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ON FUNCTIONAL-DIFFERENTIAL INCLUSIONS WITH STATE CONSTRAINTS

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Abstract. In This paper we examine differential inclusions with memory and state constrains.We prove two existence theorem. One with nonconvex valued orientor field and the other with a convex valued one. Finally we consider also the problem with no state constraints.

1. Introduction.

In this paper we examine functional-differential inclusions with state constraints, defined in a separable Hilbert space X. So the multivalued Cauchy problem under consideration has the following form:

$$\begin{cases} -\dot{x}(t) \in N_{K(t)}(x(t)) + F(t, x_t, \dot{x}_t) \ a.e.\ on\ T=[0,b] \\ x(u) = \phi(u) \qquad \text{for all } u \in T_0 = [-r, 0] \end{cases} \tag{$*$}$$

Here $K(t)$ is the time varying state constraint set which will be convex valued and $N_{K(t)}(x(t))$ is the normal cone to $K(t)$ at $x(t)$ (see Aubin-Cellina [2]). Recall that for all $x \in K(t)$ $N_{K(t)}(x) = \partial\delta_{K(t)}(x)$, where $\partial\delta_{K(t)}(\cdot)$ denotes the convex subdifferential of the indicator function $\delta_{K(t)}(\cdot)$ ($\delta_{K(t)}(x) = 0$ if $x \in K(t)$, $+\infty$ otherwise). Also $F(\cdot,\cdot,\cdot)$ is a multivalued perturbation with values in the Hilbert space X. Given $x : \hat{T} = [-r, b] \to X$, by $x_t(\cdot)$ we will denote the map describing the history from $t-r$ up to time t of $x(\cdot)$. So $x_t : [-r, 0] \to X$ is defined by $x_t(s) = x(t+s) s \in [-r, 0]$.

We will prove two existence theorems. One with nonconvex valued perturbation and the other with convex valued $F(\cdot,\cdot,\cdot)$. Also we consider the case where$K(t) = X \Rightarrow N_{K(t)}(x) = \{0\}$ and so the multivalued Cauchy problem has no state constraints.

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The results of this paper extend earlier works by Antosiewicz-Cellina [1],Fryszkowski [7], Kisielewicz [9], Moreau [10] and Papageorgiou [13], [14]. Also we extend to infinite dimensional systems with memory, the work on "differential variational inequalities" of Aubin-Cellina [2] (see chapter 5, section 6).

2. Preliminaries.

Let (Ω, Σ) be a measurable space and X a separable Banach space. We will be using the following notations:

$$\begin{aligned} P_{f(c)}(X) &= \{A \subseteq X : \text{ A nonempty, closed, (convex)}\} \\ \text{and } P_{(w)k(c)}(X) &= \{A \subseteq X : \text{ A nonempty, (w-)compact, (convex)}\} \end{aligned}$$

For $A \in 2^X \backslash \{\emptyset\}$, the norm $\mid A \mid$ is defined by $\mid A \mid = \sup\{\|x\| : x \in A\}$. Also a multifunction $F : \Omega \to 2^X \backslash \{\emptyset\}$ is said to be *graph measurable* if and only if $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. A $P_f(X)$-valued multifunction is said to be *measurable* if and only if for every $z \in X \omega \to d(z, F(\omega)) = \inf\{\|z - x\| : x \in F(\omega)\}$ is measurable. For a $P_f(X)$-valued multifunction, measurability implies graph measurability and the converse is true if there exists a complete, σ-finite measure $\mu(\cdot)$ on (Ω, Σ).

