta\eta(Y)g(W, Z) = 0,$$

for $Y \in \Gamma(\mathfrak{D})$ and $W, Z \in \Gamma(\mathfrak{D}^\perp)$.
Since, the shape operator is symmetric, above equation can be written as

$$g(A_{FZ}Y, W) + \beta\eta(Y)g(W, Z) = 0.$$

From this we get

$$A_{FZ}Y + \beta\eta(Y)Z \in \Gamma(\mathfrak{D}), \tag{4.2}$$

for any $Y \in \Gamma(\mathfrak{D})$ and $Z, W \in \Gamma(\mathfrak{D}^\perp)$.
As $\mathfrak{D}$ is totally geodesic in $\mathcal{M}$, we have for $X, Y \in \Gamma(\mathfrak{D})$ that

$$\begin{aligned} g(A_{FZ}Y + \beta\eta(Y)Z, X) &= g(h(X, Y), FZ) = -g(\phi h(X, Y), Z) \\ &= -g\left(\phi\bar{\nabla}_X Y - \phi\nabla_X Y, Z\right) \\ &= g\left(\phi\bar{\nabla}_X Y, Z\right) \\ &= g\left(\bar{\nabla}_X \phi Y, Z\right) = 0, \end{aligned}$$

for any $X, Y \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(\mathfrak{D}^\perp)$.
This means

$$A_{FZ}Y + \beta\eta(Y)Z \in \Gamma(\mathfrak{D}^\perp). \tag{4.3}$$

Thus form (4.2) and (4.3) we get

$$A_{FZ}Y + \beta\eta(Y)Z = 0.$$

Conversely, equation (3.25) gives

$$g(h(Y, W), FZ) + g(\beta\eta(Y)Z, W) = 0,$$

for any $Y \in \Gamma(\mathfrak{D}), W, Z \in \Gamma(\mathfrak{D}^\perp)$, which means by virtue of Proposition 3.2 that the leaf $\mathcal{M}^T$ of $\Gamma(\mathfrak{D})$ is totally geodesic in $\mathcal{M}$.
Next, suppose $\mathcal{M}^\perp$ be the leaf of $\Gamma(\mathfrak{D})$. Then from (2.3) and (4.1), we have

$$\begin{aligned} g(\nabla_X Y, Z) = g\left(\tilde{\nabla}_X Y, Z\right) &= g\left(\phi\tilde{\nabla}_X Y, \phi Z\right) = g\left(\tilde{\nabla}_X \phi Y - (\tilde{\nabla}_X \phi)Y, \phi Z\right) \\ &= g(h(X, \phi Y), \phi Z) = g(A_{\phi Z}X, \phi Y) \\ &= -\beta\eta(X)g(\phi Y, Z) = 0, \end{aligned}$$

for any $X, Y \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(\mathfrak{D}^\perp)$, i.e. the leaf $\mathcal{M}^\top$ of $\mathfrak{D}$ is totally geodesic in $\mathcal{M}$. Thus the submanifold $\mathcal{M}$ is a contact CR-product. □

Theorem 4.2. *A CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ is a contact CR-product if and only if T is parallel.*

Proof. Let T be parallel, then from (3.18), we get

$$A_{FY}X + th(X, Y) + \beta[g(X, Y)\xi - \eta(Y)X] = 0,$$

for any vector fields X, Y tangent to $\mathcal{M}$. If the vector field Y is in $\Gamma(\mathfrak{D})$, then using the fact that $FY = 0$, above equation becomes

$$th(X,Y) + \beta[g(X,Y)\xi - \eta(Y)X] = 0.$$

From this, we obtain

$$g(th(X,Y),Z) + g(\beta[g(X,Y)\xi - \eta(Y)X],Z) = 0,$$

for any $X \in T\mathcal{M}$, $Y \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(\mathfrak{D}^{\perp})$.
This equation means that

$$g\left(A_{FZ}Y, X\right) + \beta\eta(Y)g(X,Z) = 0,$$

which gives

$$A_{FZ}Y + \beta\eta(Y)Z = 0,$$

for any $Y \in \Gamma(\mathfrak{D})$, $Z \in \Gamma(\mathfrak{D}^{\perp})$. Thus by virtue of Theorem 4.1, above equation tells us that the submanifold $\mathcal{M}$ is a contact CR-product.
Conversely, in a contact CR-product of SQ-Sasakian manifold then from Theorem 4.1 we get (4.1). From (4.1) it can be easily shown that the endomorphism T is parallel. □

