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NEW ANALYSIS OF WITTAKER FUNCTIONS

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Summary. Integrals involving products of two Whittaker functions and Bessel functions are evaluated in §§ 3,4. Also the integrals

$$\int_0^\infty t^{\rho-1}W_{k_1m}(t)W_{-k_1m}(t)W_{\mu,\nu}\left(\frac{2iz}{t}\right)W_{\mu,\nu}\left(\frac{-2iz}{t}\right)dt$$

and

$$\int_0^\infty t^{\rho-1}W_{k_1m}(t)-W_{-k_1m}(t)W_{\mu,\nu}(2zt)W_{-\mu,\nu}(2zt)dt$$

are evaluated in § 5 while in § 6 integrals involving the product of three Whittaker functions are established.

§ Introductory.

Here integrals involging products of Whittaker functions will be established in §§ 4 and 5. In § 3 integrals involving Whittaker function will be established. They will be derived from subsidiary formulae which will be stated and proved in § 2. These also will be deduced from the theory of MAC-ROBERTS' E-functions whose definitions and properties are given in [1] pp. 348-358). A brief account of the E-functions is given in ([2] pp 393). This is because that the Whittaker function is essentially ([1] p. 406) an E-function in virtue of the formula

$$E(\frac{1}{2}-k+m,\frac{1}{2}-k-m:z)=\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)z^{-k}e^{\frac{1}{2}z}W_{k_1m}(z) \quad (1)$$

where $W_{k_1m}(z)$ is the known WHITTAKER function.

The following formulae are required in the proofs: ([3], p. 432):

$$\int_0^\infty t^{p-1}W_{k_1m}W_{-k_1m}(t)E(p;\alpha_r:q;\rho_s:z/t^2)dt$$
$$=2^{\rho-1}\pi^{-\frac{1}{2}}E\begin{pmatrix}\alpha_1,\dots,\alpha_p,\frac{1}{2}+\frac{1}{2}\rho-m,\frac{1}{2}+\frac{1}{2}\rho+m,\frac{1}{2}+\rho,1+\frac{1}{2}\rho:z/4\\ \rho_1,\dots,\rho_q,1+\frac{1}{2}\rho-k,1+\frac{1}{2}\rho+k.\end{pmatrix} \quad (2)$$

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where

$$R(\rho \pm 2m) > -1,\ R(\rho) > -1,\ |\text{amp } z| < \frac{1}{2}(p-q+1)\pi,\ \textit{and}\ p \geq q+1.$$

([3], p. 430):

$$E(\alpha, \beta :: z)E(1-\alpha, 1-\beta :: z) = \Gamma(\alpha)\Gamma(\beta)\Gamma(1-\alpha)\Gamma(1-\beta)\pi^{-\frac{1}{2}}$$

$$\left(\frac{1}{2}z\right)^{1-\alpha-\beta} \times \frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E(\alpha, \beta, \frac{1}{2}\alpha+\frac{1}{2}\beta, \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2} : \alpha+\beta : e^{i\pi}\frac{z^2}{4}) \qquad (3)$$

([1], p. 395):

$$W_{k_1m}(iz)W_{k_1m}(-iz) = \frac{z^{2k}}{2^{2k}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)} \times E\left[\frac{1}{2}-k+m, \frac{1}{2}-k, 1-k : 1-2k : \frac{1}{4}z^2\right], \qquad (4)$$

([1], p. 406 ex. 28):

$$\frac{z^{\alpha+\beta-1}e^{-z}}{\Gamma(1-\alpha)\Gamma(1-\beta)}E(1-\alpha, 1-\beta :\ 'z) = \frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E(\alpha, \beta :: e^{i\pi}z), \qquad (5)$$

where the symbol $\Sigma_{i,-i}$ means that in the expression following it i is to be replaced by $-i$ and the two espression are to be added.
([4], p, 304):

$$z^{-\lambda}\sum_{i,-i}\frac{1}{i}E(P; \alpha_r+\lambda,\ 1:q;\ \rho_s+\lambda : e^{i\pi}z) = \sum_{i,-i}\frac{1}{i}E(P; \alpha_r,\ 1:q;\ \rho_s : e^{i\pi}z) \qquad (6)$$