Now assume (Ω, Σ, μ) is a σ-finite measure space. For any multifunction $F : \Omega \to 2^X \backslash \{\emptyset\}$, let $S_F^1 = \{f \in L^1(X) : f(\omega) \in F(\omega) \mu - a.e.\}$. If $F(\cdot)$ is graph measurable, then using Aumann's selection theorem, it is easy to check that $S_F^1 \neq \emptyset$ if and only if $\omega \to \inf\{\|x\| : x \in F(\omega)\} \in L_+^1$. Also if $F(\cdot)$ is $P_f(X)$-valued, then S_F^1 is strongly closed in the Lebesgue-Bochner space $L^1(X)$. A multifunction $F : \Omega \to P_f(X)$ is said to be *integrably bounded* if and only if $F(\cdot)$ is measurable and $\omega \to \mid F(\omega) \mid = \sup\{\|x\| : x \in F(\omega)\} \in L_+^1$. Clearly for such a multifunction $S_F^1 \neq \emptyset$.

Suppose that Y, Z are Hausdorff topological spaces and $F : Y \to 2^Z \backslash \{\emptyset\}$. We say that $F(\cdot)$ is *upper semicontinuous* (u.s.c.) (resp. lower semicontinuous (l.s.c.)) if for all $V \subseteq Z$ open, the set $\{y \in Y : F(y) \subseteq V\}$ (resp. $\{y \in Y : F(y) \cap V \neq \emptyset\}$) is open in Y. Other equivalent definitions of upper and lower semicontinuity can be found in Delahaye-Denel [4].

On $P_f(X)$ we can define a (generalized) metric $h(\cdot, \cdot)$ by setting

$$h(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

where $d(a, B) = \inf\{\|a - b\| : b \in B\}$ and $d(b, A) = \inf\{\|b - a\| : a \in A\}$. Recall that $(P_f(X), h)$ is a complete metric space. If $\Omega = [0, b]$, a multifunction $F : \Omega \to P_f(X)$ is said to be *h-absolutely continuous with modulus* $r(\cdot) \in L_+^1$ if and only if $h(F(t), F(t')) \leq \int_t^{t'} r(s) ds$ for all $t, t' \in \Omega = [0, b]$.

Finally if $\{A_n\}_{n \geq 1}$ is a sequence of nonempty subsets of X, we write $w - \overline{\lim} A_n = \{x \in X : x = w - \lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \ldots < n_k < \ldots\}$ and $s - \underline{\lim} A_n = \{x \in X : x =$

$s - \lim x_n, x_n \in A_n, n \geq 1\}$. Here s-denotes the strong topology on X and w-the weak topology.

3. Existence theorems.

Let $T_0 = [-r, 0]$, $T = [0, b]$, $\hat{T} = [-r, b]$, $r > 0$ and X is a separable Hilbert space. We will make the following hypotheses concerning the data of problem (*).

$H(K)$: $K : T \to P_{kc}(X)$ is an h-absolutely continuous multifunction with modulus $r(\cdot) \in L^1_+$.

$H(F)$: $F : TxC(T_0, X) \times L^1(T_0, X) \to P_f(X)$ is a multifunction s.t.

(1) $(t, y, h) \to F(t, y, h)$ is graph measurable,

(2) $(y, h) \to F(t, y, h)$ is l.s.c. from $C(T_0, X) \times L^1(T_0, X)_w$ into X

(3) $| F(t, y, h) | \leq a(t) + b(t)\|y\|_\infty$ a.e. with $a(\cdot)$, $b(\cdot) \in L^1_+$,

H_0: $x_0 : T_0 \to X$ is absolutely continuous, $x_0(0) \in K(0)$.

Theorem 3.1. *If hypothesis $H(K)$, $H(F)$ and H_0 hold, then (*) admits a solution.*

Proof. First we will determine an a priori bound for the solutions of (*). So let $x(\cdot) \in C(T, X)$ be such a solution. From Daures [3] (see also Moreau [10]), we know that

$$\|\dot{x}(t)\| \leq r(t) + | F(t, x_t, \dot{x}_t) | \leq r(t) + a(t) + b(t)\|x_t\|_\infty \ a.e.$$

$$\Longrightarrow \|x(t)\| \leq \|x_0\| + \int_0^t (r(s) + a(s) + b(s)\|x_s\|_\infty)ds$$

$$\Longrightarrow \|x_t\|_\infty \leq \|x_0\| + \int_0^t (r(s) + a(s) + b(s)\|x_s\|_\infty)ds$$

