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CUBIC MECHANICAL METHOD FOR THE NONLINEAR SYSTEM OF SINGULAR INTEGRAL EQUATIONS

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Many applied problems in the theory of elasticity can be reduced to the solution of singular integral equations either linear or nonlinear.

In this paper we shall study a nonlinear system of singular integral equations which appear on the closed Lipanouv surface in an ideal medium [4].

We shall find a cubic mechanical method which corresponds to the system and prove its convergence; we obtained a discrete operator which corresponds to this system and study its properties and then a solution to the resulting system of the nonlinear equations which leads to an approximate solution for the original system and its convergence.

1. Introduction

Consider the two dimensional nonlinear singular integral equations system

$$\begin{cases} U(P) - \lambda AU - \lambda BV = f(P), \\ V(P) - \lambda CU - \lambda DV = g(P), \end{cases} \tag{1}$$

where A, B, C and D are nonlinear singular operators of the type

$$\int_S \frac{K(P,Q)}{r^2(P,Q)} \Phi(U(Q))\, dS_Q,$$

where $S \in \ell_1(\delta)$ be the closed Lipanouv surface in R^3,

$$\delta \in (0,1];\ P(x_1,x_2,x_3),\ Q(y_1,y_2,y_3) \in S,\ r(P,Q) = \left[\sum_{i=1}^{3}(x_i - y_i)^2\right]^{\frac{1}{2}}$$

Let $L_i(i = 0,1,2)$=constants> 0, such that for all P, P_1, P_2, Q, Q_1 and $Q_2 \in S$ the following relations are satisfied:
(a) $\mid K(P,Q) \mid \leq L_0$;

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(b) $| K(P_1, Q) - K(P_2, Q) | \leq L_1 r(P_1, P_2)\rho_Q^{-1}(P_1, P_2)$;
(c) $| K(P, Q_1) - K(P, Q_2) | \leq L_2 r^{\delta}(Q_1, Q_2)\rho_P^{-\delta}(Q_1, Q_2)$;
where $\rho_Q(P_1, P_2) = \max\{r(P_1, Q), r(P_2, Q)\}$, $\rho_P(Q_1, Q_2) = \max\{r(P, Q_1), r(P, Q_2)\}$,

$\Phi(t)$ is any function defined on the open interval $(-a, a)$ with the following conditions:
(1) $\Phi(0) = 0$;
(2) the derivative $\frac{d\Phi}{dt}$ exists, and is bounded on $(-a, a)$;
(3) $| \Phi'(t_1) - \Phi'(t_2) | \leq L_3 | t_1 - t_2 |$;
λ is a numerical parameter,
$U(P)$ and $V(P)$ are unknown functions from the Hölder class $H_\alpha(S)$; where

$$H_\alpha(S) = \{Z(P) \in C(S) : \ H(Z, \alpha) < \infty, \ \alpha \in (0, 1)\},$$

$$H(Z, \alpha) = \max_{Q_1, Q_2 \in S} | Z(Q_1) - Z(Q_2) | r^{-\alpha}(Q_1, Q_2);$$

$f(P)$ and $g(P) \in H_\alpha(S)$ are given functions.
$H_\alpha(S)$ is a B-space with the norm

$$||Z||_\alpha = ||Z||_c + H(Z; \alpha).$$

(In what follows we consider $\alpha < \delta$.)

In the integral

$$W(U; P) = \int_S \frac{K(P, Q)}{r^2(P, Q)} U(Q) dS_Q, \tag{2}$$

We take the principal value (P.V.P).

