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A GENERALIZATION OF SOME COMMUTATIVITY THEOREMS FOR RINGS I

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Abstract. In this paper we generalize some well-known commutativity theorems for rings as follows: Let $m > 1$, and n, k be non-negative integers. Let R be an s-unital ring satisfying the polynomial identity $[x^n y - y^m x^k, x] = 0$, for all $x, y \in R$. Then R is commutative.

1. Introduction.

Throughout the present paper, R will represent an associative ring (may be without unity 1). Let $Z(R)$denote the center of R, N' the set of all zero divisors of R, N the set of all nilpotent elements of R, and $C(R)$ the commutator ideal of R. For any $x, y \in R$, we set as usual $[x, y] = xy - yx$.

A ring R is called *left* (resp. *right*) *s-unital* if $x \in Rx$ (resp. $x \in xR$) for every $x \in R$. Further, R is called *s-unital* if R is both left and right s-unital, that is $x \in Rx \cap xR$, for every $x \in R$. As stated in [6] and [13], if R is s-unital (resp. left or right s-unital), then for any finite subset F of R there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e will be called a *pseudo-identity* (resp. *pseudo left identity* or *pseudo right identity*) of F in R.

A theorem of Bell [3] has been generalized by Quadri and Khan [14] as follows: If R is a ring with unity 1 and $m > 1$, $k \geq 1$ be integers such that for all $x, y \in R$, $[xy - y^m x^k, x] = 0$, then R is commutative. The commutativity of a left s-unital ring satisfying $[xy - y^m x^k, x] = 0$, $(m > 1, k \geq 1)$, for all $x, y \in R$ has been proved in [13] by Quadri and Khan. In [10] Psomopoulos has shown that an s-unital ring R in which $[x^n y - y^m x, x] = 0$, $(m > 1, n \geq 1)$ holds for all $x, y \in R$ must be commutative.

In this paper, motivated by the above polynomial identities and the polynomial identity $x^n[x, y] = [x, y^m]$ considred by Komatsu [7], we intend to prove a result on the commutativity of an s-unital ring satisfying the following property:

(P) "there exist integers $m > 1$, $n \geq 0$, and $k \geq 0$ such that $[x^n y - y^m x^k, x] = 0$, for all x, y in R".

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2. Preliminaries.

In preparation for the proof of our result, we first state the following well-known results.

Lemma 1. *([8, Lemma 3]). Let $x, y \in R$. If $[x,[x,y]] = 0$, then for any positive integer k, $[x^k, y] = kx^{k-1}[x,y]$.*

Lemma 2.([9, *Lemma]). Let R be a ring with unity 1, and let x, y be elements of R. If for some integer $k \geq 1$, $x^k y = 0 = (x+1)^k y$, then necessarily $y = 0$.*

Lemma 3. *([13, Lemma 3]). Let R be a ring with unity 1. If $(1-y^k)x = 0$, then $(1-y^{km})x = 0$, for any positive integers m, and k.*

Lemma 4. *([6, Proposition 2]). Let f be a polynomial in noncommuting indeterminates $x_1, x_2, \ldots, x_n$ with integer coefficients. Then the following statements are equivalent:*

1) *For any ring R satisfying $f = 0$, $C(R)$ is a nil ideal.*
2) *Every semiprime ring satisfying $f = 0$ is commutative.*
3) *For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.*

3. Main theorem and its corollaries.

The main result of this paper is the following:

Theorem. *Let R be an s-unital ring satisfying the property (P), then R is commutative.*

Let us pause to notice that for any $x, y \in R$, the property (P) can be expressed as:

$$x^n[x,y] = [x,y^m]x^k. \tag{1}$$

Then for any positive integer t, we have

$$\begin{aligned} x^{tn}[x,y] &= x^{(t-1)n}(x^n[x,y]) \\ &= x^{(t-1)n}[x,y^m]x^k \\ &= x^{(t-2)n}(x^n[x,y^m])x^k \\ &= x^{(t-2)n}[x,y^{m^2}]x^{2k} \\ &= \cdots \end{aligned}$$

By repeating the above process and using (1), we obtain

$$x^{tn}[x,y] = [x,y^{m^t}]x^{tk}. \tag{2}$$

Now, we prove the following lemmas which will be used in the proof of our main theorem.

