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EVALUATION OF THE E-FUNCTION WHEN TWO OF THE UPPER PARAMETERS ARE EQUAL OR DIFFER BY AN INTEGER

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Introduction.

In a former paper [1] in this journal. I evaluated infinite series involving MacRoberts' E-functions, the definitions and properties of which are to be found in ([2]; pp; 348-352) and in ([3], pp; 203-206). The E-function is defined ([2], p. 409) thus:
If $p \geq q+1$, then

$$E(p,\alpha_r;,q;\rho_s:z) = \sum_{r=1}^{p}\Big[\Pi_{s=1}^{p}{}'\Gamma(\alpha_s-\alpha_r)\Big]\Big[\Pi_{t=1}^{q}\Gamma(\rho_t-\alpha_r)\Big]^{-1}$$

$$\Gamma(\alpha_r)Z^{\alpha_r}F\left\{\begin{matrix}\alpha_r,\alpha_r-\rho_1+1,\dots,\alpha_r-\rho_q+1;(-1)p-q_z\\ \alpha_r-\alpha_1+1,\dots,*\dots,\alpha_r-\alpha_{\rho+1}\end{matrix}\right\}$$

$$= \sum_{r=1}^{p}\Big[Z^{\alpha r}\sum_{n=0}^{\infty}\Gamma(\alpha_r+n)\frac{\Pi_{t=1}^{p}{}'\Gamma(\alpha_t-\alpha_r-n)}{n!\Pi_{s=1}^{q}\Gamma(\rho_s-\alpha_r-n)}(-z)^n\Big], \tag{1}$$

where $| \, arg\ z \, | < \frac{1}{2}(p-q-1)\pi$, the prime in the product sign signifies that the factor for which $s=r$ is left and the asterisk in the E-function means that the parameter $\alpha_r-\alpha_r+1$ is omitted
If $p \leq q$, then ([2]; p. 352).

$$E(p,\alpha_r;q;\rho_s:z) = \frac{\Gamma(\alpha_1)\dots\Gamma(\alpha_p)}{\Gamma(\rho_1)\dots\Gamma(\rho_q)}F\left\{\begin{matrix}\alpha_1,\dots,\alpha_p,\frac{-1}{z}\\ \rho_1,\dots,\rho_q\end{matrix}\right\}, \tag{2}$$

Now, if two of the $\alpha's$ are equal or differ by integral values, some of the series on the right of (1) cease to exist. For instance, if $\alpha_1=\alpha+\ell$, $\alpha_2=\alpha$, where ℓ is a positive integer, the first two series are nom existent. Here it will be shown that they can be replaced by the expression

$$(-1)^{\ell}z^{\alpha+\ell}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+\ell+n)\Pi_{t=1}^{p}\Gamma(\alpha_t-\alpha-\ell-n)}{n!(\ell+n)!\Pi_{s=1}^{q}\Gamma(\rho_s-\alpha-\ell-n)}\Delta_n Z^n$$

$$+z^{\alpha}\sum_{n=0}^{\ell-1}\frac{[\Gamma(\alpha+n)](\ell-n-1)!\Pi_{t=3}^{p}\Gamma(\alpha_t-\alpha-n)}{n!\Pi_{s=1}^{q}(\rho_s-\alpha-n)}(-z)^n, \tag{3}$$

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where

$$\Delta_n = \psi(\ell+n)+\psi(n)-\psi(\alpha+\ell+n-1)-\log z$$
$$+\sum_{t=3}^{p}\psi(\alpha_t-\alpha-\ell-n-1)+\sum_{s=1}^{q}\psi(\rho_s-\alpha-\ell-n-1)$$

and

$$\psi(z) = \frac{d}{dz}\Big\{\log\Gamma(z+1)\Big\} = \frac{\Gamma'(z+1)}{\Gamma(z+1)}.$$

The proof of (3) will be given in §2, whils two subsidiary theorems will be stated and proved in §3. Also the value of $K_0(z)$, where $K_0(z)$ is the modified Bessel function of the second kind, will be derived in §3; while new infinite integrals will be deduced as particular cases in §4.

