TAMKANG JOURNAL OF MATHEMATICS
Volume 21, Number 3, Autumn 1990

# THE SEQUENCE SPACE $\mathcal{C}(p)$ AND RELATED MATRIX TRANSFORMATIONS:

S. PEHLIVAN AND Ö. ÇAKAR

**Abstract.** In this paper we define the sequence space $\mathcal{C}(p)$ defined in an incomplete seminormed space $(X,g)$, namely

$$\mathcal{C}(p) = \{(x_k) \subset X : \sup_{r\geq 1} g(x_k - x_{k+r})^{p_k} \to 0,\ k \to \infty\}$$

where $p = (p_k)$ is a sequence of positive numbers. Then we investigated some of its fundamental properties and some of related matrix transformations.

## 1. Introduction and definitions

Let $A = (a_{nk})$ be an infinite matrix of complex numbers $a_{nk}(n,k = 1,2,\cdots)$ and let $e,f$ be two non-empty subset of the space $s$ of all sequences. For every $x = (x_k) \in e$ and every integer $n$, we write $A_n(x) = \Sigma_k a_{nk}x_k$. The sequence $Ax = (A_n(x))$, if it exists, is called the *transformation of $x$ by the matrix $A$*. The sum without limits is always taken from $k = 1$ to $k = \infty$. We say that $A \in (e,f)$ if and only if $Ax \in f$ whenever $x \in e$.

If $p = (p_k)$ is a sequence of strictly positive numbers then we define (see Maddox [4]),

$$c(p) = \{x : \mid x_k - \ell \mid^{p_k} \to 0 \text{ for some } \ell\},$$
$$c_0(p) = \{x : \mid x_k \mid^{p_k} \to 0\}.$$

Let $\ell_\infty$, $c$ and $c_0$ be the spaces of bounded, convergent and null sequences, respectively. When all the terms of $(p_k)$ are constant and all equal to $p > 0$ we have $c(p) = c$ and $c_0(p) = c_0$.

Now let $e(p)$ be a nonempty subset of $s$. Then we shall denote the generalized Köthe-Toeplitz dual of $e(p)$ by $e(p;1)$, i.e.

$$e(p;1) = \{a : \Sigma_k a_k x_k \text{ converges for every } x \in e(p)\}.$$

Received May 30, 1989.

*Key Words and phrases* : Sequence space, Matrix transformations, Incomplete space, Köthe-Toeplitz dual

$\phi$ will denote the space of finitely non-zero sequences of complex numbers and $R$ will denote the set of row-finite matrices.

The following inequality will be used frequently throughout the paper:

$$| a_k + b_k |^{p_k} \leq K(| a_k |^{p_k} + | b_k |^{p_k}) \tag{1}$$

where $a_k, b_k \in \mathbf{C}$, $0 < p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$, [3].

## 2. The space $\mathcal{C}(p)$

Let $p = (p_k)$ be a sequence of strictly positive numbers and $X = (X, g)$ an incomplete seminormed complex linear space with the seminorm $g$ and zero $\Theta$. Then we define

$$\mathcal{C}(p) = \{(x_k) \subset X : \sup_{r \geq 1} g(x_k - x_{k+r})^{p_k} \to 0, \ k \to \infty\}$$

$$C(p) = \{(x_k) \subset X : \text{there exists } \ell \in X \text{ such that } g(x_k - \ell)^{p_k} \to 0 \ (k \to \infty)\}$$

We will denote the sequence spaces by $\mathcal{C}, C$ and $C_0$ namely the space of Cauchy sequences, convergent sequences and null sequences respectively defined in $X = (X, g)$.

We now give some properties of the above classes of sequences.

**Lemma 1.** *If $p \in \ell_\infty$ then $\mathcal{C}(p)$ is a linear space.*

**Proof.** Suppose $x, y \in \mathcal{C}(p)$ and $\lambda, \mu \in \mathbf{C}$. Whenever $H = \sup p_k < \infty$, we have $| \lambda |^{p_k} \leq \max(1, | \lambda |^H)$. Then from (1) and $| \mu |^{p_k} \leq \max(1, | \mu |^H)$ it follows that

$$\begin{aligned}\sup_{r \geq 1} g(\lambda x_k + \mu y_k - (\lambda x_{k+r} + \mu y_{k+r}))^{p_k} &\leq K \sup | \lambda |^{p_k} \sup_{r \geq 1} g(x_k - x_{k+r})^{p_k} \\ &+ K \sup | \mu |^{p_k} \sup_{r \geq 1} g(y_k - y_{k+r})^{p_k}\end{aligned}$$

This implies that $\lambda x + \mu y \in \mathcal{C}(p)$ which completes the proof.

