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INTEGRABILITY OF TRIGONOMETRIC SERIES

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Abstract. Generalization of the theorems of Taljakovskii [7] and Sing and Sharma [5] have been obtained.

1. A sequence $< a_n >$ of positive numbers is said to be *quasi-monotone* if $\Delta a_n \geq - \propto a_n/n$ for some positive $\propto$. It is obvious that every null monotonic decreasing sequence is quasi-monotone. A sequence $< a_n >$ is said to be *δ-quasi-monotone* if $a_n \to 0$, $a_n > 0$ ultimately and $\Delta a_n \geq -\delta_n$, where $< \delta_n >$ is a sequence of positive numbers. Clearly a null quasi monotone sequence is *δ-quasi-monotone* with $\delta_n = \propto a_n/n$.

We say that a sequence $< a_n >$ of numbers satisfies condition S or $a_n \in S$, if $a_n \to 0$ as $n \to \infty$ and there exists a sequence of numbers $< A_k >$ such that

(a) $A_k \downarrow 0$,

(b) $\Sigma_{k=1}^{\infty} A_k < \infty$, (1.1)

and

(c) $| \Delta a_k | \leq A_k$, *for all* k.

By replacing the condition (a) of (1.1) only by:

(a′) $< A_k >$ is quasi-monotone

(a″) $< A_k >$ is δ-quasi-monotone and $\Sigma k\delta_k < \infty$, we say that $< a_n > \in S(\propto)$ and $< a_n > \in S(\delta)$ respectively.

Thus, in view of the above definitions it is obvious that $S \subset S(\propto) \subset S(\delta)$. And all these three are the generalization of quasi-convex sequence. Our condition $S(\delta)$ is weaker than the conditions S and $S(\propto)$ of Sidon [4] and that of Sing and Sharma [5] respectively.

2. Let

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

and

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

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be the trigonometric series.

Integrability of the above series has been discussed by several authors, for example Young [8], Kolmogorov [3] and Sidon [4]. In 1973, Teljakoveskii [7] has proved the following Theorems by taking a set of weaker conditions of Sidon [4] than those of the earlier authors.

Theorem A. *Let the coefficient of the series $f(x)$ satisfy the condition S. Then the series is a Fourier series and the following relation holds*

$$\int_0^\pi |f(x)|\,dx \le C\sum_{n=0}^{\infty} A_n,$$

where C is an absolute constant.

Theorem B. *Let the coefficient of series $g(x)$ satisfy the condition S. Then the following relation holds for $p = 1, 2, \ldots$*

$$\int_{\pi/p+1}^{\pi} |g(x)|\,dx \le \sum_{n=1}^{p} \frac{|a_n|}{n} + O\Big(\sum_{n=1}^{\infty} A_n\Big)$$

In particular $g(x)$ is a Fourier series iff $\Sigma_{n=1}^{\infty} \frac{a_n}{n} < \infty$.

Very recently Sing and Sharma [5] have proved Theorem A and B for the class $S(\propto)$. In this paper we generalize Theorem A and B for the class $S(\delta)$, so as to get the above mentioned generalization of Taljakoveskii [7] and Sing and Sharma [5] as a special cases.

3. We prove the following theorem.

Theorem 1. *Let the coefficient of the series $f(x)$ satisfy the condition $S(\delta)$. Then the series is a Foureir series and the following relation holds*

$$\int_0^\pi \Big|\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \Big|\,dx \le C\sum_{n=0}^{\infty} A_n,$$

where C is an absolute constant

Theorem 2. *Let the coefficient of series $g(x)$ satisfy the condition $S(\delta)$. Then the series coverges to a function and the following relation holds for $p = 1, 2, 3\ldots$*

$$\int_{\pi/p+1}^{\pi} \Big|\sum_{n=1}^{\infty} a_n \sin nx \Big|\,dx \le \sum_{n=1}^{p} \frac{|a_n|}{n} + O\Big(\sum_{n=1}^{\infty} A_n\Big).$$