Lemma 5. *Let R be a ring with unity which satisfies the property (P), then $N \subseteq Z(R)$.*

Proof. Let $u \in N$. Then by (2) for any $x \in R$, and a positive integer $t \geq 1$, we have $x^{tn}[x,u] = [x, u^{m^t}]x^{tk}$. But since u is nilpotent, then $u^{m^t} = 0$, for sufficiently large t and we get $x^{tn}[x,u] = 0$ for all x in R. But $(x+1)^{tn}[x,u] = 0 = x^{tn}[x,u]$, for all $x \in R$, then by Lemma 2, this yields $[x,u] = 0$. Therefore $u \in Z(R)$, and hence $N \subseteq Z(R)$.

Lemma 6. *Let R be a ring with unity 1 which satisfies the property (P), then $C(R) \subseteq Z(R)$.*

Proof. In view of Lemma 4, $C(R)$ is a nil ideal, since $x = e_{22}$ and $y = e_{21}$ fail to satisfy (1) in $(GF(p))_2$, for a prime p. Hence by Lemma 5, we obtain $C(R) \subseteq Z(R)$, where $e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Remark 1. In view of Lemma 6, it is guaranteed that Lemma 1 holds for any pair of elements x and y in a ring with unity 1 which satisfies the property (P).

Lemma 7. *Let R be a ring with unity 1 satisfying the property (P). Then R is commutative.*

Proof. The ring R is isomorphic to subdirect sum of subdirectly irreducible rings R_i, each of which as a homomorphic image of R satisfies the property placed on R. Thus R itself can be assumed to be subdirectly irreducible ring. Let S be the intersection of all its non-zero ideals, then $S \neq (0)$.

Now, if $n = k = 0$, then we have $[y - y^m, x] = 0$, for all $x, y \in R$ and consequently by [5, Theorem 18] R is commutative. Let $n = 0$ and $k = 1$ in (1), then replacing x by $(x+1)$ we obtain $[x, y^m] = 0$, for all $x, y \in R$. Thus $[x,y] = [x, y^m]x = 0$, for all $x, y \in R$. Therefore R is commutative.

Next, suppose that $n \geq 1$ or $k \geq 0$. Let $q = 2^m - 2$ be a positive integer. Then by (1) we have

$$\begin{aligned} qx^n[x,y] &= 2^m x^n[x,y] - 2x^n[x,y] \\ &= 2^m[x,y^m]x^k - x^n[x,2y] \\ &= [x,(2y)^m]x^k - [x,(2y)^m]x^k \\ &= 0. \end{aligned}$$

By Lemma 2, we get $q[x,y] = 0$, for all x, y in R. Now combining Lemma 6 with Lemma 1, we obatin $[x^q, y] = qx^{q-1}[x,y] = 0$. Thus

$$x^q \in Z(R), \text{ for all } x, y \in R. \tag{3}$$

Replace y by y^m in (1), then we get

$$x^n[x,y^m] = [x,(y^m)^m]x^k. \tag{4}$$

Also by Lemma 6 and Lemma 1, we have

$$\begin{aligned} x^n[x,y^m] &= [x,y^m]x^n \\ &= my^{m-1}[x,y]x^n \\ &= my^{m-1}x^n[x,y] \\ &= my^{m-1}[x,y^m]x^k. \end{aligned}$$

and,

$$\begin{aligned} [x,(y^m)^m]x^k &= m(y^m)^{m-1}[x,y^m]x^k \\ &= my^{m^2-m}[x,y^m]x^k \\ &= my^{m-1}y^{(m-1)^2}[x,y^m]x^k. \end{aligned}$$

