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GENERALIZED HILBERT INTEGRAL OPERATORS ON THE HERZ SPACES

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Abstract. This paper gives some necessary and sufficient conditions for the generalized Hilbert integral operators to be bounded on the Herz spaces. The corresponding new operator norm inequalities are obtained.

1. Introduction

Considerable attention has been given to the classical Hilbert operator T defined by

$$T(f,x) = \int_0^\infty \Big(\frac{1}{x+y}\Big) f(y)dy \tag{1.1}$$

and the classical Hilbert inequality

$$\|Tf\|_p \le \frac{\pi}{\sin(\pi/p)}\|f\|_p, \quad \text{for} \quad 1 < p < \infty, \tag{1.2}$$

where $\|f\|_p = (\int_0^\infty |f(x)|^p dx)^{1/p}$ and the constant factors $\pi/\sin(\pi/p)$ is the best value (see [1]). In view of the mathematical importance and applications, considerable attention has also been given to various improvements, refinements and extensions of many inequalities by various authors (see e.g., [3, 4, 5, 6, 7, 8, 9, 10, 11] and the references cited therein).However, hardly any work was done on inequalities on the Herz spaces. It is well-known that the Herz spaces play an important role in characterizing the properties of functions and multipliers on the classical Hardy spaces. In recent years, a series of papers have paid more attention to the study of the Herz spaces themselves (see [12, 13]). The aim of this paper is to establish some new inequalities related to the generalized Hilbert integral operator

$$T(f,x) = \int_0^\infty K(x,y)f(y)dy \tag{1.3}$$

with the general kernel $K(x,y)$. We obtain some necessary and sufficient conditions for the generalized Hilbert integral operator T to be bounded on the Herz spaces. The corresponding new operator norm inequalities are obtained.

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2. Definitions and statement of the main results

Definition 1. Let $\alpha \in R$, $0 < p, q < \infty$, $B_k = \{x \in R^n : |x| \le 2^k\}$, $D_k = B_k - B_{k-1}$ $(k \in Z)$, $\varphi_k = \varphi_{D_k}$ denote the characteristic function of the set D_k.

(1) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(R^n)$ is defined by [12]:

$$\dot{K}_q^{\alpha,p}(R^n) = \Big\{f \in L_{loc}^q(R^n - \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}}(R^n) < \infty\Big\}, \tag{2.1}$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(R^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\varphi_k\|_q^p\right\}^{1/p}; \tag{2.2}$$

(2) The homogeneous Herz type space $\dot{K}_q^{\alpha,p}(\omega)$ is defined by

$$\dot{K}_q^{\alpha,p}(\omega) = \Big\{f \in L_{loc}^q(R^n - \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\omega)} < \infty\Big\}, \tag{2.3}$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\omega)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\varphi_k\|_{q,\omega}^p\right\}^{1/p}; \tag{2.4}$$

We can similarly define the non-homogeneous Herz space $K_q^{\alpha,p}(R^n)$ and $K_q^{\alpha,p}(\omega)$. It is easy to see that when $p = q$, we have $\dot{K}_p^{0,p}(R^n) = K_p^{0,p}(R^n) = L^p(R^n)$, and $\dot{K}_p^{\alpha/p,p}(R^n) = K_p^{\alpha/p,p}(R^n) = L^p(|x|^\alpha dx)$. Throughout this paper, we write

$$\|f\|_{p,\omega} = \left(\int_{R^n} |f(x)|^p \omega(x)dx\right)^{1/p}.$$

Our main results are the following two theorems:

Theorem 1. *Let $\alpha \in R$, $0 < p < \infty$, $\lambda > 0$ and $1 \le q < \infty$, $\omega(x) = x^{(1-\lambda)q}$. Let $F(p) = \int_0^\infty t^{(\lambda-1-1/q)p}K^p(1,t)dt$ and $K(x,y)$ be a nonnegative measurable function on $(0,\infty) \times (0,\infty)$ which satisfies the following conditions:*

(1) *$K(tx,ty) = t^{-\lambda}K(x,y)$ for all $t > 0$;*

(2) *There exist the constants $C_1(p), C_2(p) > 0$, such that*

$$F(1) \le C_1(p)\{F(p)\}^{1/p}, \quad \text{for} \quad 1 \le p < \infty; \tag{2.5}$$
$$F(1) \le C_2(p)\{F(p)\}^{1/p}, \quad \text{for} \quad 0 < p < 1. \tag{2.6}$$