**Lemma 2.** *If $0 < p_k \leq q_k < \infty$ for all $k$ then $\mathcal{C}(p) \subset \mathcal{C}(q)$.*

**Proof.** If $x \in \mathcal{L}(p)$ then, for all sufficiently large $k$, we write $g(x_{k+r} - x_k)^{p_k} < 1$ and so $g(x_{k+r} - x_k)^{q_k} \leq g(x_{k+r} - x_k)^{p_k} \to 0 \ (k \to \infty)$. Thus $x \in \mathcal{C}(q)$. Therefore $\mathcal{C}(p) \subset \mathcal{C}(q)$.

**Lemma 3.** *Let $p_k > 0$, $q_k > 0$. If $\underline{\lim}\, p_k/q_k > 0$ then $\mathcal{C}(q) \subset \mathcal{C}(p)$.*

Now we will conclude the relation between $\mathcal{C}$ and $\mathcal{C}(p)$. Let $p \in \ell_\infty$, $r \geq 1$. If $x \in \mathcal{C}(p)$ then there exists an integer $n_0 > 0$ such that $g(x_k - x_{k+r})^{p_k} < 1$ for every $k > n_0$ and if $x \in \mathcal{C}$ then there exists an integer $m_0 > 0$ such that $g(x_k - x_{k+r}) < 1$ for every $k > m_0$. Let us suppose that $0 < p_k \leq 1$ for every $k$ and write $g(x_k - x_{k+r}) \leq g(x_k - x_{k+r})^{p_k}$ for every $k > N = \max(n_0, m_0)$. Thus $\mathcal{C}(p) \subset \mathcal{C}$. If $1 < p_k \leq H < \infty$ then $g(x_k - x_{k+r})^{p_k} \leq g(x_k - x_{k+r})$ for every $k > N$ and so $\mathcal{C} \subset \mathcal{C}(p)$. Then we have the following

**Theorem 1.** *If* $\inf p_k > 0$ *then* $\mathcal{C} \subset \mathcal{C}(p)$.

**Proof.** The proof follows from an argument similar to the one in Lemma 2.

**Corollary 1.** *If* $0 < \inf p_k \leq \sup p_k < \infty$ *then* $\mathcal{C}(p) = \mathcal{C}$.

**Theorem 2.** *If* $p = (p_k)$ *is a decreasing sequence then* $C(p) \subset \mathcal{C}(p)$.

**Proof.** Using the inequality (1) we have, for every $x \in C(p)$

$$g(x_{k+r} - x_k)^{p_k} \leq K\{g(x_{k+r} - \ell)^{p_k} + g(x_k - \ell)^{p_k}\}$$

since $p$ is a decreasing sequence. This completes the proof.

Let now $Q$ be the set of all $p = (p_k)$ for which there exists $N > 1$ such that $\Sigma_k N^{-1/p_k} < \infty$, [4]. If $p = (p_k)$ decreases then $p \in Q$.

We shall denote the generalized Köthe-Toeplitz dual of $\mathcal{C}(p)$ by $\mathcal{C}(p, 1)$.

It is easy to see that if $p \in Q$ and $X$ is complete then $C(p) = \mathcal{C}(p)$. Three known methods, due to Maddox [5], Cakar [1], Stieglitz [6], may be modified in an incomplete space $\mathcal{C}(p)$. Then we give the following lemmas.

**Lemma 4.** *Let* $p \in Q$. *Then* $\mathcal{C}(p,1) = \{a : \Sigma_k a_k x_k$ *converges for every* $x \in \mathcal{C}(p)\} = \phi$.

Let $E \in \{\mathcal{C}, C, C_0\}$ We shall use the usual notation $e$ for corresponding spaces $E$ when $X = \mathbb{C}$, the set of complex numbers.

**Lemma 5.** *Let* $p \in \ell_\infty$. *If* $A \in (C(p), E)$ *then* $A \in (c(p), e)$.

The conditions for $A \in (c(p), c)$ are given by Lascarides, [2].

**Lemma 6.** *Let* $p \in \ell_\infty$. *Then* $A \in (c(p), c)$ *if and only if*

(i) *There exists an absolute constant* $B > 1$ *such that*

$$M_B = \sup_n \sum_k | a_{nk} | B^{-1/p_k} < \infty.$$

(ii) $\lim_n \Sigma_k a_{nk} = \alpha$ *exists.*

(iii) $\lim_n a_{nk} = \alpha_k$ *exists for every fixed* $k$.

## 3. Matrix transformations

We are going to characterize some of matrix classes which transform the sequence spaces $\mathcal{C}(p)$ into $\mathcal{C}, C$ and $C_0$.

