

SOLUTION OF SECOND AND THIRD ORDER LINEAR DIFFERENTIAL EQUATIONS BY FRACTIONAL CALCULUS

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Abstract. The method of fractional calculus is used to obtain particular solutions of some second and third order linear differential equations. Corresponding homogeneous equations are considered. Some results given recently by Nishimoto and Nishimoto & Kalla follow as particular cases of our theorems.

1. Introduction.

Fractional calculus deals with derivatives and integrals arbitrary orders, called differintegrals [1,2,3,4,5]. Many results about differintegrals of complex order ν are motivated by the corresponding results of ordinary derivatives and integrals of integer order. There are several definitions of an integral of fractional order, one of the simplest is the Riemann-Liouville operator of fractional integration [2].

$$R_x^\nu f = I_x^\nu = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt \quad \text{for } \operatorname{Re}(\nu) > 0 \quad (1)$$

$$= \frac{d^n}{dx^n} R_x^{\nu+n} f; \quad \text{for } \operatorname{Re}(\nu) < 0 \quad (2)$$

Grunwald [1], has defined another form of the differintegral of order ν

$$\frac{d^\nu f}{[d(x-a)]^\nu} = \lim_{N \rightarrow \infty} \left\{ \frac{\left[\frac{x-a}{N} \right]^{-\nu}}{\Gamma(-\nu)} \sum_{j=0}^{N-1} \frac{\Gamma(j-\nu)}{\Gamma(j+1)} f\left(x - j \left[\frac{x-a}{N} \right]\right) \right\} \quad (3)$$

where ν is arbitrary.

Nishimoto [3,5], defines the differintegral in the following form

$$f_\nu = {}_c f_\nu(z) = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{\nu+1}} d\xi \quad (4)$$

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where $f(z)$ is a regular function and it has no branch points inside and on $C(C = \{C_-, C_+\})$, C_- is an integral curve along the cut joining two points z and $-\infty + i \operatorname{Im}(z)$ and C_+ is an integral curve along cut joining two points z and $\infty + i \operatorname{Im}(z)$ where $\nu \notin Z^-; \nu \in R$ and

$$f_{-n} = \lim_{\nu \rightarrow -n} f_\nu \quad (n \in Z^+) \tag{5}$$

(Z^- and Z^+ are the set of negative and positive integers)

$$\xi \notin z, \quad -\pi \leq \arg(1 - z) \leq \pi \quad \text{for } C_-$$

and

$$0 \leq \arg(\xi - z) \leq 2\pi \quad \text{for } C_+$$

$f_\nu (\nu > 0)$ is the fractional derivative of order ν and $f_\nu (\nu < 0)$ is the fractional integral of order ν , If f_ν exists, the principal value of f will be considered for many valued functions.

Various authors [6,7,8,9] have defined and studied differintegral operators and their applications.

Recently [9] has studied a generalized m -dimensional operators of fractional integration which are defined by means of a convolution type integrals with a suitable Meijer's G -function as a kernel, viz. $G_{m,m}^{m,0}, m \geq 1$.

They can be considered also as compositions of arbitrary numbers ($m \geq 1$) of commuting Erdelyi-Kober fractional integrals and most of the known differintegral operators becomes as special cases.

Many physical applications of operators are mentioned in [1,4]. Kalla-Ross [10], Kalla-Al-Saqabi [11] have applied fractional calculus in the summation of series.

K. Nishimoto [12,13,5,14] has obtained solutions of several differential equations by using the fractional calculus. Recently Nishimoto and Kalla [15] have applied fractional calculus to solve a third order linear ordinary differential equation.

Fractional differintegration of product uv [3]: Let $u(z)$ and $v(z)$ be analytic and are one valued functions respectively, then $\{uv\}_\nu$ is fractional derivative of product uv for $\operatorname{Re}(\nu) > 0$, original product uv for $\nu = 0$ and fractional integral of product uv for $\operatorname{Re}(\nu) < 0$, if u_ν and v_ν exist, where

$$\begin{cases} \{uv\}_\nu = \{(uv)_\nu, (vu)_\nu\} \\ \begin{cases} (uv)_\nu = \sum_{n=0}^{\infty} P(\nu, n) u_{\nu-n} v_n \\ (vu)_\nu = \sum_{n=0}^{\infty} P(\nu, n) v_{\nu-n} u_n \\ \text{and } P(\nu, n) = \frac{\Gamma(\nu + 1)}{\Gamma(\nu - n + 1)\Gamma(n + 1)} \\ z, \nu \in C \end{cases} \end{cases} \tag{6}$$

In this paper, we obtain particular solutions to some second and third order linear ordinary differential equations by invoking the fractional calculus technique. Corresponding homogeneous cases are considered. Some known results follow as particular cases of our theorems established here.