Proposition 4.1. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$. Then, $\mathcal{M}$ is contact CR-product if and only if the following assertions is satisfied.*

(i) $\nabla_X Y \in \Gamma(\mathfrak{D}) \oplus \{\xi\}, \quad X \in T\mathcal{M}, \quad Y \in \Gamma(\mathfrak{D})$,

(ii) $th(X,Y) = 0, \quad X \in T\mathcal{M}, \quad Y \in \Gamma(\mathfrak{D})$,

(iii) $h(X,\phi Y) = fh(X,Y), \quad X \in T\mathcal{M}, \quad Y \in \Gamma(\mathfrak{D})$.

Proof. We suppose that $\mathcal{M}^n$ is a CR -product locally represented by $\mathcal{M}^{\top} \times \mathcal{M}^{\perp}$. Then $\mathcal{M}^{\top}$ and $\mathcal{M}^{\perp}$ are totally geodesic in $\mathcal{M}^n$. Thus, the Gauss formula implies:

$$\nabla_X Y \in \Gamma(\mathfrak{D}) \oplus \{\xi\}, \tag{4.4}$$

for any $X, Y \in \Gamma(\mathfrak{D}) \oplus \{\xi\}$ and

$$\nabla_Z W \in \Gamma(\mathfrak{D}^{\perp}), \tag{4.5}$$

for any $Z, W \in \Gamma(\mathfrak{D}^{\perp})$. Now, using (4.5) and (2.1), we obtain

$$g\left(\nabla_Z Y, W\right) = -g\left(Y, \nabla_Z W\right) = 0,$$

$$g\left(\nabla_\xi Y, W\right) = -g\left(-\beta\phi Y + [\xi, Y], W\right) = 0,$$

for any $Y \in \Gamma(\mathfrak{D})$ and Z, $W \in \Gamma(\mathfrak{D}^{\perp})$. Thus, from (4.4) we get that $\nabla_X Y \in \Gamma(\mathfrak{D}) \oplus \{\xi\}$, $X \in T\mathcal{M}$ and $Y \in \Gamma(\mathfrak{D}) \oplus \{\xi\}$. In the same way, $\nabla_X Z \in \Gamma(\mathfrak{D}^{\perp})$, $X \in T\mathcal{M}$, and $Z \in \Gamma(\mathfrak{D}^{\perp})$. Thus (i) is satisfied.
Next we prove other parts, from (3.1), (3.2), it follows that

$$\nabla_X TY = T\nabla_X Y + \beta\{g(X,Y)\xi - \eta(Y)X\} + th(X,Y), \tag{4.6}$$

and

$$h(X,TY) = F\nabla_X Y + fh(X,Y), \tag{4.7}$$

for any $X \in T\mathcal{M}$ and $Y \in \Gamma(\mathfrak{D})$.
Now taking inner product of (4.6) with $W \in \Gamma(\mathfrak{D}^{\perp})$, we get

$$g(th(X, Y), W) = 0. \tag{4.8}$$

Since $\nabla_X Y \in \Gamma(\mathfrak{D})$, from (4.7) we have

$$h(X, \phi Y) = fh(X, Y). \tag{4.9}$$

Thus, from (4.8) and(4.9), we get that assertions (ii) and (iii) are satisfied. Conversely, suppose that $(i) - (iii)$ holds. Thus, the distribution $\Gamma(\mathfrak{D}) \oplus \{\xi\}$ is integrable, since we have:

$$\begin{aligned} [X, Y] &= \nabla_X Y - \nabla_Y X \in \Gamma(\mathfrak{D}) \oplus \{\xi\} \\ [X, \xi] &= -\beta\phi X - \nabla_\xi X \in \Gamma(\mathfrak{D}) \oplus \{\xi\}, \end{aligned}$$

for any $X, Y \in \Gamma(\mathfrak{D})$. Moreover, if $\mathcal{M}^{\top}$ is a leaf of $\Gamma(\mathfrak{D}) \oplus \{\xi\}$, then, from (i) and the Gauss formula for the immersion of $\mathcal{M}^{\top}$ in $\mathcal{M}$, it follows that $\mathcal{M}^{\top}$ is totally geodesic in $\mathcal{M}$. Also, from (i) we get that $\nabla_X Z \in \Gamma(\mathfrak{D}^{\perp})$ for any $X \in T(\mathcal{M})$ and $Z \in \Gamma(\mathfrak{D}^{\perp})$. Using again the Gauss formula for a leaf $\mathcal{M}^{\perp}$ of $\Gamma(\mathfrak{D}^{\perp})$ we obtain that $\mathcal{M}^{\top}$ is totally geodesic in $\mathcal{M}$. So $\mathcal{M}$ is a CR-product. □