The following formulae will be required in the proofs: ([2]; p.141)

$$\psi(z+n) = \psi(z) + \sum_{r=1}^{n}\frac{1}{z+r}; \tag{4}$$

$$\psi(-z-1) = \psi(z) + \pi\cot(\pi z), \tag{5}$$

$$\psi(0) = -\gamma, \tag{6}$$

$$\psi(n) = \phi(n)-\gamma; \tag{7}$$

where

$$\gamma = \lim_{m\to\infty}\Big[\Big(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{m}\Big)-\log m\Big], \tag{8}$$

$$\phi(n) = \frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}, \qquad \phi(0) = 0 \tag{9}$$

If m is a positive integer, then ([2], p.154, ex.5).

$$\Pi_{t=0}^{m-1}\Big\{\Gamma(Z+\frac{t}{m})\Big\} = (2\pi)^{\frac{1}{2}m-\frac{1}{2}}m^{\frac{1}{2}-mz}\Gamma(mz) \tag{10}$$

Also ([2], p.207).

$$K_\mu(z) = \frac{\pi}{2\sin\mu\pi}\Big\{I_\mu(z)-I_\mu(z)\Big\}; \tag{11}$$

where

$$I_\mu(z) = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+\mu+1)}(\frac{z}{2})^{\mu+2n} \tag{12}$$

If m is a positive integer and if $R(k)>0$, then ([2], p.466 ex. 30).

$$\int_0^\infty e^{-t}t^{k-1}E(p;\alpha_r;q,\rho_s:\frac{z}{t^m})dt = m^{k-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}m}E(p+m,\alpha_r:q:\rho_s:\frac{z}{m^m}), \tag{13}$$

where $\alpha_{p+\nu} = \frac{k+\nu-1}{m}$; $\nu = 1, 2, \cdots, m-1$. If x is real and positive and $R(m \pm \mu) > 0$, then ([2]; p 395, ex. 109).

$$4\int_0^\infty t^{m-1} K_\mu(2t) E(p, \alpha_r; p; \rho_s : \frac{x}{t^2}) dt \;=\; E(p+2; \alpha_r; q; \rho_s : x) \tag{14}$$

where

$$\alpha_{p+1} \;=\; \frac{1}{2}(m+\mu), \qquad \alpha_{p+2} = \frac{1}{2}(m-\mu).$$

§2. Proof of the main theorem (3).

If $\alpha_1 = \alpha + \ell$, $\alpha_2 = \alpha + \epsilon$, where ℓ is zero or a positive integer and ϵ is small, the sum of the first two series on the right of (1), can be written

$$Z^{\alpha+\ell} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\ell+n)\Gamma(-\ell-n+\epsilon)\Pi_{t=3}^{p}\Gamma(\alpha_t-\alpha-n-\ell)}{n!\Pi_{s=1}^{q}\Gamma(p_s-\alpha-\ell-n)}(-z)^n$$

$$+\, Z^{\alpha+\epsilon} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+\epsilon)\Gamma(\ell-n-\epsilon)\Pi_{t=3}^{p}\Gamma(\alpha_t-\alpha-n-\epsilon)}{n!\Pi_{s=1}^{q}\Gamma(\rho_s-\alpha-n-\epsilon)}(-z)^n.$$

$$= (-1)^\ell Z^{\alpha+\ell} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\ell+n)\Pi_{t=3}^{p}\Gamma(\alpha_t-\alpha-\ell-n)}{n!\Gamma(1+\ell+n-\epsilon)\Pi_{s=1}^{q}\Gamma(\rho_s-\alpha-\ell-n)}\left(\frac{\pi}{\sin \epsilon\pi}\right) Z^n$$

$$-\, (-1)^\ell Z^{\alpha+\epsilon} \sum_{n=\ell}^{\infty} \frac{\Gamma(\alpha+n+\epsilon)\Pi_{t=3}^{p}\Gamma(\alpha_t-\alpha-n-\epsilon)}{n!\Gamma(1-\ell+n+\epsilon)\Pi_{s=1}^{q}\Gamma(\rho_s-\alpha-n-\epsilon)}\left(\frac{\pi}{\sin \epsilon\pi}\right) Z^n$$

$$+\, Z^{\alpha+t} \sum_{n=0}^{\ell-1} \frac{\Gamma(\alpha+n+\epsilon)\Gamma(\ell-n-t)\Pi_{t=3}^{p}\Gamma(\alpha_t-\alpha-n-\epsilon)}{n!\Pi_{s=1}^{q}\Gamma(\rho_s-\alpha-n-\epsilon)}(-Z)^n.$$