Then the generalized Hilbert integral operator T defined by (1.3) is a bounded operator from $\dot{K}_q^{\alpha,p}(\omega)$ into $\dot{K}_q^{\alpha,p}(R_+^1)$ if and only if

$$\int_0^\infty t^{\lambda-\alpha-1-(1/q)}K(1,t)dt < \infty. \tag{2.7}$$

Moreover, when (2.7) *holds, the operator norm* $\|T\|$ *of* T *on* $\dot{K}_q^{\alpha,p}(\omega)$ *satisfies the following inequality:*

$$\int_0^\infty t^{\lambda-\alpha-1-(1/q)}K(1,t)dt \le \|T\| \le C(p,\alpha)\int_0^\infty t^{\lambda-\alpha-1-(1/q)}K(1,t)dt, \tag{2.8}$$

where

$$C(p,\alpha) = \begin{cases} 2^{1/p-1}\{C_1(1/p)\}^{1/p}C_2(p)(1+2^{|\alpha|}), & 0<p<1 \\ 2^{1-1/p}C_1(p)\{C_2(1/p)\}^{1/p}(1+2^{|\alpha|}), & 1\le p<\infty \end{cases}. \tag{2.9}$$

Corollary 2. *If* $t^{\lambda-1-1/q}K(1,t)$ *is concave function on* $(0,\infty)$ *and* $K(1,t)$ *has compact support: supp* $K(1,t)\subset[0,b]$, *then by* (2.5), (2.6), *Hölder inequality and Theorem* 24 *in* [2, §43, pp.43-44], *we obtain* $C_1(p)=b^{1-1/p}$ *and* $C_2(p)=2^{-1}(1+p)^{1/p}b^{1-1/p}$, *thus by* (2.9), we get

$$C(p,\alpha) = \begin{cases} 2^{1/p-2}(1+p)^{1/p}(1+2^{|\alpha|}), & 0<p<1 \\ 2^{1-2/p}(1+1/p)(1+2^{|\alpha|}), & 1\le p<\infty \end{cases}. \tag{2.10}$$

Theorem 3. *Let* $\alpha\in R$, $1\le p<\infty$, $\lambda>0$ *and* $K(x,y)$ *be a nonnegative measurable function defined on* $(0,\infty)\times(0,\infty)$. *Which satisfies the following condition:* $K(tx,ty)=t^{-\lambda}K(x,y)$ *for all* $t>0$. *Let* $\omega_1(x)=x^\alpha$ *and* $\omega_2(x)=x^{(1-\lambda)p+\alpha}$.*Then the generalized Hilbert integral operator* T *is defined by* (1.3) *is a bounded operator from* $L^p(\omega_2)$ *into* $L^p(\omega_1)$ *if and only if*

$$\int_0^\infty t^{\lambda-1-(\alpha+1)/p}K(1,t)dt<\infty. \tag{2.11}$$

Moreover, when (2.11) *holds, the operator norm* $\|T\|$ *of* T *on* $L^p(\omega_2)$ *satisfies*

$$\|T\| = \int_0^\infty t^{\lambda-1-(\alpha+1)/p}K(1,t)dt. \tag{2.12}$$

If $K(x,y)=\frac{1}{x}\varphi_{(0,x)}(y)$, $\lambda>\frac{\alpha+1}{p}$, *then by* (1.3) *and* (2.12), *we reduced the classical Hardy operator*

$$T_0(f,x)=\frac{1}{x}\int_0^x f(y)dy$$

and

$$\left\{\int_0^\infty\left|\frac{1}{x}\int_0^x f(y)dy\right|^p x^\alpha dx\right\}^{1/p} \le \frac{p}{\lambda p-\alpha-1}\left(\int_0^\infty |f(x)|^p x^{(1-\lambda)p+\alpha}dx\right)^{1/p}. \tag{2.13}$$

where $\|T_0\|=p/(\lambda p-\alpha-1)$ *is the best possible constant. In pacticular, when* $\alpha=0$, $\lambda=1$, (2.13) *reduce to the classical Hardy inequality* [3, 4]:

$$\left(\int_0^\infty\left|\frac{1}{x}\int_0^x f(t)dt\right|^p dx\right)^{1/p} \le \frac{p}{p-1}\left(\int_0^\infty |f(x)|^p dx\right)^{1/p}. \tag{2.14}$$

Hence, our main results imply many useful inequalities with the best constant factors.