Invoking Gronwall's inequality we get that

$$\|x_t\|_\infty \ \leq \ (\|x_0\| + \|r\|_1 + \|a\|_1) \exp \|b\|_1 \ = \ M$$

Then let $\hat{F} : T \times C(T_0, x) \times L^1(T_0, X) \to P_f(X)$ be defined by

$$\hat{F}(t, y, h) \ = \ \begin{cases} F(t, y, h) & if \ \|y\|_\infty \leq M \\ F(t, \dfrac{My}{\|y\|}, h) & if \ \|y\|_\infty > M. \end{cases}$$

So we have that $\hat{F}(t, y, h) = F(t, p_M(y), h)$, where $p_M(\cdot)$ is the M-radial retraction. Recall that $p_M(\cdot)$ is Lipschitz continuous. So the map $u : T \times C(T_0, X) \times L^1(T_0, X) \times X \to T \times C(T_0, X) \times L^1(T_0, X) \times X$ defined by $(t, y, h, v) \xrightarrow{u} (t, p_M(y), h, v)$ is measurable. Then observe that $Gr\hat{F} = \{(t, y, h, v) \in T \times C(T_0, X) \times X : v \in \hat{F}(t, y, h) = F(t, p_M(y), h)\} = u^{-1}(GrF)$. But by hypothesis $H(F)$ (1), $GrF \in B(T) \times B(C(T_0, X)) \times B(L^1(T_0, X))$. So $\hat{F}$ is graph measurable. Also $(y, h) \to \hat{F}(t, y, h)$ is clearly l.s.c. Finally note that

$|\hat{F}(t,y,h)| \leq a(t)+b(t)M = c(t)$ a.e. $c(\cdot) \in L^1_+$. In the sequel we will consider (*) with the orientor field $F(t,y,h)$ replaced by $\hat{F}(t,y,h)$.

Let $W \subseteq C(\hat{T},X)$ be defined by :

$$W = \{y \in C(\hat{T},X) : y(t) = x_0 + \int_0^t g(s)ds,\ t \in T \| g(t)\| \leq r(t)+c(t)\ a.e.,$$
$$y(u) = \phi(u), u \in T_0\}.$$

Clearly W is equicontinuous, convex and closed. Let $V = \{y \in W : y(t) \in K(t)\ t \in T\}$. Then from Daures [3] and Moreau [10], we know that $V \neq \emptyset$ and $V(t) = \{v(t) : v \in V\} = W(t) \cap K(t) \in P_k(X)$. So invoking the Arzela-Ascoli theorem, we deduce that V is compact in $C(\hat{T},X)$. Also let $Q = \{h \in L^1(\hat{T},X) : \|h(t)\| \leq r(t)+c(t)$ a.e. and $h(u) = \dot{x}_0(u)$ a.e. on $T_0\}$ (note that since X is a Hilbert space and by hypothesis H_0, $x_0 : T_0 \to X$ is absolutely continuous, $\dot{x}_0(\cdot)$ exists a.e. on T_0). Clearly Q with the relative weak $L^1(\hat{T},X)$-topology is compact and since $L^1(\hat{T},X)$ is separable, Q is metrizable too. In the sequel we will consider Q with this relative weak $L^1(\hat{T},X)$-topology.

Next consider the multifunction $G : V \times Q \to P_f(L^1(X))$ defined by

$$G(y,h) = S^1_{\hat{F}(\cdot,y,h)}$$

Since $\hat{F}(\cdot,\cdot,\cdot)$ is graph measurable, $t \to \hat{F}(t,y_t,h_t)$ is measurable and also almost everywhere bounded by $c(t)$. So $G(y,h) \neq \emptyset$ for all $(y,h) \in V \times Q$. Also invoking theorem 4.1 of [12] we get that $G(\cdot,\cdot)$ is l.s.c. from $V \times Q$ into $L^1(X)$. Apply Fryszkowski's selection theorem [6] to get $g : V \times Q \to L^1(X)$ continuous s.t. $g(y,h) \in G(y,h)$. For each $(y,h) \in V \times Q$ consider the equation

$$\begin{cases} -\dot{x}(t) \in N_{K(t)}(x(t)) + g(y,h)(t) \text{ a.e. on } T \\ x(u) = x_0(u)\ for\ u \in T_0 \end{cases} \qquad (*)(y,h).$$