4. For the proof of the above theorem we require the following lemmas.

Lemma 1[2]. *If the sequence of numbers $<\propto>$ satisfies the condition $|\propto_i| \leq 1$, then*

$$\int_0^{\pi} | \sum_{i=0}^{k} \propto_i \frac{\sin(i+1/2)x}{2\sin x/2} | \, dx \; \leq \; C(k+1),$$

and

$$\int_{\pi/p+1}^{\pi} | \sum_{i=0}^{k} \propto_i \frac{\cos(i+1/2)x}{2\sin x/2} | \, dx \; \leq \; C(k+1),$$

Where C is a positive absolute constant.

Lemma 2 [1]. *If $< a_n >$ is δ-quasi-monotone with $\Sigma n^v \delta_n < \infty$ $v \neq 0$ then the convergence of $\Sigma n^{v-1} a_n$ implies that $n^v a_n \to 0$, $n \to \infty$.*

Lemma 3. *Let $< a_n >$ be a δ-quasi-monotone sequence with*

$$\sum_{n=1}^{\infty} n\delta_n \; < \; \infty. \textit{ If } \sum_{n=1}^{\infty} a_n \; < \; \infty, \textit{ then } \sum_{n=1}^{\infty}(n+1) \mid \Delta a_n \mid < \; \infty.$$

Proof. By partial summation we have

$$\sum_{k=1}^{n} a_k \; = \; \sum_{k=1}^{n-1}(k+1)\Delta a_k + (n+1)a_n - a_1.$$

Since $< a_n >$ is δ-quasi-monotone sequence and $\Sigma_{k=1}^{\infty} a_k < \infty$, we have $na_n = o(1)$, by Lemma 2. Therefore, by taking the limit we have,

$$\sum_{k=1}^{\infty} a_k \; = \; \sum_{k=1}^{\infty}(k+1)\Delta a_k - a_1.$$

From which it is clear that $\Sigma_{k=1}^{\infty}(k+1)\Delta a_k < \infty$.

Now,

$$\begin{aligned}
\sum_{k=1}^{\infty}(k+1) \mid \Delta a_k \mid \; &= \; \sum_{k=1}^{\infty}(k+1) \mid a_k - a_{k+1} + \delta_k - \delta_k \mid \\
&\leq \; \sum_{k=1}^{\infty}(k+1)(a_k - a_{k+1} + \delta_k) + \sum_{k=1}^{\infty} \delta_k(k+1) \\
&= \; \sum_{k=1}^{\infty}(k+1)\Delta a_k + 2\sum_{k=1}^{\infty} \delta_k(k+1) \\
&< \; \infty,
\end{aligned}$$

by virtue of the hypothesis.

5. Proof of Theorem 1. By virtue of hypothesis $\Delta a_n \geq -\delta_n$, we have

$$| \Delta a_n | \leq \Delta a_n + 2\delta_n.$$

Also the convergence of $\Sigma_{k=1}^{\infty} k\delta_k$ implies that $\Sigma_{k=1}^{\infty}\delta_k < \infty$. Therefore, using the condition that $a_n \to 0$, we have

$$\sum_{n=1}^{\infty} | \Delta a_n | \leq \sum_{n=1}^{\infty} \Delta a_n + 2\sum_{n=1}^{\infty} \delta_n < \infty.$$