There are some similar results for the non-homogeneous Herz spaces. We omit the details here.

3. Proofs of Theorems 1 and 3

In what follows, we shall write simply $\dot{K}_q^{\alpha,p}(R_+^1)$ and $\dot{K}_q^{\alpha,p}(\omega)$ to denote K and $K(\omega)$, respectively.

Proof of Theorem 1. Using Minkowski inequality for integrals and setting $y = tx$, we get

$$\begin{aligned}\|(Tf)\varphi_k\|_q &= \left\{ \int_{D_k} \left| \int_0^\infty K(x,y)f(y)dy \right|^q dx \right\}^{1/q} \\ &\le \int_0^\infty \left(\int_{D_k} |f(tx)|^q x^{(1-\lambda)q} dx \right)^{1/q} K(1,t)dt \\ &= \int_0^\infty \left(\int_{2^{k-1}t<y\le 2^k t} |f(y)|^q y^{(1-\lambda)q} dy \right)^{1/q} t^{\lambda-1-1/q} K(1,t)dt.\end{aligned}$$

For each $t \in (0,\infty)$, there exists an integer m such that $2^{m-1} < t \le 2^m$. Setting

$$A_{k,m} = \{y \in (0,\infty) : 2^{k+m-1} < y \le 2^{k+m}\},$$

we obtain

$$\begin{aligned}\|(Tf)\varphi_k\|_q &\le \int_0^\infty \left(\int_{A_{(k-1),m}} |f(y)|^q \omega(y)dy + \int_{A_{k,m}} |f(y)|^q \omega(y)dy \right)^{1/q} t^{\lambda-1-1/q} K(1,t)dt \\ &\le \int_0^\infty \Big(\|f\varphi_{k+m-1}\|_{q,\omega} + \|f\varphi_{k+m}\|_{q,\omega} \Big) t^{\lambda-1-1/q} K(1,t)dt.\end{aligned}$$

It follows that

$$\|Tf\|_K \le \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left[\int_0^\infty \Big(\|f\varphi_{k+m-1}\|_{q,\omega} + \|f\varphi_{k+m}\|_{q,\omega} \Big) t^{\lambda-1-1/q} K(1,t)dt \right]^p \right\}^{1/p}. \tag{3.1}$$

Now, we consider two cases for p:

Case 1. $0 < p < 1$. In this case, it follows from (3.1), (2.6) and (2.5) that

$$\|Tf\|_K \le C_2(p) \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \int_0^\infty \Big(\|f\varphi_{k+m-1}\|_{q,\omega}^p + \|f\varphi_{k+m}\|_{q,\omega}^p \Big)$$

$$\times t^{(\lambda-1-1/q)p} K^p(1,t)dt\Big\}^{1/p}$$

$$\leq 2^{1/p-1} C_2(p)\bigg\{\bigg[\sum_{k=-\infty}^{\infty} 2^{(k+m-1)\alpha p}\|f\varphi_{k+m-1}\|_{q,\omega}^p$$

$$\times \int_0^\infty 2^{-(m-1)\alpha p} t^{(\lambda-1-1/q)p} K^p(1,t)dt\bigg]^{1/p}$$

$$+\bigg[\sum_{k=-\infty}^{\infty} 2^{(k+m)\alpha p}\|f\varphi_{k+m}\|_{q,\omega}^p \int_0^\infty 2^{-m\alpha p} t^{(\lambda-1-1/q)p} K^p(1,t)dt\bigg]^{1/p}\bigg\}$$

$$\leq 2^{1/p-1}\{C_1(1/p)\}^{1/p} C_2(p)\|f\|_{K(\omega)} \int_0^\infty (2^{-(m-1)\alpha} + 2^{-m\alpha}) t^{\lambda-1-1/q} K(1,t)dt$$

$$\leq 2^{1/p-1}\{C_1(1/p)\}^{1/p} C_2(p)(1+2^{|\alpha|})\|f\|_{K(\omega)} \int_0^\infty t^{\lambda-\alpha-1-1/q} K(1,t)dt. \tag{3.2}$$