As before from Daures [3] and Moreau [10]. we deduce that $(*)(y,h)$ has a unique solution $s(g(y,h))(\cdot) \in C(\hat{T},X)$. For any $u \in L^1(X)$, let $\hat{s}(u) = (s(u),\dot{s}(u))$. From [13] (see also the lemma below) we know $s(\cdot)$ is continuous, while $\dot{s}(\cdot)$ has closed graph, hence is continuous on $L^1(X)$. Therefore $\hat{s}(\cdot)$ is continuous from $L^1(X)$ into $V \times Q$. Let $q = \hat{s}og$. Then $q : V \times Q \to V \times Q$ is continuous and so from the Schauder-Tichonov fixed point theorem, we get $(y,h) \in V \times Q$ s.t. $q(y,h) = (y,h)$. Then $y = s(g(y,h))(\cdot) \in C(\hat{T},X)$ is the solution of (*) with orientor field $\hat{F}(t,y,h)$. Note that

$$\begin{aligned} \|\dot{y}(t)\| &\leq r(t) + |\hat{F}(t,y_t,\dot{y}_t)|\ a.e. \\ &\leq r(t)+a(t)+b(t)\|y_t\|_\infty\ a.e. \text{ (recall the definition of } \hat{F}(\cdot,\cdot,\cdot)) \\ \Longrightarrow \|y\|_\infty &\leq M \\ \Longrightarrow \hat{F}(t,y_t,\dot{y}_t) &= F(t,y_t,\dot{y}_t) \\ \Longrightarrow y(\cdot)\ &solves\ (*). \end{aligned}$$

Q.E.D.

The next existence theorem involves convex valued perturbations. To prove it we will need a continuous dependence result, which is actually interesting by itself. So consider the following evolution inclusion.

$$\begin{cases} -\dot{x}(t) \in N_{K(t)}(x(t)) + f(t) \ a.e. \\ x(0) = x_0 \end{cases} \quad (**)$$

We know that given $x_0 \in K(0)$ and $f \in L^1(X)$, the multivalued Cauchy problem (**) has a unique solution $s(f)(\cdot) \in C(T,X)$. The next proposition examines the map $f \to s(f)(\cdot)$.

Proposition 3.1. *If hypothesis $H(K)$ holds, $x_0 \in K(0)$ and $f \in W \in P_{wk}(L^1(X))$, then $f \to s(f)$ is continuous from W_w into $C(T,X)$.*

Proof. Recall that W_w is compact, metrizable. Also W is uniformly integrable. Since $\|\dot{s}(f)(t)\| \leq r(t) + \|f(t)\|$ a.e. (see Daures [3]), we deduce that $\{s(f)(\cdot)\}_{f\in W}$ is equicontinuous. Also for all $f \in W$ and all $t \in T$, $s(f)(t) \in K(t) \in P_{kc}(X)$. Hence from the Arzela-Ascoli theorem we deduce that $\{s(f)(\cdot)\}_{f\in W}$ is relatively compact in $C(T,X)$.

Next let $f_n \xrightarrow{w} f$ in $W \subseteq L^1(X)$. Set $x_n = s(f_n)$ and $x = s(f)$. Recalling that $N_{K(t)}(\cdot) = \partial\delta_{K(t)}(\cdot)$ and exploiting the monotonicity of the convex subdifferential, we have:

$$(-\dot{x}_n(t) + \dot{x}(t), -x_n(t) + x(t)) \leq (f_n(t) - f(t), -x_n(t) + x(t)) \ a.e.$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|x_n(t) - x(t)\|^2 \leq (f_n(t) - f(t), -x_n(t) + x(t)) \ a.e.$$

$$\Rightarrow \|x_n(t) - x(t)\|^2 \leq 2\int_0^t (f_n(s) - f(s), -x_n(s) + x(s))ds.$$