2. Solutions of second order linear differential equations.

Theorem 1. *If $f_\nu (\neq 0)$ exists, then the non-homogeneous second order linear ordinary differential equation*

$$\phi_2 \cdot (z^2 - \mu z) + \phi_1(2\nu z - \mu\nu + \mu) + \phi\nu(\nu - 1) = f \tag{7}$$

has a particular solution of the form

$$\phi = [(f_{1-\nu} \frac{z - \mu}{z})_{-1} \frac{1}{(z - \mu)^2}]_{\nu-2} \tag{8}$$

where $\phi = \phi(z)$, $f = f(z)$, $z \in C$ and μ, ν are constants.

Proof. Putting

$$\phi = W_{\nu-1} = W_{\nu-1}(z) \tag{9}$$

yields

$$\phi_1 = W_\nu \tag{10}$$

and

$$\phi_2 = W_{\nu+1} \tag{11}$$

The differential equation (7) can be written as follows after using (9), (10), (11), and (6).

$$(w_1 \cdot z^2)_\nu - \mu(w_2 \cdot z)_{\nu-1} = f \tag{12}$$

By making the differintegral of order $(1 - \nu)$ of both sides of (12), we get

$$w_2(z^2 - \mu z) + w_1 \cdot 2z = f_{1-\nu}. \tag{13}$$

Setting

$$w_1 = u(z) \tag{14}$$

Equation (13) can be written as

$$\frac{du}{dz} + \frac{2}{(z - \mu)}u = \frac{f_{1-\nu}}{(z^2 - \mu z)}. \tag{15}$$

The first order differential equation (15) has a solution.

$$u = \left(f_{1-\nu} \frac{z-\mu}{z}\right)^{-1} \frac{1}{(z-\mu)^2} \quad (16)$$

Hence, from (11), (14) and (16), $\phi(z)$ can be written as given by (8).

Theorem 2. *The homogeneous second order linear ordinary differential equation*

$$\phi_2(z^2 - \mu z) + \phi_1(2\nu z - \mu\nu + \mu) + \phi\nu(\nu - 1) = 0 \quad (17)$$

has a particular solution of the form

$$\phi = A[(z - \mu)^{-1}]_{\nu-1} \quad (18)$$

where $\phi = \phi(z)$, $z \in C$ and μ, ν and A are constants.

Proof. Let

$$\phi = w_{\nu-1}(z) \quad (19)$$

This gives

$$\phi_1 = w_\nu \quad (20)$$

and

$$\phi_2 = w_{\nu+1} \quad (21)$$

Using (19), (20), (21) and (6) in (17), we get

$$(w_2(z^2 - \mu z) + w_1 \cdot 2z)_{\nu-1} = 0 \quad (22)$$

or it can be written as

$$w_2(z^2 - \mu z) + w_1 2z = 0 \quad (23)$$

solution of (23) can be written as

$$w_1 = \frac{A_1}{(z - \mu)^2} \quad (24)$$

A_1 is constant, hence, from (19) and (24), the particular solution of (17) is given by (18).

For $\mu = 1$, the theorems 1 and 2 reduces to the results given recently by Nishimoto [5, p.58].

Example of Theorem 1.

Let $\nu = \frac{1}{2}$, $\mu = 1$ and $f(z) = \frac{1}{z}$

We have from (7) and (8) respectively

$$\phi_2(z^2 - z) + \phi_1\left(z + \frac{1}{2}\right) - \frac{\phi}{4} = \frac{1}{z} \quad (25)$$

and

$$\phi = | \left(\left(\frac{1}{z} \right)_{\frac{1}{2}} \frac{z-1}{z} \right)_{-1} \frac{1}{(z-1)^2} |_{-\frac{3}{2}} \tag{26}$$

From [3], we can easily show that

$$\left(\frac{1}{z} \right)_{\frac{1}{2}} = \frac{-i}{2} \sqrt{\pi} z^{-\frac{3}{2}}$$

and

$$\left(\frac{-i\sqrt{\pi}}{2} z^{-\frac{5}{2}} (z-1) \right)_{-1} = i\sqrt{\pi} \left(z^{-\frac{1}{2}} - \frac{1}{3} z^{-\frac{3}{2}} \right)$$

and since

$$z^{-2} \left(1 - \frac{1}{z} \right)^{-2} = \sum_{k=1}^{\infty} k z^{-(k+1)}. \tag{27}$$

Therefore (26) can be written as

$$\phi = \sqrt{\pi} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})} \left(1 - \frac{2k}{6k+9} \left(\frac{1}{z} \right) \right) z^{-k} \tag{28}$$

Theorem 3. *If $f_{\nu} (\neq 0)$ exists, then the non-homogeneous second order linear ordinary differential equation*

$$\begin{aligned} \phi_2(z^2 - \mu z) + \phi_1(2\nu z - \mu\nu - \mu) + \phi\nu(\nu - 1) &= f \\ z \neq 0, \mu \end{aligned} \tag{29}$$

has a particular solution of the form

$$\phi = \left[\left(f_{-\nu} \frac{1}{(z-\mu)^2} \right)_{-1} \left(\frac{z-\mu}{z} \right) \right]_{\nu-1} \tag{30}$$

where $\phi = \phi(z)$, $z \in C$, μ and ν are constants.