5 Pseudo parallel CR-submanifold of SQ-Sasakian manifold

Definition 3. [14] A CR-submanifold $\mathcal{M}$ of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ is called Chaki-pseudo parallel if h satisfies

$$(\bar{\nabla}_X h)(Y, Z) = 2\alpha(X)h(Y, Z) + \alpha(Y)h(X, Z) + \alpha(Z)h(X, Y) \tag{5.1}$$

for all $X, Y, Z \in T\mathcal{M}$, where α is a nowhere vanishing 1-form.

In particular if $\alpha(X) = 0$ then $\mathcal{M}$ is said to be parallel submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$.
We now prove the following:

Theorem 5.1. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$. Then $\mathcal{M}$ is mixed totally geodesic if $\mathcal{M}$ is Chaki-pseudo parallel provided $F\nabla_X Y + \alpha(Y)FX = 0$ and $\alpha(\xi) \neq 0$.*

Proof. Putting $Z = \xi$ in (5.1) and taking $X \in \Gamma(\mathfrak{D}^{\perp})$, $Y \in \Gamma(\mathfrak{D})$ and using Theorem 3.2 we get

$$-h(\xi, \nabla_X Y) = \alpha(\xi)h(X, Y) + \alpha(Y)h(X, \xi). \tag{5.2}$$

Again by using Theorem 3.2, we get

$$\beta F\nabla_X Y = -\alpha(Y)\beta FX + \alpha(\xi)h(X, Y). \tag{5.3}$$

If $\beta F\nabla_X Y + \alpha(Y)\beta FX = 0$ and $\alpha(\xi) \neq 0$ then from (5.3) we get $h(X, Y) = 0$, where $X \in \Gamma(\mathfrak{D}^{\perp})$ and $Y \in \Gamma(\mathfrak{D})$, i.e., $\mathcal{M}$ is mixed totally geodesic. This proves the theorem. □

Definition 4. [4, 5, 11] A CR-submanifold M of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ is said to be Deszcz-pseudo parallel if h satisfies

$$(\bar{R}(X, Y) \cdot h)(Z, U) \quad = \quad R^{\perp}(X, Y)h(Z, U) - h(R(X, Y)Z, U) \tag{5.4}$$

$$\begin{aligned} &- h(Z, R(X,Y)U) \\ &= L_h Q(g,h)(Z,U;X,Y), \end{aligned}$$

where L_h is some function on $W = \{x \in \mathcal{M} : (h - Hg)_x \neq 0\}$ for all vector fields X, Y tangent to $\mathcal{M}$ and the tensor Q is defined by

$$\begin{aligned} Q(g,h)(Z,U;X,Y) &= g(Y,Z)h(X,U) - g(X,Z)h(Y,U) \\ &+ g(Y,U)h(X,Z) - g(X,U)h(Y,Z). \end{aligned} \tag{5.5}$$

In particular, if $L_h = 0$ then $\mathcal{M}$ is said to be semi-parallel submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$.

Theorem 5.2. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$. If $\mathcal{M}$ is Deszcz-pseudo parallel then it is mixed totally geodesic provide $Z\beta = 0$, $h(A_{h(Y,Z)}\xi, \xi) = h(A_{-\beta FY}\xi, Z)$ and $L_h + \beta^2 \neq 0$.*

Proof. From (5.4) and (5.5) we have

$$\begin{aligned} &R^{\perp}(X,Y)h(Z,U) - h(R(X,Y)Z,U) - h(Z,R(X,Y)U) \\ &= L_h[g(Y,Z)h(X,U) - g(X,Z)h(Y,U) + g(Y,U)h(X,Z) - g(X,U)h(Y,Z)]. \end{aligned} \tag{5.6}$$

