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ON SOME PROPERTIES OF MONIC POLYNOMIALS

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1. Introduction

Let

$$f(z) = z^k + a_1 z^{k-1} + \cdots + a_k, \qquad k \geq 2$$

be a monic polynomial of degree k.

Let $E\{z :| f(z) |\leq R\}$ where $R > 0$ and $d(E)$ be the diameter of the set E. Then we prove

Theorem 1.

$$d(E) \geq 2R^{1/k}$$

The theory of iteration of a rational or entire function of a complex variable z deals with the sequence of natural interates $f_n(z)$ defined by

$$f_0(z) = z, \; f_{n+1}(z) = f(f_n(z)), \; n = 0, 1, \cdots .$$

In the theory developed by Fatou [3,4] and Julia [5], the central object of study is the *Fatou Set* of those points of the complex plane in no neighbourhood of which the sequence of natural interates $\{f_n(z)\}$ forms a normal family in the sense of Montel. Unless $f(z)$ is a rational function of order 0 or 1, the set $F(f)$ is a nonempty perfect set. For a survey of the main properties of the *Fatou Set* we refer the reader to the excellent paper of Brolin [2]. Bhattacharyya and Arumaraj [1] considered the question of bounds of the diameter of the set $F(f)$ where $f(z)$ is a monic polynomial. They conjectured that if $f(z)$ is a monic polynomial of degree ≥ 2, then $d(F(f)) \geq 2$.

We will show that this conjecture follows from our Theorem 1. We thus have

Theorem 2. *If $f(z)$ is a monic polynomial of degree $k \geq 2$, then $d(F(f)) \geq 2$.*

We also prove

Theorem 3. *If $d(F(f)) = 2$ where f is a monic polynomial of degree $k \geq 2$, then $f(z) = (z - \alpha)^k$, $k \geq 2$ for some α which can be zero.*

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If α is zero then it is obvious that $d(F(z^k)) = 2$. This shows that the bounds obtained in theorem 1 and 3 are sharp.

We prove our theorems in section 2.

2. Proof of the theorems

We first prove theorem 2, assuming theorem 1 which we shall prove next.

Proof of theorem 2. Let $\{f_n(z)\}$ be the natural iterates of $f(z)$ where

$$f(z) = z^k + a_1 z^{k-1} + \cdots + a_k, \; k \geq 2.$$

We use the construction employed by Bhattacharyya and Arumaraj [1] for $F(f)$. We choose $R > 0$ such that

$$| f(z) | > R \text{ for } | z | > R.$$

We set

$$D_{-n} = \{z : | f_n(z) | \leq R\}$$

and define

$$D = \cap_{n=0}^{\infty} D_{-n}.$$

Then $F(f) = \partial D$.

From theorem 1 we have

$$d(D_{-n}) \geq 2R^{1/k^n} \geq 2 \text{ for every } n.$$

Thus $d(D) \geq 2$, i.e. $d(F(f)) \geq 2$.
This proves the theorem.

Proof of theorem 1. Take any constant η such that $| \eta | = 1$. We shall find this η later. The equation

$$\eta f(z) = f(\omega) \tag{2.1}$$

defines a member of the function elements of $z = \infty$ which have the form

$$\omega = \eta^{1/k} z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \tag{2.2}$$

one for each k-th root of η. Note that ω is not linear. Each such function element is a branch of an algebraic function ω of z which satisfies (2.1). All possible branches which arise from analytic continuation are analytic for all finite z, except for a finite number of algebraic singularities and at these singularities and at these singularities the value of ω is finite since $\omega = \infty$ implies $z = \infty$.