The limit when $\epsilon \longrightarrow 0$ of the first two terms in the last expression is obtained by removing the factor $\frac{\pi}{\sin \epsilon\pi}$, then differentiating with respect to ϵ, and finally making $\epsilon \longrightarrow 0$. On replacing n by $\ell + n$ in the second series of the last expression, formula (3) is obtained. Thus (3) is proved.

§3. Derivation of $K_0(z)$.

The first subsidiary theorem to be proved is

$$K_\mu(z) \;=\; \frac{1}{4\pi}\sum_{i,-i}\frac{1}{L}E(1, \frac{1}{2}\mu, \frac{-1}{2}\mu; e^{i\pi}\frac{z^2}{4}) \tag{15}$$

where $K_\mu(z)$ is the modified Bessel function of the second kind defined by (11) and (12) and the symbol $\Sigma_{i,-i}$ means that in the expression following it i is to be replaced by $-i$ and the two expressions are to be added.

To prove (15), expand the E-function on the right of (15) by means of (1), substract the corresponding terms and so obtain (15) by a second application of (12). To find the value of $K_0(z)$, put $\mu = 0$ in (15) so getting

$$K_0(z) = \frac{1}{4\pi}\sum_{i,-i}\frac{1}{i}E(0,0,1 :: \frac{z^2}{4}).$$

From (3) with $\alpha = \ell = 0$, $p = 3$, $q = 0$, this becomes,

$$\begin{aligned}(4i\pi)\cdot K_0(z) &= \sum_{\nu=0}^{\infty}\frac{\pi}{(\nu!)^2\sin\nu\pi}(\frac{z^2}{4})^{\nu}\cdot(2i\sin\pi\nu) \\ &\Big[2\psi(\nu)-\psi(\nu-1)+\psi(-\nu)\Big]+\sum_{\nu=0}^{\infty}\frac{\pi}{(\nu!)^2\sin\nu\pi}\Big(\frac{z^2}{4}\Big)^{\nu} \\ &\Big[e^{-i\pi^{\nu}}(-i\pi+2\log(\frac{z}{2})-e^{i\pi\nu}\cdot(i\pi+2\log\frac{z}{2})\Big] \\ &= \sum_{\nu=0}^{\infty}\frac{2i\pi}{(\nu!)^2}\Big[2\psi(\nu)-\psi(\nu-1)+\psi(-\nu)\Big](\frac{z^2}{4})^{\nu} \\ &+\sum_{\nu=0}^{\infty}\frac{\pi}{(\nu!)^2\sin\pi\nu}(2i\sin\nu\pi) \\ &\Big[-i\pi(2\cos\nu\pi)-2\log\frac{z}{2}\Big]\cdot(\frac{z^2}{4})^{\nu}.\end{aligned}$$

Now apply (5) and (7) and get.

$$\begin{aligned}4i\pi K_0(z) &= \sum_{\nu=0}^{\infty}\frac{2i\pi}{(\nu!)^2}(\frac{z^2}{4})^{\nu}\Big[2\phi(\nu)-2\gamma\Big] \\ &\quad -2(\log\frac{z}{2})\cdot\sum_{\nu=0}^{\infty}\frac{2i\pi}{(\nu!)^2}(\frac{z^2}{4})^{\nu}.\end{aligned}$$

$$K_0(z) = \sum_{\nu=0}^{\infty}\frac{1}{(\nu!)^2}\Big(\frac{z^2}{4}\Big)^{\nu}\phi(\nu)-\Big[\gamma+\log\frac{z}{2}\Big]I_0(Z) \qquad (16)$$

which is a known result ([2], p.268).
The second theorem to be proved in this section is

$$\sum_{t=1}^{m-1}\psi(\frac{t}{m}-n-1) = m\phi(mn)-\phi(n)-(m-1)\gamma-m\log m; \qquad (17)$$

where m and n are positive integres.