Substituting $X = U = \xi$ in (5.6) and using (2.5), (2.7) and Theorem 3.2, we get

$$h(A_{h(Y,Z)}\xi, \xi) - h(A_{-\beta FY}\xi, Z) = (L_h + \beta^2)h(Y,Z), \tag{5.7}$$

where $Z \in \Gamma(\mathfrak{D})$ and $Y \in \Gamma(\mathfrak{D})^{\perp}$. By virtue of (5.6) we get the result. □

Corollary 5.3. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$. If $\mathcal{M}$ is semi-parallel then it is mixed totally geodesic provide $Z\beta = 0$, $h(A_{h(Y,Z)}\xi, \xi) = h(A_{-\beta FY}\xi, Z)$.*

Definition 5. [14] A submanifold $\mathcal{M}$ of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ with respect to semi-symmetric metric connection is called Chaki-pseudo parallel if $\tilde{h}$ satisfies

$$(\tilde{\nabla}_X \tilde{h})(Y,Z) = 2\alpha(X)\tilde{h}(Y,Z) + \alpha(Y)\tilde{h}(X,Z) + \alpha(Z)\tilde{h}(X,Y) \tag{5.8}$$

for all $X, Y, Z \in T\mathcal{M}$, where α is a nowhere vanishing 1-form.

Theorem 5.4. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ with semi-symmetric metric connection. Then $\mathcal{M}$ is mixed totally geodesic if $\mathcal{M}$ is Chaki-pseudo parallel provided $F\nabla_X Y + \alpha(Y)FX = 0$ and $\alpha(\xi) + 1 \neq 0$, where $X \in \Gamma(\mathfrak{D}^{\perp})$ and $Y \in \Gamma(\mathfrak{D})$.*

Proof. Putting $Z = \xi$ in (5.8) and taking $X \in \Gamma(\mathfrak{D}^{\perp})$, $Y \in \Gamma(\mathfrak{D})$ and using Theorem 3.2 we get

$$-h(\xi, \nabla_X Y) - \eta(Y)h(X,\xi) - h(X,Y) = \alpha(\xi)h(X,Y) + \alpha(Y)h(X,\xi). \tag{5.9}$$

Again by using Theorem 3.2, we get

$$\beta F\nabla_X Y = -\alpha(Y)\beta FX + (\alpha(\xi) + 1)h(X,Y). \tag{5.10}$$

If $\beta F\nabla_X Y + \alpha(Y)\beta FX = 0$ and $(\alpha(\xi) + 1) \neq 0$ then from (5.10) we get $h(X,Y) = 0$, where $X \in \Gamma(\mathfrak{D}^{\perp})$ and $Y \in \Gamma(\mathfrak{D})$, i.e., $\mathcal{M}$ is mixed totally geodesic. This proves the theorem. □

Corollary 5.5. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ with semi-symmetric metric connection. Then $\mathcal{M}$ is mixed totally geodesic if $\mathcal{M}$ is parallel provided $\nabla_X Y \in \Gamma(\mathfrak{D})$.*

Definition 6. [4, 5, 11] A CR-submanifold $\mathcal{M}$ of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ with semi-symmetric metric connection is said to be Deszcz-pseudo parallel if $\tilde{h}$ satisfies

$$\begin{aligned}(\tilde{\bar{R}}(X,Y)\cdot\tilde{h})(Z,U) &= R^{\perp}(X,Y)\tilde{h}(Z,U) - \tilde{h}(R(X,Y)Z,U) \\ &- \tilde{h}(Z,R(X,Y)U) \\ &= L_{\tilde{h}}Q(g,\tilde{h})(Z,U;X,Y),\end{aligned} \tag{5.11}$$

for all $X, Y, Z, W \in T\mathcal{M}$, where $L_{\tilde{h}}$ is some function on $W = \{x \in \mathcal{M} : \ (\tilde{h} - Hg)_x \neq 0\}$. In particular, if $L_{\tilde{h}} = 0$ then $\mathcal{M}$ is said to be semi-parallel submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$.