We shall use the characterstic function $f(P, \theta)$ which is obtained in [1], and is defined by: $f(P, \theta) = \lim_{r \to 0} K(P, r, \theta)$, where θ is the angle between the plane passing through the interval r and the normal to the surface S at the point P and another fixed plane passing through the same normal; as in [1,3], the singular integral (2) can be written in the form

$$W(U; P) = W_1(U; P) + W_2(U, P), \ where$$
$$W_1(U, P) = \int_S \frac{K(P, Q) - f(P, \theta)}{r^2(P, Q)} U(Q) dS_Q,$$
$$W_2(U; P) = \int_S \frac{f(P, \theta)}{r^2(P, Q)} (U(Q) - U(P)) dS_Q.$$

It is clear that the first integral W_1 has a weak singularity, and the second W_2 is singular Define, the functions $M(P, Q)$ and $f(P, \theta)$ such that $M(P, Q) \stackrel{\text{def}}{=} K(P, Q)$-$f(P, \theta)$, where $M(P, Q)$ and $f(P, \theta)$ satisfies the following conditions; there exists $L_1(i = \overline{4, 9})$ = constants > 0, for all P, P_1, P_2, Q, Q_1 and $Q_2 \in S$,
(i) $| M(P, Q) | \leq L_4 r^{\delta}(P.Q)$;

(ii) $| M(P_1, Q_1) - M(P_2, Q_2) |\leq L_5 r^\delta(P_1, P_2) + L_6 r^\delta(Q_1, Q_2)$;
(iii) $| f(P, \theta) |\leq L_7$;
(iv) $| f(P_1, \theta) - f(P_2, \theta) |\leq L_8 r(P_1, P_2)\rho_Q^{-1}(P_1, P_2)$;
(v) $| f(P, \theta_1) - f(P, \theta_2) |\leq L_9 r^\delta(Q_1, Q_2)\rho_P^{-\delta}(Q_1, Q_2)$.
where θ, θ_1 and θ_2 corresponding to Q, Q_1 and Q_2 respectively

2. The cubic formula for the singular integral (2) and its convergence.

To find the cubic formula and prove its convergency we use the results obtained in [5]:

Pratitioning S: Let the equation of the closed Lipanouv surface $S \in \ell_1(\delta)$ in the parametric form as follows:
$x_i = x_i(U, V)$, $i = 1, 2, 3$; where $x_i(U, V)$ are continuous and differentiable functions defined on $\triangle$, where $\triangle$ is the closed range of the variables U and V.

In the same way S can be divided by the linear coordinates $U(n)$ and $V(n)$ into subdomains $S(P_k), k = 1, 2, \cdots, N_n$; with support points P_k where n is the parameter of division, $n \geq n_0$. Moreover we defined the following:
for all $k, k_1, k_2 \in \{1, 2, \cdots, N_n\}$,

$$r_k(n) = R_k(n), \qquad r_{k_1}(n) = r_{k_2}(n),$$

where

$$\begin{aligned} r_K(n) &= \inf_{Q\in\partial S(P_K)} \{r(P_K, Q)\}; \\ R_K(n) &= \sup_{Q\in S(P_K)} \{r(P_K, Q)\}; \\ \lim_{n\to\infty} r_k(n) &= \lim_{n\to\infty} R_K(n) = 0; \end{aligned}$$

we choose n_0 such that for all k, $R_K(n) < d$, where d is the standard radius of Lipanouv sphere.
$a_n = b_n$ means that there exists $\ell_1, \ell_2 =$ constants > 0, such that for all $n > n_0$.

$\ell_1\, a_n \leq b_n \leq \ell_2\, a_n$.

Now we construct the cubic formula for the equation (2).

Let

$$W_1^{(n)}(U; P_\ell) \stackrel{\text{def}}{=} \sum_{k=1, k\neq\ell}^{N_n} \frac{K(P_\ell, P_K) - f(P_\ell, \theta P_K)}{r^2(P_\ell, P_K)} U(P_K)\ mes\ S(P_K); \tag{3}$$

$$W_2^{(n)}(U; P_\ell) \stackrel{\text{def}}{=} \sum_{k=1, k\neq\ell}^{N_n} \frac{f(P_\ell, \theta P_K)}{r^2(P_\ell, P_K)} (U(P_K) - U(P_\ell))\ mes\ S(P_K); \tag{4}$$