Proof. Setting

$$\phi = w_{\nu}(z) \tag{31}$$

yields

$$\phi_1 = w_{\nu+1} \tag{32}$$

$$\phi_2 = w_{\nu+2} \tag{33}$$

Using (31), (32), (33) and (6) in (29), we obtain

$$(w_2 \cdot z^2)_{\nu} - \mu(w_2 \cdot z + w_1)_{\nu} = f \tag{34}$$

Making the differintegral of both sides of (34) of order $(-\nu)$, we get

$$w_2(z^2 - \mu z) - \mu w_1 = f_{-\nu} \quad (35)$$

This equation can be written as

$$(w_1 \frac{z}{z - \mu})_1 = \frac{f_{-\nu}}{(z - \mu)^2} \quad (36)$$

and

$$w_1 = (\frac{f_{-\nu}}{(z - \nu)^2})_{-1} (\frac{z - \mu}{z}) \quad (37)$$

then

$$w = [(\frac{f_{-\nu}}{(z - \mu)^2})_{-1} (\frac{z - \mu}{z})]_{-1} \quad (38)$$

and ϕ can be written from (31) as

$$\phi = [(\frac{f_{-\nu}}{(z - \mu)^2})_{-1} (\frac{z - \mu}{z})]_{\nu-1}.$$

Theorem 4. *The homogeneous second order linear ordinary differential equation*

$$\begin{aligned} \phi_2(z^2 - \mu z) + \phi_1(2\nu z - \mu\nu - \mu) + \phi\nu(\nu - 1) &= 0 \\ z \neq 0, \mu \end{aligned} \quad (39)$$

where $\phi = \phi(z)$, $z \in C$, μ and ν are constants, has a particular solution

$$\phi = B(z - \mu/\ln z)_\nu \quad (40)$$

where B is constant

Proof. Letting

$$\phi = w_\nu(z) \quad (41)$$

yields

$$\phi_1 = w_{\nu+1} \quad (42)$$

$$\phi_2 = w_{\nu+2} \quad (43)$$

We can obtain as in proof of theorem 3, the following

$$(w_2 z^2)_\nu - \mu(w_2 z + w_1)_\nu = 0 \quad (44)$$

By making the differintegral of both sides of (44) of order $(-\nu)$, we have

$$w_2 = \frac{\mu}{z^2 - \mu z} w_1 \quad (45)$$

The solution of (45) is given by

$$w = B(z - \ln z) \tag{46}$$

From (41), the solution of (39) can be given in the form of (40).

The results given by Nishimoto [5, p.45] follow from the theorems 3 and 4, for $\mu = 1$.

Theorem 5. *If $f_\nu (\neq 0)$ exists, then the non-homogeneous second order linear ordinary differential equation*

$$\begin{aligned} \phi_2(z + b) + \phi_1(\nu + 1 - ab - az) - \phi a\nu &= f \\ z &\neq -b \end{aligned} \tag{47}$$

has a particular solution

$$\phi = [(f_{-\nu} e^{-a(z+b)})_{-1} \frac{e^{a(z+b)}}{(z+b)}]_{\nu-1} \tag{48}$$

where $\phi = \phi(z)$, $z \in C$ and ν , a and b are constants.

Proof. Letting

$$\phi = w_{\nu-1}(z) \tag{49}$$

yields

$$\phi_1 = w_\nu \tag{50}$$

$$\phi_2 = w_{\nu+1} \tag{51}$$

Hence, we can write (47) after using (49), (50), (51) and (6) in the following form

$$(w_1(z + b))_\nu + (w(1 - ab))_\nu - a(wz)_\nu = f \tag{52}$$

This equation becomes

$$w_1(z + b) + w((1 - ab) - az) = f_{-\nu}, \tag{53}$$

by making the differintegral of order $(-\nu)$ to both sides of (52).

Integrating (53), we can easily get

$$w = (f_{-\nu} e^{-a(z+b)})_{-1} \frac{e^{a(z+b)}}{(z+b)} \tag{54}$$

Finally, we can obtain the particular solution of (47) in the form given by (48) by use of (49).