To prove (17) take logarithms of both sides of (10), differentiate with respect to Z, so getting

$$\sum_{t=1}^{m-1} \psi(z+\frac{t}{m}-1) = m\psi(mz-1) - m\log m - \psi(z).$$

Now apply (5), and get

$$\sum_{t=1}^{m-1} \psi(z+\frac{t}{m}-1) = m\psi(-mz) - \psi(-z) - m\log m + \pi\cot\pi z - m\pi\cot(\pi mz).$$

But, when $z \longrightarrow -n, \pi\cot\pi z - m\pi\cot\pi(mz) \longrightarrow 0.$

From this (17) follows. Thus (17) is proved.

§4. Evaluation of certain infinite integrals.

We are now in a position to evaluate a large number of infinite new integrals by applying formula (3).

For example, if m is a positive integer, $\mu = 0$; then (14) gives

$$\int_0^\infty \lambda^{m-1} K_0(2\lambda) e^{-\frac{\lambda^2}{z}} d\lambda = \frac{1}{4} E(\frac{m}{2}, \frac{m}{2} :: z). \tag{18}$$

When $m = 1$, from (3) with $\ell = 0$, $\alpha = \frac{1}{2}$, $p = 2$, $q = 0$ this becomes

$$\frac{1}{4}\sqrt{z}\sum_0^\infty \frac{\Gamma(\frac{1}{2}+n)}{(n!)^2} Z^n \left[2\psi(n) - \psi(n-\frac{1}{2}) - \log z\right].$$

Now apply (7) and get

$$\begin{aligned}\int_0^\infty e^{-\frac{\lambda^2}{z}} K_0(2\lambda) d\lambda &= \frac{1}{4}\sqrt{\pi z}\sum_{n=0}^\infty \frac{(\frac{1}{2};n)}{(n!)^2}\left[2\phi(n) - \psi(n-\frac{1}{2})\right] Z^n \\ &\quad - \frac{1}{4}\sqrt{z\pi}(2\gamma + \log z)\ {}_1F_1(\frac{1}{2};1;z)\end{aligned} \tag{19}$$

When $m = 2$, (18), (3), (7) and (8) give

$$\int_0^\infty e^{-\frac{\lambda^2}{z}} \lambda K_0(2\lambda) d\lambda = \frac{1}{4} z \sum_{n=0}^\infty \frac{\phi(n)}{n!} Z^n - \frac{1}{4} Z(\gamma + \log z) e^z. \tag{20}$$

Again, if $m = 1$, $\mu = 0$, $\alpha_1 = 1$, then (14) gives

$$\int_0^\infty \frac{K_0(2\lambda)}{(\lambda^2 + z)} d\lambda = \frac{1}{4z} E(\frac{1}{2}, \frac{1}{2}, 1 :: z).$$

From (3) with $\ell = 0$, $\alpha_1 = \alpha_2 = \frac{1}{2}$, $\alpha_3 = 1$, $p = 3$, $q = 0$, this becomes

$$\frac{1}{4\sqrt{z}}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{2}+n)\Gamma(\frac{1}{2}-n)}{(n!)^2}z^n\Big[2\psi(n)-\log z+\psi(-n-\frac{1}{2})-\psi(n-\frac{1}{2})\Big]$$
$$-\frac{1}{4}\sum_{n=0}^{\infty}\Big[\Gamma(-\frac{1}{2}-n)\Big]^2(-z)^n.$$

Now from (5), (6), (7), (8) and (9), this becomes.