We take

$$W^{(n)}(U; P_\ell) = W_1^{(n)}(U; P_\ell) + W_2^{(n)}(U; P_\ell)$$

$$= \sum_{k=1,k\neq\ell}^{N_n} \frac{K(P_\ell,P_K)}{r^2(P_\ell,P_K)} U(P_K)\ mes\ S(P_K)$$
$$- U(P_\ell) \sum_{k=1,k\neq\ell}^{N_n} \frac{f(P_\ell,\theta P_K)}{r^2(P_\ell,P_K)}\ mes\ S(P_K); \qquad (5)$$

as the cubic formula for equation (2) at the support points P_ℓ, $\ell = 1,2,\cdots,N_n$. Now we extend this formula at all points of the surface S, setting

$$W^{(n)}(U;P) = W^{(n)}(U;P_\ell),\ if\ P \in S(P_\ell). \qquad (6)$$

So, the remainder value of the formula (5),(6):

$$\begin{aligned} R^{(n)}(U;P) &\stackrel{\text{def}}{=} W(U;P) - W^{(n)}(U;P) \\ &= R_1^{(n)}(U;P) + R_2^{(n)}(U;P) \\ &= (W_1(U;P) - W_1^{(n)}(U;P)) + (W_2(U;P) - W_2^{(n)}(U;P)). \end{aligned}$$

Let

$$R(n) = \max_{k=1,N_n} \{R_K(n)\}, r(n) = \min_{k=1,N_n} \{r_k(n)\}.$$

We obtain the following theorems which are analogs to the results in [5].

Theorem 1. *For all $n \geq n_0$, we have:*

$$\underline{\max}_{\ell=l,N_n} \mid R^{(n)}(U;P_\ell) \mid \leq C_0 \|U\|_\alpha R^\alpha(n)\ Ln \frac{l}{r(n)}.$$

Theorem 2. *For all $n \geq n_0$, we have:*

$$Sup_{P\in S} \mid R^{(n)}(U;P) \mid \leq C_l \|U\|_\alpha R^\alpha(n)\ Ln \frac{l}{r(n)}.$$

where in general C_i, $i = 0,1,2,\cdots$, are constants independently of n.

3. The discrete operator and its properties.

Let R_{N_n} - be a vector space with dimension N_n whose elements are of the form

$$U_{N_n} = (U(P_1), U(P_2), \ldots, U(P_{N_n})). \qquad (7)$$

We define the nommin this space as follows

$$\begin{aligned}
\|U_{N_n}\|_\alpha^{(N_n)} &= \underline{\max}_{i=1,N_n} \mid U(P_i) \mid + H^{(N_n)}(U_{N_n};\alpha), \ where \\
H^{(N_n)}(U_{N_n};\alpha) &= \sup_{\rho>0} \rho^{-\alpha}\overline{\omega}^{(N_n)}(U_{N_n};\rho), \\
\overline{\omega}^{(N_n)}(U_{N_n};\rho) &= \rho \sup_{\xi\geq\rho} \xi^{-1}\omega^{(N_n)}(U_{N_n},\xi)
\end{aligned}$$

is a majorant for the continuity modlus of the vector U_{N_n},

$$\omega^{(N_n)}(U_{N_n};\rho) = \max_{r(P_k,P_\ell)<\rho,k,\ell=\overline{1,N_n}} \mid U(P_K) - U(P) \mid, \ 0 < \rho < diam\ S$$

and $\omega^{(N_n)}(U_{N_n};\rho) = 0$ if $\rho < \min_{k\neq\ell}\{r(P_K,P_\ell)\}$.