Thus $\frac{a_0}{2} + \Sigma_{n=1}^{\infty} a_n \cos nx$ converges to $f(x)$ for all x except possibly $x = 0$.
By summation by parts, we have

$$\begin{aligned} f(x) &= \lim_{n\to\infty} \left[\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx\right] \\ &= \lim_{n\to\infty} \left[\frac{a_0}{2} + \sum_{k=1}^{n-1} D_k(x)\Delta a_k + a_n D_n(x) - \frac{a_0}{2}\right] \\ &= \lim_{n\to\infty} \left[\sum_{k=1}^{n-1} D_k(x)\Delta a_k + a_n D_n(x)\right] \\ &= \sum_{k=1}^{\infty} \Delta a_k D_k(x), \end{aligned}$$

by the fact that $\lim_{n\to\infty} a_n D_n(x) = 0$ if $x \neq 0$ where $D_n(x) = 1/2 + \cos x + \cos 2x + \ldots + \cos nx$.
Now applications of Abel's transformation and Lemma 1 and 2 yield,

$$\begin{aligned} &\int_0^{\pi} | \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx | \, dx \\ &= \int_0^{\pi} | \sum_{k=0}^{\infty} \Delta a_k D_k(x) | \, dx \\ &= \int_0^{\pi} | \sum_{k=0}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) | \, dx \\ &\leq \sum_{k=0}^{\infty} | \Delta A_k | \int_0^{\pi} | \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) | \, dx \\ &\leq C \sum_{k=0}^{\infty} (k+1) | \Delta A_k |, \end{aligned}$$

since $| \frac{\Delta a_i}{A_i} | = |\alpha_i| \leq 1$.
Then by Lemma 3.

$$\int_0^\pi | \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx | < \infty,$$

and satisfies the following inequality

$$\int_0^\pi | \frac{a_0}{2} + \sum_{k=1}^\infty a_k \cos kx | \leq C \sum_{k=0}^\infty (k+1) | \Delta A_k | \leq C \sum_{k=0}^\infty A_k.$$

This proves the theorem.

Proof of Theorem 2. Since $\Delta a_n \geq -\delta_n$, we have $| \Delta a_n | \leq a_n + 2\delta_n$. The convergence of the series $\Sigma_{k=1}^\infty k\delta_k < \infty$ implies that $\Sigma_{k=1}^\infty \delta_k < \infty$. Therefore, by using the condition that $a_n \to 0$, we have

$$\sum_{n=1}^\infty | \Delta a_n | \leq \sum_{n=1}^\infty \Delta a_n + 2 \sum_{n=1}^\infty \delta_n$$

Thus, $\Sigma_{k=1}^\infty a_n \sin nx$ converges to $g(x)$ for every x.
We suppose that $a_0 = 0$ and $A_0 = \max(| a_1 |, A_1)$, we see that $A_0 \leq \Sigma A_k$. Putting

$$\overline{D}_0(x) = \frac{-\cos x/2}{2 \sin x/2} \; for \; k \geq 1$$

$$\begin{aligned} \overline{D}_k &= \overline{D}_0(x) + \sin x + \sin 2x + \ldots + \sin kx \\ &= -\frac{\cos(k+1/2)x}{2 \sin x/2} \end{aligned}$$

Then

$$\begin{aligned} &\int_{\pi/p+1}^\pi | \sum_{k=1}^\infty a_k \sin kx | \, dx \\ = &\int_{\pi/p+1}^\pi | \sum_{k=0}^\infty \Delta a_k D_k^-(x) | \, dx \\ = &\sum_{j=1}^p \int_{\pi/j+1}^{\pi/j} | \sum_{k=0}^{j-1} \Delta a_k D_k^-(x) | \, dx \\ &+ O\Big(\sum_{j=1}^p \int_{\pi/j+1}^{\pi/j} | \sum_{k=j}^\infty \Delta a_k D_k^-(x) | \, dx \Big) \qquad I_1 + I_2, \text{ say} \end{aligned}$$

Application of Abel's transformation and lemma 2 yield:

$$\begin{aligned} \sum_{k=j}^\infty \Delta a_k D_k^-(x) &= \sum_{k=j}^\infty A_k \frac{\Delta a_k}{A_k} D_k^-(x) \\ &= \sum_{k=j}^\infty \Delta a_k \sum_{i=0}^k \frac{\Delta a_i}{A_i} D_i^-(x) - A_j \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i} D_i^-(x) \end{aligned}$$