From the first part of the proof, we know that we may assume without any loss of generality, that $x_n \to \hat{x}$ in $C(T,X)$. Then we have:

$$\|x_n(t) - x(t)\| \leq 2\int_0^t (f_n(s) - f(s), -x_n(s) + \hat{x}(s))ds + 2\int_0^t (f_n(s) - f(s), -\hat{x}(s) + x(s))ds$$

Since $x_n \to \hat{x}$ in $C(T,X)$ and $f_n \xrightarrow{w} f$ in $W \subseteq L^1(X)$, we have

$$\int_0^t (f_n(s) - f(s), -x_n(s) + \hat{x}(s))ds \to 0$$

$$\textit{and} \int_0^t (f_n(s) - f(s), -\hat{x}(s) + x(s))ds \to 0.$$

So $\|x_n(t) - x(t)\|^2 \to 0 \Longrightarrow x = \hat{x} \Longrightarrow x_n = s(f_n) \to x = s(f)$ *in* $C(T,X)$.

Q.E.D.

Now we are ready for the existence result, when the set valued perturbation is convex valued.

We will need the following hypotheses on the orientor field $F(t,y,h)$.

$H(F)_1$: $F: T \times C(T_0,X) \times L^1(T_0,X) \to P_{fc}(X)$ is a multifunction s.t.

(1) $(t,y,h) \to F(t,y,h)$ is graph measurable,

(2) $(y,h) \to F(t,y,h)$ is u.s.c. from $C(T_0,X) \times L^1(T_0,X)_w$ into X_w,

(3) $| F(t,y,h) | \leq a(t) + b(t)\|y\|_\infty$ a.e. with $a(\cdot), b(\cdot) \in L^1_+$.

Theorem 3.2. *If hypotheses $H(K), H(F)_1$ and H_0 hold, then (*) admits a solution.*

Proof. As in the proof of theorem 3.1 we can check that every solution $x(\cdot) \in C(\hat{T},X)$ satisfies $\|x_t\|_\infty \leq M$. Then we define $\hat{F}(t,y,h) = F(t,p_M(y),h)$. Clearly $\hat{F}(\cdot,\cdot,\cdot)$ has the same measurability and continuity properties as $F(\cdot,\cdot,\cdot)$. In addition observe that $| \hat{F}(t,y,h) | \leq a(t) + b(t)M = c(t)$ a.e. $c(\cdot) \in L^1_+$. In what follows we will consider the multivalued Cauchy problem (*) with $\hat{F}(\cdot,\cdot,\cdot)$ being the set valued perturbation.

Let $V = \{g \in L^1(X) : \|g(t)\| \leq a(t) + r(t) \ a.e.\}$ From Dunford's compactness theorem (see Diestel-Dhl [5], theorem 1, p.101), we have that V is weakly compact in $L^1(X)$. Let $R: V \to 2^{L^1(X)}$ be defined by $R(g) = S^1_{\hat{F}(\cdot,s(g)_\cdot,\dot{s}(g)_\cdot)}$. It is easy to see that $R(g) \in P_{fc}(V)$. We claim that $R(\cdot)$ is u.s.c. on V with the relative weak $L^1(X)$-topology. Because of the weak compactness of V in $L^1(X)$ and since the weak $L^1(X)$-topology on V is metrizable ($L^1(X)$ being separable), we know that it is enough to check that $R(\cdot)$ has closed graph. So let $\{(g_n,f_n)\}_{n\geq 1} \subseteq GrR$ and assume $(g_n,f_n) \xrightarrow{w\times w} (g,f)$ in $V \times V$. From proposition 3.1, we know that $s(g)_n \to s(g)$ in $C(\hat{T},X)$. Also by passing to a subsequence if necessary, we may assume that $\dot{s}(g)_n \xrightarrow{w} \dot{s}(g)$ in $L^1(X)$. Invoking theorem 3.1 of [12], we have:

$$f(t) \in \overline{conv}\, w - \overline{\lim}\hat{F}(t,s(g_n)_t,\dot{s}(g_n)_t) \ a.e.$$