$$\int_0^{\infty}\frac{K_0(2\lambda)}{(z+\lambda^2)}d\lambda = \frac{\pi}{2\sqrt{z}}\sum_{n=0}^{\infty}\frac{(-z)^n}{(n!)^2}\phi(n)-\frac{\pi}{4\sqrt{z}}(2\gamma+\log z)$$
$$\overset{J}{0}(2\sqrt{z})+\pi\,{}_1F_2(1;\frac{3}{2},\frac{3}{2};-z) \tag{21}$$

where $\overset{J}{n}(z)$ is the Bessel function of the first kind. Also, if $m = 2$, $\mu = 0$, $p = 1$, with $\alpha_1 = \frac{1}{2}$, $q = 0$; then (14) gives

$$\int_0^{\infty}\frac{\lambda K_0(2\lambda)}{\sqrt{z+\lambda^2}}d\lambda = \frac{1}{4\sqrt{\pi z}}E(1,1,\frac{1}{2}::z).$$

From (3) with $\ell = 0$, $\alpha = 1$, $p = 3$, $q = 0$, $\alpha_3 = \frac{1}{2}$; this becomes

$$\int_0^{\infty}\frac{\lambda K_0(2\lambda)}{\sqrt{z+\lambda^2}}d\lambda = \frac{\sqrt{z}}{2}(\gamma+\log z)\frac{\sin(2\sqrt{z})}{2\sqrt{z}} - \frac{\sqrt{z}}{2}$$
$$\cdot\sum_{n=0}^{\infty}\frac{\phi(n)-\psi(n+\frac{1}{2})}{n!(\frac{3}{2};n)}(-z)^n+\frac{\pi}{4}\cos(2\sqrt{z}). \tag{22}$$

Again, if m is any positive integer, then (13) with $k = 1$ becomes

$$\int_0^{\infty}\frac{e^{-t}}{z+t^m}dt = (2\pi)^{\frac{1}{2}-\frac{m}{2}}m^{-\frac{1}{2}}z^{-1}E(1,1,\frac{1}{m},\frac{1}{m},\frac{2}{m},\ldots,\frac{m-1}{m}::\frac{z}{m^m})$$

From (3) with $\ell = 0$, $\alpha = 1$, $p = 1+m$, $q = 0$, this becomes.

$$\int_0^\infty \frac{e^{-t}}{z+t^m}dt = (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\frac{1}{2}-m} \sum_{n=0}^{\infty} \frac{1}{(n!)}\Big[\Gamma(\frac{t}{m}-1-n)\Big]$$
$$\cdot(\frac{z}{m^m})^n\Big[\psi(n)-\log(\frac{z}{m^m})+\sum_{t=1}^{m-1}\psi(\frac{t}{m}-2-n)\Big]$$
$$+(2\pi)^{\frac{1}{2}-\frac{m}{2}} m^{-\frac{1}{2}} z^{-1}\sum_{t=1}^{m-1}(\frac{z}{m^m})^{\frac{t}{m}}\sum_{n=0}^{\infty}\frac{1}{n!}\Gamma(\frac{t}{m}+n)\Big[\Gamma(1-\frac{t}{m}-n)\Big]^2$$
$$\Big[\Pi_{s=1}^{m=1}{}'\Gamma(\frac{s-t}{m}-n)\Big]\Big(\frac{-z}{m^m}\Big)^n.$$

Here apply (17) with $(n+1)$ in place of n noting that

$$m\phi(mn+m) - \phi(n+1) = \phi(mn+m-1) - \phi(n),$$

then the last expression becomes

$$\int_0^\infty \frac{e^{-t}}{z+t^m}dt = \frac{(-1)^{m+1}}{m}\sum_{n=0}^{\infty}\frac{\{(-1)^{m+1}z\}^n}{(mn+m-1)!}\Big[m\phi(mn+m-1)-m\gamma-\log z)\Big]$$
$$-\frac{\pi}{mz}\sum_{t=1}^{m-1}\frac{\left(-z^{\frac{1}{m}}\right)^t}{\sin(\frac{\pi t}{m})}\sum_{n=0}^{\infty}\frac{\{(-1)^{m+1}z\}^n}{(mn+t-1)!} \tag{23}$$

Many particular cases can be derived from the main theorems (21), (22) and (23) by specializing the values of the parameters p, q and m. For example, if $m = 1$, (23) becomes.

$$\int_0^\infty \frac{e^{-t}}{z+t}dt = -(\gamma+\log z)e^z + \sum_{n=0}^{\infty}\frac{\phi(n)}{n!}Z^n \tag{24}$$

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