It is clear that:

$$\left(R_{N_n}, \|\cdot\|_\alpha^{(N_n)}\right) \equiv H_\alpha^{(N_n)} \text{ is a B-space.}$$

consider now the vector

$$W^{(n)}U_{N_n} = (W^{(n)}(U_{N_n};P_1), W^{(n)}(U_{N_n};P_2), \ldots, W^{(n)}(U_{N_n};P_{N_n})), \tag{8}$$

with

$$W^{(n)}(U_{N_n};P_i) = W^{(n)}(U;P_i), \qquad i = \overline{1,N_n}$$

Equation (8) can be writen by using (3) and (4) as:

$$\begin{aligned}
W_1^{(n)}U_{N_n} &= (W_1^{(n)}(U_{N_n},P_1), W_1^{(n)}(U_{N_n}),P_2), \cdots, W_1^{(n)}(U_{N_n};P_{N_n})); & (9)\\
W_2^{(n)}U_{N_n} &= (W_2^{(n)}(U_{N_n};P_1), W_2^{(n)}(U_{N_n});P_2), \cdots, W_2^{(n)}(U_{N_n};P_{N_n})); & (10)\\
W_1^{(n)}(U_{N_n};P_i) &= W_1^{(n)}(U;P_i) \ and \ W_2^{(n)}(U_{N_n};P_i) \ = \ w_2^{(n)}(U;P_i), \ i = \overline{1,N_n},
\end{aligned}$$

Definition 1. The correspondence $W_1^{(n)}$ of every vector U_{N_n} in the form (7) with the vector in the form (9) is called a *discrete operator with weak singularity.*

Definition 2. The correspondence $W_2^{(n)}$ of every vector U_{N_n} in the form (7) with the vector in the form (10) is called a *discrete singular operator.*

As [1] we can obtain the following Lemma.

Lemma 1.

$$\|W_l^{(n)}U_{N_n}\|_\alpha^{(N_n)} \leq C_2\|U_{N_n}\|_\alpha^{(N_n)}.$$

the proof is given if we put $h_0' = \frac{(1/2)\mathrm{diam}S}{2+1/C_0'}$ *in lemma 2' [5] instead of* h_0[1].

Lemma 2.

$$||W_2^{(n)}U_{N_n}||_\alpha^{(N_n)} \ \le \ C_3||U_{N_n}||_\alpha^{(N_n)}.$$

From Lemmas 1 and 2 we have the following theorem.

Theorem 3.

$$||W^{(n)}U_{N_n}||_\alpha^{(N_n)} \ \le \ C_4||U_{N_n}||_\alpha^{(N_n)}.$$

4. The solution of the nonlinear algebraic system of equations.

We will write two lemmas which were proved in [2].

Lemma 3[2]. *Let* $||U_{N_n}||_\alpha^{(N_n)} < a$. *Then*

$$||\phi(U_{N_n})||_\alpha^{(N_n)} \ \le \ C_5||U_{N_n}||_\alpha^{(N_n)}.$$

Lemma 4[2]. *Let* $||U_{N_n}^{(l)}||_\alpha^{(N_n)} < a$ *and* $||U_{N_n}^{(2)}||_\alpha^{(N_n)} < a$. *then*

$$||\phi(U_{N_n}^{(l)}) - \phi(U_{N_n}^{(2)})||_\alpha^{(N_n)} \ \le \ C_6||U_{N_n}^{(l)} - U_{N_n}^{(2)}||_\alpha^{(N_n)}.$$

By applying the cubic formulae (5) and (6) to the system of the integral equations (1), then they will be written as

$$\begin{aligned} &U(P) - \lambda W^{(n)}(\phi_A(U);P) - \lambda R^{(n)}(\phi_A(U);P) \\ &- \lambda W^{(n)}(\phi_B(V),P) - \lambda R^{(n)}(\phi_B(V),P) = f(P); \end{aligned}$$

$$\begin{aligned} &V(P) - \lambda W^{(n)}(\phi_C(U);P) - \lambda R^{(n)}(\phi_C(U);P) \\ &- \lambda W^{(n)}(\phi_D(V),P) - \lambda R^{(n)}(\phi_D(V);P) = g(P) \end{aligned}$$

Setting $P = P_i$, $i = \overline{1, N_n}$ and neglecting the remainder terms, we get the following system of nonlinear algebraic equations:

$$\begin{cases} w(P_i) - \lambda W^{(n)}(\phi_A(w);P_i) - \lambda W^{(n)}(\phi_B(Z),P_i) = f(P_i); \\ Z(P_i) - \lambda W^{(n)}(\phi_C(w);P_i) - \lambda W^{(n)}(\phi_D(Z),P_i) = g(P_i); \end{cases} \tag{11}$$

with $w(P_i)$ and $Z(P_i)$, $i = \overline{1, N_n}$ the approximated values of $U(P_i)$ and $V(P_i)$.