Observe that $s(g_n)_t \to s(g)_t$ in $C(T_0,X)$ and $\dot{s}(g_n)_t \xrightarrow{w} \dot{s}(g)_t$ in $L^1(T_0,X)$. So because of $H(F)_1$ and the definition of $\hat{F}(\cdot,\cdot,\cdot)$ we have, $w - \overline{\lim}\hat{F}(t,s(g_n)_t,\dot{s}(g)_t) \subseteq \hat{F}(t,s(g)_t,\dot{s}(g)_t)$ a.e. Hence

$$\begin{aligned} & f \in S^1_{\hat{F}(\cdot,s(g)_\cdot,\dot{s}(g)_\cdot)} \\ \Longrightarrow\ & (g,f) \in GrR \\ \Longrightarrow\ & R(\cdot) \text{ is u.s.c. on } V_w. \end{aligned}$$

Apply the Kakutani-KyFan fixed point theorem, to get $f \in V$ s.t. $f \in R(f)$. Clearly then $s(f)(\cdot)$ is the desired solution of (*) with $\hat{F}(\cdot,\cdot,\cdot)$ being the orientor field. Then using the definition of $\hat{F}(t,y,h)$ and Gronwall's inequality, we can show that $\|s(f)_t\|_\infty \leq M$ $t \in T \Rightarrow \hat{F}(t,s(f)_t,\dot{s}(f)_t) = F(t,s(f)_t,\dot{s}(f)_t) \Rightarrow s(f)(\cdot)$ is the desired solution of (*)

Q.E.D.

Finally we will consider the case where no state constraints are present i.e. $K(t) = X$ for all $t \in T$ and so $N_{K(t)}(x) = X$. Our theorem extends theorem 3.1$'$ of [14] as well as the finite dimensional results of Kisielewicz [9] (theorems 1 and 2).

So the multivalued Cauchy problem under consideration is now the following

$$\begin{cases} \dot{x}(t) \in F(t, x_t, \dot{x}_t) \; a.e. \; on \; T \\ x(u) \;=\; x_0(u) \; u \in T_0 \end{cases} \quad (**).$$

Here X ia any separable Banach space. We will need the following hypotheses on the data of (**).

$H(F)_2$: $F : T \times C(T_0, X) \times L^1(T_0, X) \to P_f(X)$ is a multifunction s.t.

(1) $(t, y, h) \to F(t, y, h)$ is graph measurable,

(2) $(y, h) \to F(t, y, h)$ is l.s.c. from $C(T_0, X) \times L^1(T_0, X)_w$ into X

(3) $| F(t, y, h) | \leq (1 + \|y\|_\infty)G(t)$ a.e., with $G : T \to P_{kc}(X)$ integrably bounded.

H_{01}: $x_0(\cdot) \in C(T_0, X)$

Theorem 3.3. *If hypotheses $H(F)_2$ and H_{01} hold, then (**) admits a solution.*

Proof. Again, exploiting the growth hypothesis $H(F)_2(3)$ and using Gronwall's inequality, we get that $\|x_t\|_\infty \leq (\|x_0\|_\infty + \| \; | G(\cdot) | \; \|_1) \exp(\| \; | (G(\cdot) | \; \|_1) = M$. Define $\hat{F}(t, y, h) = F(t, p_M(y), h)$. We know (see the proof of theorem 3.1) that $\hat{F}(\cdot, \cdot, \cdot)$ has the same measurability and continuity properties as $F(\cdot, \cdot, \cdot)$. Also $| \hat{F}(t, y, h) | \leq (1 + M)$ $G(t) = \hat{G}(t)$ a.e. with $\hat{G} : T \to P_{kc}(X)$ integrably bounded.