Theorem 5.6. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ with semi-symmetric metric connection. If $\mathcal{M}$ is Deszcz-pseudo parallel then it is mixed totally geodesic provide $Z\beta = 0$, $h(A_{h(Y,Z)}\xi,\xi) + \beta h(\phi Y, Z) = h(A_{-\beta FY}\xi, Z)$ and $L_h + \beta^2 \neq 0$.*

Proof. From (5.11), we have

$$\begin{aligned}&R^{\perp}(X,Y)h(Z,U) - h(R(X,Y)Z,U) - h(Z,R(X,Y)U) \\ &= L_h[g(Y,Z)h(X,U) - g(X,Z)h(Y,U) + g(Y,U)h(X,Z) - g(X,U)h(Y,Z)].\end{aligned} \tag{5.12}$$

Substituting $X = U = \xi$ in (5.12) and using (2.20), (2.21) and Theorem 3.2, we get

$$h(A_{h(Y,Z)}\xi,\xi) + \beta h(\phi Y, Z) - h(A_{-\beta FY}\xi, Z) = (L_h + \beta^2)h(Y,Z), \tag{5.13}$$

where $Z \in \Gamma(\mathfrak{D})$ and $Y \in \Gamma(\mathfrak{D}^{\perp})$. By virtue of (5.13) we get the result. □

Corollary 5.7. *Let $\mathcal{M}$ be a CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ with semi-symmetric metric connection. If $\mathcal{M}$ is Deszcz-pseudo parallel then it is mixed totally geodesic provided $Z\beta = 0$, $h(A_{h(Y,Z)}\xi,\xi) + \beta h(\phi Y, Z) = h(A_{-\beta FY}\xi, Z)$.*

6 Almost Ricci soliton and Almost Yamabe solitons with torse-forming vector field on CR-submanifold of a SQ-Sasakian manifold

A Riemannian manifold $(\mathcal{M}^n, g)$ is said to be a Ricci soliton if there exists a vector field V on $\mathcal{M}$ satisfying the following equation [21]

$$\frac{1}{2}L_V g(X,Y) + S(X,Y) = \lambda g(X,Y), \tag{6.1}$$

where L_V is the Lie derivative with respect to V, $S(X,Y)$ is the Ricci tensor of $(\mathcal{M}^n, g)$ and λ is constant. If λ is a smooth function on $\mathcal{M}$ then $(\mathcal{M}^n, g, V, \lambda)$ is said to be an almost Ricci soliton. A Riemannian manifold $(\mathcal{M}^n, g)$ is said to be a Yamabe soliton if there exists a vector field V on $\mathcal{M}$ satisfying the following equation [7]

$$\frac{1}{2}L_V g(X,Y) = (\delta - \lambda)g(X,Y), \tag{6.2}$$

where L_V is the Lie derivative with respect to V, δ is scalar curvature on (M^n, g) and λ is constant. If λ is a smooth function on $\mathcal{M}$ then $(\mathcal{M}^n, g, V, \lambda)$ is said to be an almost Yamabe soliton. The soliton is called expanding, steady or shrinking according as $\lambda > 0, \lambda = 0$ or $\lambda < 0$, respectively.
A vector field V on a Riemannian manifold $(\mathcal{M}^n, g)$ is known as a torse-forming vector field [34] if it satisfies

$$\nabla_X V = \psi X + \theta(X)V, \tag{6.3}$$

where ψ is some smooth function on $\mathcal{M}$ and θ is a 1-form. The torse-forming vector field is

- concircular if θ vanishes identically,
- concurrent if $\psi = 1$ and $\theta = 0$,
- recurrent if $\psi = 1$,
- parallel if $\psi = \theta = 0$.

We now consider $\mathcal{M}$ be CR-submanifold of a SQ-Sasakian manifold $\bar{\mathcal{M}}$ with respect to semi-symmetric metric connection $\tilde{\bar{\nabla}}$. Thus we have the following decomposition for any $X, Y \in T\mathcal{M}$

$$X = PX + QX, \tag{6.4}$$

where P and Q are orthogonal projections on horizontal and vertical distribution $\mathfrak{D}$ and $\mathfrak{D}^\perp$ respectively.
Since $X, \xi \in T\mathcal{M}$, by equating the horizontal, vertical and normal components of (2.16) we get

$$P\tilde{\nabla}_X Y = P\nabla_X Y + \eta(Y)PX - g(X,Y)P\xi, \tag{6.5}$$

$$\tilde{h}(X,Y) = h(X,Y) \tag{6.6}$$

and

$$Q\tilde{\nabla}_X Y = Q\nabla_X Y + \eta(Y)QX - g(X,Y)Q\xi. \tag{6.7}$$

In this section, we discuss almost Ricci solitons and almost Yamabe solitons whose potential field is torse forming on CR-submanifold of SQ-Sasakian manifold with respect to semi-symmetric metric connection. Here, we denote V^t and V^n as tangential and normal components of such vector field.

