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MULTIPLIERS FOR $|C,\delta|_k$ SUMMABILITY OF FOURIER SERIES

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Abstract. Recently PANDEY[2] proved a result on $|C,1|$ summability of Fourier series improving the conditions of some previously known theorems on the absolute $(C,1)$ summability factors of Fourier series. In the present paper we extend that result.

1. Introduction

Let Σa_n be a given infinite series with partial sums s_n. Let σ_n^δ and t_n^δ denote the n^{th} Cesáro mean of order $\delta(\delta > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$, respectively. The series Σa_n is said to be *absolutely summable* (C,δ) with index $k \geq 1$, or simply *summable* $|C,\delta|_k$, if

$$\sum_{n=1}^{\infty} n^{k-1} \mid \sigma_n^\delta - \sigma_{n-1}^\delta \mid^k < \infty$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-1} \mid t_n^\delta \mid^k < \infty.$$

Let $f(t)$ be Lebesgue integrable periodic function with period 2π and $\Sigma_{n=0}^{\infty} A_n(t)$ denote its Fourier series. Then for $n \geq 1$,

$$\pi A_n(x) = \int_0^\pi \phi(t) \cos nt \, dt$$

where $\phi(t) = f(x+t) + f(x-t) - 2f(x)$.

We write

$$\varphi(t) = \int_t^\gamma u^{-1} \mid \phi(u) \mid du, \qquad 0 < \gamma \leq \pi$$

$$\mu_n = \left(\Pi_{v=1}^{\ell-1} \log^v n\right)(log^\ell n)^{1+\varepsilon}, \ \log^\ell n_0 > 0, \ \varepsilon > 0, \ \ell \geq 2,$$

where $\log^\ell n = \log(\log^{\ell-1} n), \cdots, \log^2 n = \log\log n$.

Recently, PANDEY[2] has proved the following theorem including some previous results

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Theorem A. *If*

$$\varphi(t) = O\{(\log^{\ell}(1/t))^{\eta}\} \text{ as } t \to +0,$$

then the series

$$\sum_{n=n_0}^{\infty} A_n(x)/\mu_n$$

is summable $|C,1|$ *for* $0 < \eta < \varepsilon$.

We give the following extension of theorem A.

Theorem B. *Let* $\{\lambda_n\}$ *be any sequence of constants. Let* $g(u)$ *and* $h(u)$ *be positive functions such that* $H(u) = uh(u)$, $u^{\beta}g(1/u)$ *are both nondecreasing for some* β, $0 < \beta < 1$. *Suppose for* $k \geq 1$

$$\varphi(t) = O\{g(1/t)\}, \qquad t \to 0;$$

$$\sum_{n=1}^{\infty} n^{2k-\delta k-1} |\lambda_n|^k [h(n)]^k [g(n)]^k < \infty;$$

and

$$\sum_{n=1}^{\infty} n^{2k-1} |\Delta\lambda_n|^k [h(n)]^k [g(n)]^k < \infty.$$

Then the series $\Sigma n\lambda_n h(n) A_n(x)$ *is summable* $|C,\delta|_k$, $0 < \delta \leq 1$.

2. The following results are needed:

Lemma 1[1]. *If* $\sigma > -1$ *and* $\sigma - \delta > 0$, *then*

$$\sum_{n=\mu}^{\infty} \frac{A_{n-\mu}^{\delta}}{nA_n^{\sigma}} = \frac{1}{\mu A_{\mu}^{\sigma-\delta-1}}.$$

Theorem 2. *The series* $\Sigma\lambda_n a_n$ *is summable* $|C,\delta|_k$, $k \geq 1$, $0 < \delta \leq 1$, *if the following holds:*

$$\sum_{n=1}^{\infty} n^{k-\delta k-1} |\lambda_n|^k |s_n|^k < \infty;$$

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta\lambda_n|^k |s_n|^k < \infty.$$

Proof. Let T_n^{δ} be the nth Cesáro mean of order δ of the sequence $\{n\lambda_n a_n\}$. Then we have

$$T_n^\delta = \frac{1}{A_n^\delta}\sum_{v=1}^{n} A_{n-v}^{\delta-1} v\lambda_v a_v,$$

where

$$A_n^\delta = \binom{n+\delta}{n} = \frac{(\delta+1)(\delta+2)\cdots(\delta+n)}{n!} \sim \frac{n^\delta}{\Gamma(\delta+1)}.$$

$$\begin{aligned} T_n^\delta &= \frac{1}{A_n^\delta}\Big[\sum_{v=1}^{n-1}\Big\{ - A_{n-v}^{\delta-1}\lambda_v s_v + (v+1)\Delta A_{n-v}^{\delta-1}\lambda_v s_v \\ &\qquad + (v+1)A_{n-v-1}^{\delta-1}\Delta\lambda_v s_v\Big\} + n\lambda_n s_n\Big] \\ &= T_{n,1}^\delta + T_{n,2}^\delta + T_{n,3}^\delta + T_{n,4}^\delta, \qquad \textit{say}. \end{aligned}$$