Thus (4) gives

$$my^{(m-1)}(1-y^{(m-1)^2})[x,y^m]x^k = 0. \tag{5}$$

Replace x by $(x+1)$ in (5), then we get $my^{m-1}(1-y^{(m-1)^2})[x,y^m](x+1)^k = 0$. So by Lemma 2, $my^{m-1}(1-y^{(m-1)^2})[x,y^m] = 0$. Then Lemma 3 gives

$$my^{(m-1)}(1-y^{q(m-1)^2})[x,y^m] = 0. \tag{6}$$

Next, we claim that $N' \subseteq Z(R)$. Let $u \in N'$, then by (3) $u^{q(m-1)^2} \in N' \cap Z(R)$, and $Su^{q(m-1)^2} = 0$. By using (6) we obtain $mu^{(m-1)}[x,u^m](1-u^{q(m-1)^2}) = 0$.

If $mu^{m-1}[x,u^m] \neq 0$, then $(1-u^{q(m-1)^2}) \in N'$, and so $S(1-u^{q(m-1)^2}) = 0$ which gives a contradiction that $S \neq (0)$. Therefore $mu^{m-1}[x,u^m] = 0$. From (1) and using Lemma 1 repleatedly we obtain

$$\begin{aligned} x^{2n}[x,u] &= x^n(x^n[x,u]) \\ &= x^n[x,u^m]x^k \\ &= [x,u^{m^2}]x^{2k} \\ &= mu^{m(m-1)}[x,u^m]x^{2k} \\ &= mu^{m-1}u^{(m-1)^2}[x,u^m]x^{2k} \\ &= mu^{m-1}[x,u^m]u^{(m-1)^2}x^{2k}. \end{aligned}$$

This implies that $x^{2n}[x,u] = 0$. Hence Lemma 2 gives $[x,u] = 0$, that is $u \in Z(R)$. Therefore $N' \subseteq Z(R)$.

Now, for any $x \in R$, x^q and $x^{qm} \in Z(R)$. Then by (1) for any $y \in R$, we have

$$\begin{aligned} (x^q - x^{qm})x^n[x,y] &= x^q(x^n[x,y]) - x^{qm}(x^n[x,y]) \\ &= x^n(x^q[x,y]) - x^{qm}[x,y^m]x^k \\ &= x^n[x,x^qy] - [x,(x^qy)^m]x^k \\ &= x^n[x,x^qy] - x^n[x,x^qy]. \end{aligned}$$

Therefore, $(x^q - x^{qm})x^n[x,y] = 0$, and

$$(x - x^{qm-q+1})x^{n+q-1}[x,y] = 0. \tag{7}$$

Now, if R is not commutative, then by [5, Theorem 18], there exists an element $x \in R$ such that $(x - x^t) \notin Z(R)$, where $t = qm - q + 1$. This also reveals $x \notin Z(R)$. Thus neither x nor $(x - x^t)$ is a zero-divisor, and so $(x - x^t)x^{n+q-1} \notin N'$. Hence (7) forces that $[x,y] = 0$, for all $x, y \in R$. Thus $x \in Z(R)$ which is a contradiction. Therefore R must be commutative. This completes the proof.

Corollary 1. *([14, Theorem]). Let R be a ring with unity 1, and let $m > 1$ and $k \geq 1$ be fixed positive integers. If $[xy - y^m x^k, x] = 0$, for all x, y in R, then R is commutative.*

Proof. Put $n = 1$, and take, $k \geq 1$ in Lemma 7. Then we obtain $[xy - y^m x^k, x]$, for all $x, y \in R$. Then by Lemma 7, R must be commutative.

Proof of the theorem. In view of Proposition 1 of [6], R is commutative, if R with unity 1 satisfying the property (P) is commutative, and this is guaranteed by Lemma 7.