Now we write system (11) in the vector form as follows

$$\begin{cases} w_{N_n} \ = \ \lambda W^{(n)}\phi_A(w_{N_n}) \ + \ \lambda W^{(n)}\phi_B(Z_{N_n}) + f_{N_n}, \\ Z_{N_n} \ = \ \lambda W^{(n)}\phi_C(W_{N_n}) + \lambda W^{(n)}\phi_D(Z_{N_n}) + g_{N_n}; \end{cases} \tag{12}$$

In turns we write the system (12) in the opperator form

$$\overline{E}_{N_n} = \lambda U \overline{E}_{N_n} + b_{N_n} \tag{13}$$

where

$$\begin{aligned} \overline{E}_{N_n} &= (w_{N_n}, Z_{N_n}), \qquad \bar{b}_{N_n} = (f_{N_n}, g_{N_n}) \qquad \textit{and} \\ U\overline{E}_{N_n} &= [W^{(n)}\phi_A(w_{N_n}) + W^{(n)}\phi_B(Z_{N_n}); W^{(n)}\phi_C(w_{N_n}) + W^{(n)}\phi_D(Z_{N_n})]. \end{aligned}$$

Theorem 4. *If* $|\lambda| < \min\{\lambda_1, \lambda_2\}$, *then the system of algebraic equations (13) has a unique solution. This solution can be found by using the method of iterated approximation. (The values of* $\lambda_1 \& \lambda_2$ *are given below).*

Proof. Let $\overline{E}_{N_n} \in B^{(N_n)}(\bar{b}_{N_n}; a_1)$ be a sphere in $H_\alpha^{(N_n)} \times H_\alpha^{(N_n)}$ with radius a_1 and center $\bar{b}_{N_n}$.

Then we have

$$\begin{aligned} &||\lambda U\overline{E}_{N_n}|| \overset{\text{def}}{=} \max\{\lambda||W^{(n)}\phi_A(w_{N_n}) + \lambda W^{(n)}\phi_B(Z_{N_n})||_\alpha^{(N_n)}; \\ &; ||\lambda W^{(n)}\phi_C(w_{N_n}) + \lambda W^{(n)}\phi_D(Z_{N_n})||_\alpha^{(N_n)}\} \leq \\ &< |\lambda| \max\{C_4C_5(A)(a_1 + ||f_{N_n}||_\alpha^{(N_n)}) + C_4C_5(B)(a_1 + ||g_{N_n}||_\alpha^{N_n)}) \\ &; C_4C_5(C)(a_1 + ||f_{N_n}||_\alpha^{(N_n)}) + C_4C_5(D)(a_1 + ||g_{N_n}||_\alpha^{(N_n)})\} < a_1, \end{aligned}$$

for

$$\begin{aligned} &|\lambda| \leq a_1/C_4[\max\{C_5(A), C_5(C)\}(a_1 + ||f_{N_n})||_\alpha^{(N_n)}) + \\ &+ \max\{C_5(B), C_5(D)\}(a_1 + ||g_{N_n}||_\alpha^{(N_n)})] \equiv \lambda_1; \\ &\quad i.e.\ (\lambda U\overline{E}_{N_n} + \bar{b}_{N_n}) \in B^{(N_n)}(\bar{b}_{N_n}; a_1). \end{aligned}$$