Case 2. $1 \leq p < \infty$. In this case, it follows from (3.1), (2.5) and (2.6) that

$$\|Tf\|_K \leq C_1(p)\bigg\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \int_0^\infty \Big(\|f\varphi_{k+m-1}\|_{q,\omega} + \|f\varphi_{k+m}\|_{q,\omega}\Big)^p$$

$$\times t^{(\lambda-1-1/q)p} K^p(1,t)dt\bigg\}^{1/p}$$

$$\leq 2^{1-1/p} C_1(p)\bigg\{\bigg[\sum_{k=-\infty}^{\infty} 2^{(k+m-1)\alpha p}\|f\varphi_{k+m-1}\|_{q,\omega}^p$$

$$\times \int_0^\infty 2^{-(m-1)\alpha p} t^{(\lambda-1-1/q)p} K^p(1,t)dt\bigg]^{1/p}$$

$$+\bigg[\sum_{k=-\infty}^{\infty} 2^{(k+m)\alpha p}\|f\varphi_{k+m}\|_{q,\omega}^p \int_0^\infty 2^{-m\alpha p} t^{(\lambda-1-1/q)p} K^p(1,t)dt\bigg]^{1/p}\bigg\}$$

$$\leq 2^{1-1/p} C_1(p) C_2(1/p)^{1/p}\|f\|_{K(\omega)} \int_0^\infty (2^{-(m-1)\alpha} + 2^{-m\alpha}) t^{\lambda-1-1/q} K(1,t)dt$$

$$\leq 2^{1-1/p} C_1(p) C_2(1/p)^{1/p}(1+2^{|\alpha|})\|f\|_{K(\omega)} \int_0^\infty t^{\lambda-\alpha-1-1/q} K(1,t)dt. \tag{3.3}$$

Hence, by (3.2) and (3.3), we get

$$\|T\| \leq C(p,\alpha) \int_0^\infty t^{\lambda-\alpha-1-1/q} K(1,t)dt. \tag{3.4}$$

where $C(p,\alpha)$ is defined by (2.9).

To prove the opposite inequality, we set for each $\varepsilon \in (0,1)$,

$$f_\varepsilon(x) = \begin{cases} 0, & 0 < x \le 1, \\ x^{\lambda-\alpha-1-1/q-\varepsilon}, & x > 1, \end{cases}$$

then

$$\|f_\varepsilon \varphi_k\|_{q,\omega}^q = \int_{2^{k-1}<x\le 2^k} x^{(\lambda-\alpha-1-1/q-\varepsilon)q} x^{(1-\lambda)q} dx = C_3 2^{-k(\alpha+\varepsilon)q}, \tag{3.5}$$

where

$$C_3 = \left| \frac{2^{(\alpha+\varepsilon)q} - 1}{(\alpha+\varepsilon)q} \right|.$$

It follows that

$$\|f_\varepsilon\|_{K(\omega)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} (C_3^{1/q} 2^{-k(\alpha+\varepsilon)})^p \right\}^{1/p} = C_3^{1/q} \frac{2^{-\varepsilon}}{(1-2^{-\varepsilon p})^{1/p}}. \tag{3.6}$$

Observe that

$$T(f_\varepsilon, x) = x^{-(\alpha+1/q+\varepsilon)} \int_{x^{-1}}^{\infty} t^{\lambda-(\alpha+1+1/q+\varepsilon)} K(1,t) dt. \tag{3.7}$$

For each $\varepsilon \in (0,1)$, there exists a positive integer l, such that $2^{l-1} \le 1/\varepsilon < 2^l$, so that