**Theorem 3.** *Let* $p \in Q$. $A \in (\mathcal{C}(p), \mathcal{C})$ *if and only if*

(i) $A \in R$

(ii) $T = \sup_{n,k} | a_{nk} |^{p_k} < \infty$

(iii) $\lim_n \Sigma_{k=i} a_{nk} = \alpha_i$ *exists.*

(iv) $\lim_n a_{nk} = \alpha_k$ *exists for every fixed* $k$.

**Proof.** For the sufficiency, let us write $H = \sup p_k$ and let $0 < \varepsilon < 1$, $r \geq 1$. Then there exist an integer $n_0 = n_0(\varepsilon, x) \geq 1$ and $B \geq \max(1, NT)$, $\Sigma N^{-1/p_k} < \infty$ such that $g(x_k - x_{k+r})^{p_k} \leq \varepsilon^H B^{-1} < 1$ for $k > n_0$. Let $k_0 > n_0$, using the conditions (i-iv) and $\Sigma_k N^{-1/p_k} < \infty$ $(N > 1)$, we have

$$g(A_m(x) - A_n(x)) \leq \sum_{k=1}^{n_0} | a_{mk} - a_{nk} | g(x_k - x_{k_0}) + \sum_{k=n_0+1} | a_{mk} | g(x_k - x_{k_0})$$
$$+ \sum_{k=n_0+1} | a_{nk} | g(x_k - x_{k_0}) + g(x_{k_0}) | \sum_k a_{mk} - \sum_k a_{nk} |$$

Then we have, for sufficiently large $n, m$, $g(A_m(x) - A_n(x)) \leq \varepsilon$, so $(A_n(x)) \in \mathcal{C}$.

The necessity of (i) is observe from Lemma 4. For the necessity of (iii) and (iv) we observe that $A \in (C(p), \mathcal{C})$ whenever $A \in (\mathcal{C}(p), \mathcal{C})$ since $(\mathcal{C}(p), \mathcal{C}) \subset (C(p), \mathcal{C})$. According to Lemma 5, $A \in (c(p), c)$. Lemma 6 gives the necessity of (iii) and (iv). For the necessity of (ii) it is enough to prove that, in the case $p \in Q$, condition (i) of Lemma 6 and condition (ii) of Theorem 3 are equivalent. If the condition (i) of Lemma 6 holds then there exists an absolute constant integer $B > 1$ such that

$$| a_{nk} |^{p_k} \leq M_B^{p_k} B \leq B \max(1, M_B^H) < \infty$$

for every $n, k$ and therefore condition (ii) of Theorem 3 holds. If on the other hand $p \in Q$ then there exists an integer $N > 1$ such that $\Sigma_k N^{-1/p_k} < \infty$ and if $T < \infty$ then for any integer $B \geq \max(1, NT)$ we have

$$| a_{nk} | \leq T^{1/p_k}$$

for every $n$, whence

$$\sum_k | a_{nk} | B^{-1/p_k} \leq \sum_k N^{-1/p_k} < \infty.$$

This completes the proof.

Finally we can give the following corollaries:

**Corollary 2.** *Let $p \in Q$. Then $A \in (\mathcal{C}(p), C)$ if and only if together with the conditions (i),(ii),(iv) of Theorem 3. we have*

(iii)′ $\lim_k \Sigma_k a_{nk} = \Sigma_n a_k$.

(v) $(a_k) \in \phi$.

**Corollary 3.** *Let $p \in Q$. Then $A \in (\mathcal{C}(p), C_0)$ if and only if conditions (i)-(iv) of Theorem 3. hold with $\alpha_i = 0$ and $\alpha_k = 0$ for every $i$ and $k$.*

## References


[1] Ö. Cakar, On matrix transformations of sequence spaces defined in an incomplete space, *Comm. de la Fac. des Sc. de | 'Universitité d'Ankara*, 22 A(1973), 107-121.

[2] C.G. Lascarides, A study of certain sequence spaces of Maddox a generalization of a theorem of I yer, *Pacific J. Math.* 38, (1971), 487-500.

[3] I.J. Maddox, Spaces of strongly summable sequences, *Quaterly J. Math. Oxford,* (2), 18, (1967), 345-355.

[4] I.J. Maddox, Paranormed sequence spaces generated by infiinite matrices, *Proc. Camb. Phil. Soc.,* 64, (1968), 335-340.

[5 ] I.J. Maddox, Matrix transformations in an incomplete spaces, *Canadian J. Math.,* 20, (1968), 727-734.

[6] M. Stieglitz, Matrixtransformationen von unvollständigen Folgenräumen *Math. Z.,* 133, (1973), 129-132.



S. Pehlívan

University of Akdeniz.

Ö. Cakar

University of Ankara.