Theorem 6.1. *An almost Ricci soliton (g, V^t, λ) on a CR-submanifold $\mathcal{M}$ of an SQ-Sasakian manifold $\bar{\mathcal{M}}$ with a semi-symmetric metric connection and V as a torse-forming vector field satisfies*

$$\begin{aligned} S(X,Y) &= (\lambda + \eta(V^n) - \psi + 2n - 1)g(X,Y) + (2n-1)\{\eta(X)\eta(Y) \\ &+ \Phi(X,Y)\} - \frac{1}{2}\{\theta(X)g(V,Y) + \theta(Y)g(X,V)\} \end{aligned} \tag{6.8}$$

for any vector fields X, Y on $\mathcal{M}$.

Proof. In view of (2.9), (6.3) and (2.15), we have

$$\begin{aligned} \psi X + \theta(X)V &= \tilde{\bar{\nabla}}_X V = \tilde{\bar{\nabla}}_X \left(V^t + V^n\right) \\ &= \nabla_X V^t + h\left(X, V^t\right) - A_{V^n}X + \nabla_X^\perp V^n + \eta\left(V^n\right)X. \end{aligned} \tag{6.9}$$

Comparing tangential and normal components of (6.9), we obtain

$$\nabla_X V^t = \psi X + \theta(X)V + A_{V^n}X - \eta(V^n)X, \tag{6.10}$$

and

$$h(X, V^t) = -\nabla^{\perp}_X V^n. \tag{6.11}$$

From the definition of Lie derivative and (6.10), we have

$$\begin{aligned} L_{V^t}g(X,Y) &= g(\nabla_X V^t, Y) + g(X, \nabla_Y V^t) \\ &= 2\psi g(X,Y) + 2g(A_{V^n}X, Y) - 2\eta(V^n)g(X,Y) \\ &+ \theta(X)g(V,Y) + \theta(Y)g(X,V). \end{aligned} \tag{6.12}$$

Using (6.12), (2.19) and (6.1), it yields

$$\begin{aligned} S(X,Y) &= (\lambda + \eta(V^n) - \psi + 2n - 1)g(X,Y) + (2n-1)\{\eta(X)\eta(Y) \\ &+ \Phi(X,Y)\} - \frac{1}{2}\{\theta(X)g(V,Y) + \theta(Y)g(X,V)\}. \end{aligned} \tag{6.13}$$

This proves our assertion. □

Corollary 6.2. *An almost Ricci soliton (g, V^t, λ) on a CR-submanifold $\mathcal{M}$ of an SQ-Sasakian manifold $\bar{\mathcal{M}}$ with a semi-symmetric metric connection and V as a concircular vector field is pseudo η-Einstein.*

Proof. Since V is concircular, i.e. $\theta = 0$ identically. Putting $\theta = 0$ in (6.8) we get the corollary. □

Theorem 6.3. *An almost Yamabe soliton (g, V^t, λ) on a CR-submanifold $\mathcal{M}$ of an SQ-Sasakian manifold $\bar{\mathcal{M}}$ with a semi-symmetric metric connection and V as a torse-forming vector field satisfies*

$$\begin{aligned} \left(\tilde{\delta} - \lambda - \psi + \eta(V^n)\right) g(X,Y) &= g(A_{V^n}X, Y) + \frac{1}{2}\{\theta(X)g(V,Y) \\ &+ \theta(Y)g(X,V)\} \end{aligned} \tag{6.14}$$

for any vector fields X, Y on $\mathcal{M}$, where $\tilde{\delta}$ is scalar curvature on (M^n, g) with respect to semi-symmetric connection.

Proof. By virtue of (6.12) and (6.2), we get (6.14). This proves the Theorem. □

Corollary 6.4. *If an almost Yamabe soliton (g, V^t, λ) on a CR-submanifold $\mathcal{M}$ of an SQ-Sasakian manifold $\bar{\mathcal{M}}$ with a semi-symmetric metric connection and V as a torse-forming vector field is minimal, then $\left(\tilde{\delta} - \lambda - \psi + \eta(V^n)\right) n = \theta(V)$ holds.*

Proof. Since M^n is minimal, then from (6.14) we get the corollary. □

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