Theorem 6. *The homogeneous second order linear ordinary differential equation*

$$\phi_2(z+b) + \phi_1(\nu+1-ab-az) - \phi_{a'} = 0 \quad (55)$$

has a particular solution

$$\phi = \left(\frac{Ke^{az}}{z+b}\right)^{\nu-1}, \quad (56)$$

where $\phi = \phi(z)$, $z \in C$, ν, a, b and K are constants.

Proof. By following the same method as in obtaining equation (53), we have

$$w_1(z+b) + w((1-ab)-az) = 0 \quad (57)$$

Integrating (57), we obtain

$$w = K \frac{e^{az}}{(z+b)}, \quad (58)$$

Hence, from (49), the particular solution of (55) is in the form of (56).

3. Solution of a third order linear ordinary differential equation.

Theorem 7. *If $f_\alpha (\neq 0)$ exists, then non-homogeneous third order linear ordinary differential equation*

$$\begin{aligned} \phi_3(z^3 - \delta^2 z) + \phi_2\{(3\alpha + \beta + \gamma)z^2 + \beta\gamma z - \alpha\delta^2\} \\ + \phi_1\{\alpha(3\alpha + 2\beta + 2\gamma - 3)z + \alpha\beta\gamma\} \\ + \phi\alpha(\alpha - 1)(\alpha + \beta + \gamma - 2) = f \\ (z \neq 0, \pm\delta) \end{aligned} \quad (59)$$

has a particular solution

$$\phi = \left((f_{-\alpha} \frac{(z-\delta)^A (z+\delta)^B}{z^3 - \delta^2 z})^{-1} \frac{1}{(z-\delta)^A (z+\delta)^B} \right)^{\alpha-2} \quad (60)$$

where

$\phi = \phi(z)$, $f = f(z)$, $z \in C$, α, β, γ and δ^2 are constants and

$$A = \frac{1}{2}(\beta + \gamma + \frac{\beta\gamma}{\delta}); \quad B = \frac{1}{2}(\beta + \gamma - \frac{\beta\gamma}{\delta}).$$

Proof. Putting

$$\phi = w_\alpha = w_\alpha(z) \quad (61)$$

yields

$$\phi_1 = w_{\alpha+1} \quad (62)$$

$$\phi_2 = w_{\alpha+2} \quad (63)$$

$$\phi_3 = w_{\alpha+3} \quad (64)$$

By substituting (61)-(64) and (6) in (59), we obtain

$$(w_3z^3)_\alpha - \delta^2(w_3z)_\alpha + (\beta + \gamma)(w_2z^2)_\alpha + \beta\gamma(w_2z)_\alpha = f \tag{65}$$

Therefore, we can write (65) as follows

$$w_3 + w_2 \frac{(\beta + \gamma)z + \beta\gamma}{z^2 - 2} = \frac{f_{-\alpha}}{z^3 - \delta^2z} \tag{66}$$

Let

$$w_2 = u(z) \tag{67}$$

we have

$$u_1 + u \cdot \frac{(\beta + \gamma)z + \beta\gamma}{z^2 - \delta^2} = \frac{f_{-\alpha}}{z^3 - \delta^2z} \tag{68}$$

A particular solution of (68) is given in the form (60).

For $\delta = 1$, theorem 7, gives the non-homogeneous third order linear differential equation considered by Nishimoto and Kalla [15].

Theorem 8. *Homogeneous third order linear ordinary differential equation of the form*

$$\begin{aligned} \phi_3(z^3 - \delta^2z) + \phi_2\{(3\alpha + \beta + \gamma)z^2 + \beta\gamma z - \alpha\delta^2\} \\ + \phi_1\{\alpha(3\alpha + 2\beta + 2\gamma - 3)z + \alpha\beta\gamma\} \\ + \phi\alpha(\alpha - 1)(\alpha + \beta + \gamma - 2) = 0 \end{aligned} \tag{69}$$

$(z \neq 0, \pm \delta)$

has a solution of the form

$$\phi = \left(\frac{1}{(z - \delta)^A(z + \delta)^B} \right)^{\alpha-2} \tag{70}$$

where $\phi = \phi(z)$ and $z \in C$.

A and B are defined in Theorem 7.

Proof. Following the same steps in proof theorem 7, we obtain

$$w_3 + w_2 \frac{(\beta + \gamma)z + \beta\gamma}{z^2 - \delta^2} = 0 \tag{71}$$

Therefore, we have

$$u_1 + u \frac{(\beta + \gamma)z + \beta\gamma}{z^2 - \delta^2} = 0 \tag{72}$$

where

$$w_2 = u(z) = u \tag{73}$$

And the solution of (72) is given by

$$\phi = w_\alpha = u_{\alpha-2} = \left(\frac{1}{(3-\delta)^A(z+\delta)^B} \right)^{\alpha-2}. \quad (74)$$

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