Therefore

$$\begin{aligned} I_2 &= \sum_{j=1}^{p}\int_{\pi/j+1}^{\pi/j} \mid \sum_{k=j}^{\infty}\Delta a_k D_k^-(x) \mid dx \\ &\le \sum_{j=1}^{p}\Big[\sum_{k=j}^{\infty} \mid \Delta A_k \mid \int_{\pi/j+1}^{\pi} \mid \sum_{i=0}^{k}\frac{\Delta a_i}{A_i} D_i^-(x) \mid dx \\ &\quad + A_j \int_{\pi/j+1}^{\pi/j} \mid \sum_{i=0}^{j-1}\frac{\Delta a_i}{A_i} D_i^-(x) \mid dx\Big] \\ &\le \sum_{k=1}^{\infty} \mid \Delta A_k \mid \int_{\pi/j+1}^{\pi} \mid \sum_{i=0}^{k}\frac{\Delta a_i}{A_i} D_i^-(x) \mid dx \\ &\quad + C\sum_{j=1}^{\infty} A_j \int_{\pi/j+1}^{\pi/j} \frac{j\ dx}{2\sin x/2} \\ &\le C\sum_{k=1}^{\infty}(k+1) \mid \Delta A_k \mid + C\sum_{j=1}^{\infty} A_j,\ \textit{by Lemma 1}, \\ &\le C\sum_{k=1}^{\infty}(k+1) \mid \Delta A_k \mid, \end{aligned}$$

Hence by Lemma 3

$$I_2 \le C\sum_{k=1}^{\infty}(k+1) \mid \Delta A_k \mid < \infty.$$

For all $x \in [0,\pi]$ $k = 0,1,2,\cdots$

$$D_k^-(x) = -\frac{1}{x} + O(k+1).$$

we have

$$\begin{aligned} I_1 &= \sum_{j=1}^{p}\int_{\pi/j+1}^{\pi/j} \mid \sum_{k=0}^{j-1}\Delta a_k D_k^-(x) \mid dx \\ &= \sum_{j=1}^{p}\int_{\pi/j+1}^{\pi/j} \mid \sum_{k=0}^{j-1}\Delta a_k \mid \frac{dx}{x} + O\Big(\sum_{j=1}^{p}\int_{\pi/j+1}^{\pi/j}\sum_{k=0}^{j-1} \mid \Delta a_k \mid (k+1)dx\Big) \\ &= \sum_{j=1}^{p}\frac{\mid a_j \mid}{j} + O\Big(\sum_{j=1}^{p}\frac{\mid a_j \mid}{j^2}\Big) + O\Big(\sum_{j=1}^{p}\sum_{k=0}^{j-1}\frac{(k+1)\mid \Delta a_k \mid}{j^2}\Big) \end{aligned}$$

But

$$\sum_{j=1}^{p}\frac{\mid a_j \mid}{j^2} \le C\max_j \mid a_j \mid \le C\sum_{k=1}^{\infty} \mid \Delta a_k \mid \le C\sum_{k=1}^{\infty} A_k$$

and

$$\sum_{j=1}^{p}\sum_{k=0}^{j-1}\frac{(k+1)}{j^2}\mid \Delta a_k \mid \leq\ C\Sigma_{k=0}^{\infty}\mid \Delta a_k\mid \leq\ C\sum_{k=1}^{\infty}A_k.$$

Therefore,

$$I_1=\sum_{j=1}^{p}\frac{\mid a_j\mid}{j}+O\Big(\sum_{k=1}^{\infty}A_k\Big).$$

Hence

$$\int_{\pi/p+1}^{\pi}\mid\sum_{k=1}^{\infty}a_k\sin kx\mid dx\ =\ \sum_{k=1}^{p}\frac{\mid a_k\mid}{k}+O\Big(\sum_{k=1}^{\infty}A_k\Big)$$

This proves Theorem 2.

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