Now we choose λ such that the operator $\lambda U\overline{E}_{N_n} + \bar{b}_{N_n}$ will be a contraction operator; for all $\overline{E}_{N_n}^{(1)}$ and $\overline{E}_{N_n}^{(2)} \in B^{(N_n)}(\bar{b}_{N_n}, a_1)$.

$$\begin{aligned} &||\lambda U\overline{E}_{N_n}^{(1)} - \lambda U\overline{E}_{N_n}^{(2)}|| \overset{\text{def}}{=} \max\{||\lambda W^{(n)}\phi_A(w_{N_n}^{(1)}) + \lambda W^{(n)}\phi_B(Z_{N_n}^{(1)}) - \\ &\lambda W^{(n)}\phi_A(w_{N_n}^{(2)}) - \lambda W^{(n)}\phi_B(Z_{N_n}^{(2)})||_\alpha^{(N_n)}; ||\lambda W^{(n)}\phi_C(W_{N_n}^{(1)}) + \\ &+ \lambda W^{(n)}\phi_D(Z_{N_n}^{(1)}) - \lambda W^{(n)}\phi_C(W_{N_n}^{(2)}) - \lambda W^{(n)}\phi_D(Z_{N_n}^{(2)})||_\alpha^{(N_n)}\} \\ &\leq |\lambda| C_4 \max\{C_6(A) + C_6(B); C_6(C) + C_6(D)\}||\overline{E}_{N_n}^{(1)} - \overline{E}_{N_n}^{(2)}|| \\ &= \gamma||\overline{E}_{N_n}^{(1)} - \overline{E}_{N_n}^{(2)}||, \end{aligned}$$

where $\gamma < 1$, for

$$|\lambda| < \frac{1}{C_4 \max\{C_6(A) + C_6(B); C_6(C) + C_6(D)\}} \equiv \lambda_2$$

5. The convergence of the approximate solution of the system (1).

In [4] a solution of the system (1) by the approximation method is given where λ is finite.

Assume that

$$\overline{E}^* = (u^*, v^*) \in B(\bar{b}; a_1) \subset H_\alpha(S) \times H_\alpha(S)$$

be the unique solution of the system (1), and

$$\overline{E}_{N_n} = (w_{N_n}, Z_{N_n}) \in B^{(N_n)}(\bar{b}_{N_n}, a_1) \subset H_\alpha^{(N_n)} \times H_\alpha^{(N_n)}$$

be unique solution of the system of algebraic equations (13). We mean by $\overline{E} = (w, Z)$ the approximation solution of the system (1) where the functions $w(P)$ and $Z(P)$ are given by

$$w(P) = w(P_\ell) \text{ and } Z(P) = Z(P_\ell),$$

if $P \in S(P_\ell)$, $\ell = \overline{1, N_n}$.

We define

$$\sup_{P\in S} \mid \overline{E}^*(P) - \overline{E}(P) \mid \stackrel{\text{def}}{=} \max\{\sup_{P\in S} \mid u^*(P) - w(P) \mid ; \sup_{P\in S} \mid v^*(P) - Z(P) \mid\}. \quad (14)$$

If we take $0 < \beta < \alpha$ we have as in [2] that

$$\sup_{P\in S} \mid \overline{E}^*(P) - \overline{E}(P) \mid < C_7 R^{\alpha-\beta}(n) \text{Ln}\frac{1}{r(n)}.$$

Theorem 5. *Under a suitable choice of* $\mid \lambda \mid$ *we have:*

(a) *The system of nonlinear singular integral equation (1) and the system of non-linear algebraic equations (11) have a unique solutions in* $H_\alpha(S)$ *and* $H_\alpha^{(N_n)}$ *respectively. This solution can be found by using the method of iterated approximation.*

(b) *The approximation solution of system (1) which is constructed by the solutions of system (11) by formula (14) converges to an exact solution with speed of order*

$$O(R^{\alpha-\beta}(n)\ \text{Ln}\frac{1}{r(n)}), \qquad 0 < \beta < \alpha.$$

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