Denote by $g(z)$ any value of one of the algebraic branches obtained as above from (2.2). Then

$$\eta\, f(z) \;=\; f(g(z))$$

so that if z is in E i.e. $\mid f(z) \mid\leq R$ then

$$\mid f(g(z)) \mid = \mid \eta f(z) \mid \leq \; R$$

i.e. $g(E) \subset E$ and $g(\partial E) \subset E$ as ∂E is $\{z :\mid f(z) \mid= R\}$. Thus

$$\Big\{d(E)\Big\}^k \;\geq\; \max_{z\in\partial E} \mid z - g(z) \mid^k$$

i.e.

$$\Big\{d(E)\Big\}^k \;\geq\; \max_{\mid f(z)\mid=R} \Big\{R \mid z - g(z) \mid^k / \mid f(z) \mid \Big\}. \tag{2.3}$$

Now

$$\phi(z) \;=\; (z - g(z))^k / f(z) \tag{2.4}$$

is analytic at ∞ with $\phi(\infty) = (1-\mu)^k$ where μ is the determination of $\eta^{1/k}$ taken for $g(z)$ in (2.2). We choose $\eta = +1$ or -1 according as k is even or odd, so that $\eta = (-1)^k$. Taking the determination $\mu = \eta^{1/k} = -1$, gives $\phi(\infty) = 2^k$ for the corresponding branch $g(z)$.

We now continue the branch of g just defined to obtain all possible analytic continuations, keeping z restricted to the domain $D = (C \cup \{\infty\})\backslash E$. This gives rise to a possibly many valued (algebraic) function which has atmost finitely many algebraic singularities at certain finite points and a number of branches, each with a simple pole at ∞, where g is as in (2.2).

The corresponding continuation of ϕ in (2.4) remains finite at finite points of D, since $f(z) \neq 0$ in D, while the values at ∞ are of the form $(1-\eta^{1/k})^k$, of which the greatest is 2^k. Let the set of these continuations be denoted by ϕ^*. If ϕ^* is not a constant, the values $\phi^*(D)$ form an open connected set by the open mapping theorem. The branches ϕ^* are continuous in $\overline{D}$, bounded at $z = \infty$, so in fact $\phi^*(D)$ is bounded, since $\phi^*(D)$ contains the value 2^k.

Set

$$\beta \;=\; \sup\Big\{x \mid x > 0,\ x \ in\ \phi^*(D)\}.$$

Then β is $\phi(\gamma)$ for some γ in $\partial D = \partial E$.
[For there exist z_n in D, branches ϕ_n, $\phi_n(z_n) \to \beta$. We can assume z_n converges in $\overline{D}$ and $z_n \not\to \infty$ since $\beta > 2^k$.

Thus $z_n \to \infty$ in $\overline{D}$ and we can assume all ϕ_n are in fact the same 'branch' of ϕ since they are but finitely many. Thus $\phi(\gamma) = \beta$ is in the boundary of $\phi^*(D)$. By the open mapping theorem $\beta \in D$, so $\beta \in \partial D = \partial E = z :\mid f(z) \mid= R]$.

Taking $z = \beta$ and fixing g so that

$$\left| \frac{\gamma - g(\gamma)}{f(\gamma)} \right|^k = \phi(\gamma) = \beta.$$

We have in (2.3)

$$d(E)^k \geq R \cdot \beta > 2^k R.$$

Thus

$$d(E) > 2R^{1/k}.$$

The proof of the theorem is now complete.

Proof of theorem 3. Take ϕ =constant 2^k where ϕ is as in the proof of theorem 1. Consider the expansions near a zero of $f(z)$. Suppose that

$$f(z) = a(z-\alpha)^r + \cdots,$$

so $r \leq k$.

Then $z = g(z)$ at $z = \alpha$, so $g(\alpha) = \alpha$. By (2.1)

$$\eta(z-\alpha)^r + \cdots = (g-\alpha)^2 + \cdots$$

so that

$$g - \alpha = \eta^{1/r}(z-\alpha) + \text{higher terms.}$$

Hence

$$z - g(z) = (z-\alpha)(1-\eta^{1/r}) + \text{higher terms of } (z-\alpha).$$

This gives

$$\begin{aligned} 2^k &= \frac{(z-g(z))^k}{f(z)} \\ &= (z-\alpha)^{k-r}(1-\eta^{k/r}) + \cdots, \quad k - r \geq 0. \end{aligned}$$

But then $k - r \leq 0$. Hence r must be equal to k. This means that f has a single zero of order k at α. Hence f must be $(z-\alpha)^k$.

From Theorem 2 it follows that
$d(F(f)) = 2$ implies $f(z) = (z-\alpha)^k$ and conversely.

References

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