$$\begin{aligned}
\|Tf_\varepsilon\|_K^p &= \sum_{k=1}^{\infty} 2^{k\alpha p} \left\{ \int_{x>1} \Big[T(f_\varepsilon, x)\varphi_k(x) \Big]^q dx \right\}^{p/q} \\
&= \sum_{k=1}^{\infty} 2^{k\alpha p} \left\{ \int_{2^{k-1}<x\le 2^k} x^{-(\alpha+1/q+\varepsilon)q} \left(\int_{x^{-1}}^{\infty} t^{\lambda-(\alpha+1+1/q+\varepsilon)} K(1,t) dt \right)^q dx \right\}^{p/q} \\
&\ge \sum_{k=l+1}^{\infty} 2^{k\alpha p} \left(\int_{\varepsilon}^{\infty} t^{\lambda-(\alpha+1+1/q+\varepsilon)} K(1,t) dt \right)^p \left(\int_{2^{k-1}<x\le 2^k} x^{-(\alpha+1/q+\varepsilon)q} dx \right)^{p/q} \\
&= \left(\int_{\varepsilon}^{\infty} t^{\lambda-(\alpha+1+1/q+\varepsilon)} K(1,t) dt \right)^p \sum_{k=l+1}^{\infty} 2^{k\alpha p} \Big(C_3 2^{-k(\alpha+\varepsilon)q} \Big)^{p/q} \\
&= C_3^{p/q} \left(\int_{\varepsilon}^{\infty} t^{\lambda-(\alpha+1+1/q+\varepsilon)} K(1,t) dt \right)^p \frac{2^{-\varepsilon p(l+1)}}{1-2^{-\varepsilon p}}.
\end{aligned} \tag{3.8}$$

Thus,

$$\|T\| \ge \frac{\|Tf_\varepsilon\|_K}{\|f_\varepsilon\|_{K(\omega)}} \ge 2^{-\varepsilon l} \int_{\varepsilon}^{\infty} t^{\lambda-(\alpha+1+1/q+\varepsilon)} K(1,t) dt. \tag{3.9}$$

Take limits as $\varepsilon \to 0$ in (3.9), we obtain

$$\|T\| \ge \int_0^{\infty} t^{\lambda-(\alpha+1+1/q)} K(1,t) dt.$$

This finishes the proof of Theorem 1.

Proof of Theorem 3. By Minkowsks inequality for integrals and setting $y = tx$, we have

$$\begin{aligned}\|Tf\|_{p,\omega_1} &\le \left\{ \int_0^\infty \left(\int_0^\infty K(x,y)|f(y)|dy \right)^p x^\alpha dx \right\}^{1/p} \\ &\le \int_0^\infty \left\{ \int_0^\infty |f(tx)|^p x^{(1-\lambda)p+\alpha} dx \right\}^{1/p} K(1,t)dt \\ &= \|f\|_{p,\omega_2} \int_0^\infty t^{\lambda-1-(\alpha+1)/p} K(1,t)dt.\end{aligned}$$

It follows that

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_{p,\omega_1}}{\|f\|_{p,\omega_2}} \le \int_0^\infty t^{\lambda-1-(\alpha+1)/p} K(1,t)dt. \tag{3.10}$$

To prove the opposite inequality , we set for any $\varepsilon \in (0,1)$,

$$f_\varepsilon(x) = \begin{cases} 0, & 0 < x \le 1, \\ x^{\lambda-1-(1+\alpha)/p-\varepsilon}, & x > 1, \end{cases}$$

then

$$\|f_\varepsilon\|_{p,\omega_2}^p = \int_1^\infty x^{-p\varepsilon-1} dx = \frac{1}{p\varepsilon}. \tag{3.11}$$

Thus,

$$\begin{aligned}\|Tf_\varepsilon\|_{p,\omega_1} &= \left(\int_{x>1} |T(f_\varepsilon, x)|^p x^\alpha dx \right)^{1/p} \\ &= \left\{ \int_{x>1} x^{-(1+p\varepsilon)} \left(\int_{x^{-1}}^\infty t^{\lambda-1-(1+\alpha)/p-\varepsilon} K(1,t)dt \right)^p dx \right\}^{1/p} \\ &\ge \left(\int_\varepsilon^\infty t^{\lambda-1-(1+\alpha)/p-\varepsilon} K(1,t)dt \right) (p\varepsilon)^{-1/p}.\end{aligned}$$

This implies

$$\|T\| \ge \frac{\|Tf_\varepsilon\|_{p,\omega_1}}{\|f_\varepsilon\|_{p,\omega_2}} \ge \int_\varepsilon^\infty t^{\lambda-1-(1+\alpha)/p-\varepsilon} K(1,t)dt. \tag{3.12}$$

Take limits as $\varepsilon \to 0$ in (3.12), we get

$$\|T\| \ge \int_0^\infty t^{\lambda-1-(1+\alpha)/p} K(1,t)dt. \tag{3.13}$$

Then by (3.12) and (3.13), we have $\|T\| = \int_0^\infty t^{\lambda-1-(1+\alpha)/p} K(1,t)dt$. The Theorem is proved.

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