Next let $W \subseteq C(\hat{T}, X)$ be defined by:

$$W = \{y \in C(\hat{T}, X) : y(t) = x_0 + \int_0^t g(s)ds, \; t \in T, \; g \in S^1_{\hat{G}}, \; y(u) = x_0(u) \; u \in T_0\}.$$

It is clear that W is an equicontinuous subset of $C(\hat{T}, X)$. Also for every $y \in W$ and every $t \in T$, we have $y(t) \in x_0 + \int_0^t \hat{G}(s)ds$. But from Radström's embedding theorem (see for example Hiai-Umegaki [8], section 3), we have that $\int_0^t \hat{G}(s)ds \in P_{kc}(X)$. So $\overline{W(t)} = cl\{y(t) : y(\cdot) \in W\} \in P_k(X)$. Hence invoking the Arzela-Ascoli theorem we deduce that W is relatively compact in $C(\hat{T}, X)$. We claim that it is compact. So we need to show that it is closed in $C(\hat{T}, X)$. To this end let $y_n \to y$ in $C(\hat{T}, X)$, $y_n \in W$. Then by definition we have:

$$\begin{aligned} y_n(t) \;&=\; x_0 \;+\; \int_0^t g_n(s)ds, \; t \in T, \; g_n \in S^1_{\hat{G}} \\ \text{and } y_n(u) \;&=\; x_0(u), \; u \in T_0. \end{aligned}$$

But note (see proposition 3.1 of [11]) that S^1_G is w-compact in $L^1(X)$. So by passing to a subsequence if necessary, we may assume that $g_n \xrightarrow{w} g \in S^1_{\hat{G}}$ in $L^1(X)$. Then $y_n(t) =$

$x_0 + \int_0^t g_n(s)ds \xrightarrow{w} x_0 + \int_0^t g(s)ds = y(t)\ t \in T,\ y(u) = x_0(u)\ u \in T_0 \Rightarrow y \in W \Rightarrow W$ is compact in $C(T, X)$.

Let $R : W \times (S^1_{\hat{G}}, w) \to P_{fc}(L^1(X))$ be defined by $R(y, h) = S^1_{\hat{F}(\cdot, y_\cdot, h_\cdot)}$. As in the proof of theorem 3.1 we can check that $R(\cdot, \cdot)$ is l.s.c.. Apply Fryszkowski's selection theorem [6] to get $r : W \times (S^1_{\hat{G}}, w) \to L^1(X)$ continuous s.t. $r(y, h) = S^1_{\hat{F}(\cdot, y_\cdot, h_\cdot)}$. Then let $W \times (S^1_{\hat{G}}, w) \to W \times (S^1_{\hat{G}, w})$ be defined by

$$k(y, h)(\cdot) = (k_1(y, h)(\cdot), r(y, h)(\cdot))$$

where $k_1(y, h)(t) = x_0 + \int_0^t r(y, h)(s)ds\ t \in T$ and $k_1(y, h)(u) = x_0(u)\ u \in T_0$. Since by construction $r(\cdot, \cdot)$ is continuous, $k(\cdot, \cdot)$ is too. So apply the Schauder-Tichonov fixed point theorem to get $(x, h) \in W \times S^1_{\hat{G}}$ s.t.

$$\begin{aligned} k(x, h) &= (x, h) \\ \Longrightarrow h(t) &= g(x, h)(t)\ a.e.\ and\ x(t) = x_0 + \int_0^t h(s)ds\ t \in T. \end{aligned}$$

Clearly then $h(\cdot) = \dot{x}(\cdot) \in S^1_{\hat{F}(\cdot, x_\cdot, \dot{x}_\cdot)}$. As before using the definition of $\hat{F}(\cdot, \cdot, \cdot)$ and Gronwall's inequality, we get that $\hat{F}(t, x_t, \dot{x}_t) = F(t, x_t, \dot{x}_t) \Rightarrow x(\cdot) \in C(\hat{T}, X)$ solves (**).

Q.E.D.

Remark. Theorem 3.3 can not be derived from theorem 3.1, since in that theorem $K(\cdot)$ was $P_{kc}(X)$-valued, while for problem (**), we need to take $K(t) = X,\ t \in T$.

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