To prove the Theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1} | T_{n,r}^\delta |^k < \infty, \qquad r = 1,2,3,4.$$

Applying Hölder's inequality, we have

$$\begin{aligned} \sum_{n=1}^{m} n^{-1} | T_{n,1}^\delta |^k &\le \sum_{n=1}^{m} \frac{1}{n(A_n^\delta)^k}\Big\{\sum_{v=1}^{n-1} A_{n-v}^{\delta-1} | \lambda_v || s_v |\Big\}^k \\ &\le \sum_{n=1}^{m} \frac{1}{nA_n^\delta}\sum_{v=1}^{n-1} A_{n-v}^{\delta-1} | \lambda_v |^k | s_v |^k \Big\{\frac{1}{A_n^\delta}\sum_{v=1}^{n-1} A_{n-v}^{\delta-1}\Big\}^{k-1} \\ &= O(1)\sum_{n=1}^{m} \frac{1}{nA_n^\delta}\sum_{v=1}^{n-1} A_{n-v}^{\delta-1} | \lambda_v |^k | s_v |^k \\ &= O(1)\sum_{v=1}^{m-1} | \lambda_v |^k | s_v |^k \sum_{n=v}^{m} \frac{A_{n-v}^{\delta-1}}{nA_n^\delta} \end{aligned}$$

hence, by lemma 1

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} | T_{n,1}^\delta |^k &= O(1)\sum_{v=1}^{\infty} v^{-1} | \lambda_v |^k | s_v |^k \\ &= O(1)\sum_{v=1}^{\infty} v^{k-\delta k-1} | \lambda_v |^k | s_v |^k \end{aligned}$$

$$\sum_{n=1}^{\infty} n^{-1} \mid T_{n,2}^{\delta} \mid^k \leq \sum_{n=1}^{m} \frac{1}{n(A_n^\delta)^k} \Big\{ \sum_{v=1}^{n-1} (v+1) \mid \Delta A_{n-v}^{\delta-1} \mid\mid \lambda_v \mid\mid s_v \mid \Big\}^k$$

$$(when\ \delta = 1,\ T_{n,2}^{\delta} = 0\ as\ \Delta A_{n-v}^{\delta-1} = 0)$$

$$\leq \sum_{n=1}^{m} \frac{1}{n(A_n^\delta)^k} \sum_{v=1}^{n-1} (v+1)^k \mid \Delta A_{n-v}^{\delta-1} \mid\mid \lambda_v \mid^k \mid s_v \mid^k \cdot \Big\{ \sum_{v=1}^{n-1} \mid \Delta A_{n-v}^{\delta-1} \mid \Big\}^{k-1}$$

$$= O(1) \sum_{n=1}^{m} \frac{1}{n(A_n^\delta)^k} \sum_{v=1}^{n-1} v^k \mid \Delta A_{n-v}^{\delta-1} \mid\mid \lambda_v \mid^k \mid s_v \mid^k$$

$$\Big(as \sum_{v=1}^{n-1} \mid \Delta A_{n-v}^{\delta-1} \mid = O\Big\{ \sum_{v=1}^{n-1} (n-v)^{\delta-2} \Big\} = O(1), \quad 0 \leq \delta < 1 \Big)$$

$$= O(1) \sum_{v=1}^{m-1} v^k \mid \lambda_v \mid^k \mid s_v \mid^k \sum_{n=v}^{m} \frac{\mid \Delta A_{n-v}^{\delta-1} \mid}{n(A_n^\delta)^k}$$

$$= O(1) \sum_{v=1}^{m} v^{k-\delta k-1} \mid \lambda_v \mid^k \mid s_v \mid^k \sum_{n=v}^{m} (n-v)^{\delta-2}$$

therefore

$$\sum_{n=1}^{\infty} n^{-1} \mid T_{n,2}^{\delta} \mid^k = O(1) \sum_{v=1}^{\infty} v^{k-\delta k-1} \mid \lambda_v \mid^k \mid s_v \mid^k$$