Corollary 2. *([10, Theorem]). Let R be an s-unital ring, and let $m > 1$ and $n \geq 1$ be fixed positive integers. If $[x^n y - y^m x, x] = 0$, for all $x, y \in R$, then R is commutative.*

Proof. Set $k = 1$, and consider $n \geq 1$ in the main theorem. Then we obtain $[x^n y - y^m x, x] = 0$ for all $x, y \in R$ and $m > 1$. Thus the commutativity of R follows from the main Theorem.

Corollary 3. *([13, Theorem]). Let R be a left s-unital ring. If there exist positive integers $m > 1$ and $k \geq 1$ such that for any $x, y \in R$, $[xy - y^m x^k, x] = 0$, then R is commutative.*

Proof. By [13, Lemma 4] R is s-unital. Then the commutativity of R follows from Proposition 1 of [6], and Corollary 1.

Corollary 4. *([7, Theorem]) Let m, n be fixed non-negative integers. Suppose that R satisfies the polynomial identity*

$$x^n[x,y] = [x,y^m]. \tag{*}$$

(a) *If R is left s-unital, then R is commutative except when $(m,n) = (1,0)$.*
(b) *If R is right s-unital, then R is commutative except for $m = 1$, and $n = 0$; and also $m = 0$ and $n > 0$.*

Proof. (a) Let R be a left s-unital ring, and let $x, y \in R$ such that $ex = x$ and $ey = y$ for some element $e \in R$. If $m = 0$, and $n > 0$, then $x[x,y] = 0$, and hence $y = ye \in yR$. Thus R is s-unital, and R is commutative by Theorem. If $m = 1$, and $n > 0$, then (*) becomes $x^n[x,y] = [x,y]$. Hence $x = x^{n+1} - x^{n+1}e + xe = x(x^n - x^n e + e) \in xR$. Thus R is s-unital, and R is commutative.

Next, we may assume that $m > 1$ and $n > 0$. By making repeated use of (*), we see that for any positive integer t

$$x^{tn}[x,y] = [x, y^{m^t}]. \qquad (**)$$

Let $u \in N$. Then for any $x \in R$, and $t \geq 1$ we have by (**), $x^{tn}[x,u] = [x, u^{m^t}]$. But since u is nilpotent, $u^{m^t} = 0$ for sufficiently large t. Thus $x^{tn}[x,u] = 0$. Since R is left s-unital, then $u = eu$ for some $e \in R$. But $e^{tn}[e,u] = 0$ which gives $u = ue$. Let $x \in R$, then there exists $e' \in R$, such that $e'x = x$. Further, for some $e'' \in R$, $e''e' = e'$, and thus $e''x = x$. Therefore $(x - xe'')^2 = 0$, that is $(x - xe'') \in N$. Since $e'(x - xe'') = x - xe''$ we have $x - xe'' = (x - xe'')e' = 0$ which gives $x = xe''$. Hence R is s-unital. Thus R is commutative by the main Theorem.

(b) Let R be a right s-unital ring, and let $y \in R$ such that $ye = y$, for some $e \in R$. If $m = 1$, and $n > 0$ then $x^n[x,y] = [x,y]$, for all $x, y \in R$. Thus $y = e^n y - e^{n+1} y + ey = (e^n - e^{n+1} + e)y \in Ry$, and hence the asseration is clear. Let $m > 1$ and $n > 0$, and let $u \in N$. Following the same argument as in (a), we have $x^{tn}[x,u] = 0$, and $u = ue$ for some $e \in R$. Thus $u = eu$. Let $x \in R$, then there exists $e' \in R$ such that $xe' = x$ and for some $e'' \in R$, $e'e'' = e'$. Thus $xe'' = x$. Therefore $(x - e''x)^2 = 0$, and hence $(x - e''x) \in N$. Thus $x - e''x = (x - e''x)e' = e'(x - e''x) = 0$ which forces $x = e''x$. Hence R is s-unital, and R is commutative by Theorem.

Remark 2. ([7, Remark]) let K be a field. Then, the non-commutative ring $R = \begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$ has a right identity element and satisfies $x[x,y] = 0$.

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