$$\sum_{n=1}^{m} n^{-1} \mid T_{n,3} \mid^k = O(1) \sum_{n=1}^{m} \frac{1}{n(A_n^\delta)^k} \Big\{ \sum_{v=1}^{n-1} v\, A_{n-v}^{\delta-1} \mid \Delta\lambda_v \mid\mid s_v \mid \Big\}^k$$

$$= O(1) \sum_{n=1}^{m} \frac{1}{nA_n^\delta} \sum_{v=1}^{n-1} v^k A_{n-v}^{\delta-1} \mid \Delta\lambda_v \mid^k \mid s_v \mid^k \Big\{ \frac{1}{A_n^\delta} \sum_{v=1}^{n-1} A_{n-v}^{\delta-1} \Big\}^{k-1}$$

$$= O(1) \sum_{n=1}^{m} \frac{1}{nA_n^\delta} \sum_{v=1}^{n-1} v^k A_{n-v}^{\delta-1} \mid \Delta\lambda_v \mid^k \mid s_v \mid^k$$

$$= O(1) \sum_{n=1}^{m-1} v^k \mid \Delta\lambda_v \mid^k \mid s_v \mid^k \sum_{n=v}^{m} \frac{A_{n-v}^{\delta-1}}{nA_n^\delta},$$

then, by Lemma 1

$$\sum_{n=1}^{\infty} n^{-1} | T_{n,3}^{\delta} |^k = O(1) \sum_{v=1}^{\infty} v^{k-1} | \Delta\lambda_v |^k | s_v |^k$$

$$\sum_{n=1}^{\infty} n^{-1} | T_{n,4}^{\delta} |^k \leq \sum_{n=1}^{\infty} \frac{n^k | \lambda_n |^k | s_n |^k}{n(A_n^{\delta})^k}$$
$$= O(1) \sum_{n=1}^{\infty} n^{k-\delta k-1} | \lambda_n |^k | s_n |^k .$$

Lemma 3. *Suppose $g(u)$ is a positive function such that $u^{\beta} g(1/u)$ nondecreasing for some β, $0 < \beta < 1$. If*

$$\varphi(t) = O\{g(1/t)\}, \qquad t \to 0,$$

then

$$\int_0^t | \phi(u) | \, du = O\{tg(1/t)\}.$$

Proof.

$$\int_0^t | \phi(u) | \, du = \int_0^t -u\varphi'(u) du$$
$$= \Big[-u\varphi(u)\Big]_0^t + \int_0^t \Big\{u^{\beta}\varphi(u)\}u^{-\beta} du$$
$$= O\{tg(1/t)\} + O\{t^{\beta} g(1/t) \int_0^t u^{-\beta} du\}$$
$$= O\{tg(1/t)\}.$$

3. Proof of Theorem B.

Let $S_n(x)$ be the n^{th} partial sum of the sequence $\{H(n)A_n(x)\}$. Then we have

$$S_n(x) = \sum_{v=1}^{n} H(v) \int_0^{\pi} \cos(vu)\phi(u) du$$
$$= \Big\{ \int_0^{n^{-1}} + \int_{n^{-1}}^{\pi} \Big\} \Big\{ \sum_{v=1}^{n} H(v) \cos(uv) \Big\} \phi(u) du$$
$$= I_1 + I_2, \qquad \textit{say}.$$

Since $H(u)$ is nonnegative, nondecreasing, thus, we have by Abel's lemma

$$
\begin{aligned}
\Big|\sum_{v=1}^{n} H(v)\cos(vu)\Big| &= O\{H(n)\max_{1\le r\le n}\Big|\sum_{v=r}^{n}\cos(vu)\Big|\} \\
&= O\Big\{sH(n)\Big\},
\end{aligned}
$$

where $s = n$ or u^{-1}.

Therefore

$$
\begin{aligned}
|I_1| &= O\Big\{n\, H(n)\int_0^{n^{-1}} |\phi(u)|\, du\Big\} = O\Big\{H(n)g(n)\Big\}, \\
|I_2| &= O\Big\{H(n)\int_{n^{-1}}^{\pi} u^{-1} |\phi(u)|\, du\Big\} = O\Big\{H(n)g(n)\Big\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|S_n(x)| &= O\Big\{H(n)\ g(n)\Big\}, \\
|S_n(x)|^k &= O[H(n)]^k[g(n)]^k.
\end{aligned}
$$

The theorem follows by making use of theorem 2.

Remark. If we are putting : $k = 1$, $\delta = 1$, $h(n) = n^{-1}$, $\lambda_n = \mu_n$ and $g(n) = (\log^{\ell} n)^{\eta}$ in theorem B, we obtain theorem A.

References

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