([5], p. 759):

$$\sum_{i,-i}\frac{1}{i}(P; \alpha_r : q; \rho_s : ze^{i\pi}) = (2\pi)^{-\frac{1}{2}(n-1)(p-q-2)}n^{\Sigma\alpha_r-\Sigma\rho_s-\frac{1}{2}(p-q-1)}$$

$$\times \sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\Delta(\alpha_1, n), \ldots, \Delta(\alpha_p; n) : (z/n^{p-q-1})^n e^{i\pi} \\ \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \Delta(\rho_1; n), \ldots, \Delta(\rho_q; n)\end{matrix}\right] \qquad (7)$$

where $p > q+1$, n is any positive integer, $|\text{amp } z| < \frac{\pi}{2}(p-q-1)$ and the symbol $\Delta(n; \alpha)$ represents the set of parameters

$$\frac{\alpha}{n}, \frac{\alpha+1}{n}, \ldots, \frac{\alpha+n-1}{n}.$$

The integrals, involving WHITTAKER functions will be deduced from the following integrals involving E-function

([1], p. 406, ex. 30):

$$\int_0^\infty e^{-\lambda}\lambda^{\rho-1}E\Big(p;\alpha_r:q,\rho_s:\frac{z}{\lambda^n})d\lambda = (2\pi)^{\frac{1}{2}-\frac{1}{2}n}n^{\rho-\frac{1}{2}}\times E(P+n;\alpha_r:q;\rho_s:z/n^n), \quad (8)$$

where n is any positiove integer, $R(\rho)>0$,

$$= \alpha_{\rho+v+1} = (\rho+v)/n,\ (v=0,1,2,\dots,n-1).$$

([6], p. 77):

$$\int_0^\infty e^{-\lambda}\lambda^{\rho-1}E(P;\alpha_r:q;\rho_s z\lambda^n)d\lambda = \pi\operatorname{cosec}(\rho\pi)(2\pi)^{\frac{1}{2}n-\frac{1}{2}}n^{\rho-\frac{1}{2}}$$

$$E\Big(P;\alpha_r:1-\frac{\rho}{n},1-\frac{\rho+1}{n},\dots,1-\frac{\rho+n-1}{n},q;\rho_s;e^{\pm in\pi}n^n z\Big) \quad (9)$$

$$+2^{\frac{1}{2}-\frac{1}{2}n}\pi^{\frac{1}{2}+\frac{1}{2}n}\sum_{v=0}^{n-1}(-1)^{v+1}\frac{n^{-\frac{1}{2}-v}z^{-(\rho+v)/n}}{\sin\Big(\frac{\rho+v}{n}\Big)\pi\Pi_{s=1}^{v}\sin\frac{s\pi}{n}\Pi_{t=1}^{n-v-1}\sin\frac{t\pi}{n}}$$

$$\times E\left[\begin{matrix}\alpha_1+\frac{\rho+v}{n},\dots,\alpha_p+\frac{\rho+v}{n}:e^{\pm n\pi i}n^n z\\ 1+\frac{\rho+v}{n},1+\frac{1}{n},\dots,1+\frac{v}{n},1-\frac{1}{n},\dots,1-\frac{n-v-1}{n},q:\rho_s+\frac{\rho+v}{n}\end{matrix}\right]$$

where n is any positive integer, $|\text{amp } z|<\pi$ and

$$R(m\alpha_r+\rho)>0,\ r=1,2,\dots,p.$$

([7], p. 258):

$$4\int_0^\infty \lambda^{P-1}K_n(2\lambda)E\Big(P;\alpha_r:q;\rho_s:\frac{z}{\lambda^z}\Big)d\lambda = E(P+2;\alpha_r:q;\rho_s;z), \quad (10)$$

where

$$R(\rho\pm n)>0,\ \alpha_{p+1}=(\rho+n)/2\ \textit{and}\ \alpha_{p+2}=(\rho-n)/2.$$

([8], p. 8):

$$4i\pi\int_0^\infty \lambda^{\rho-1}J_n(2\lambda)E(P;\alpha_r:q;\rho_s:z/\lambda^2)d\lambda \quad (11)$$
$$= i^{\rho-n}E(P+2;\alpha_r:q;\rho_s:ze^{-i\pi})-i^{n-\rho}E(P+2;\alpha_r:q;\rho_s ze^{i\pi}),$$

where

$$R(\rho+n)>0,\ R\Big(\frac{3}{2}-\rho+2\alpha_r\Big)>0,\ r=1,2,\dots,P,\ \alpha_{p+1}=\frac{\rho+n}{2}\ \textit{and}\ \alpha_{p+z}=(\rho-u)/2.$$

([9], p.52):

$$\int_0^\infty \lambda^{\rho-1}K_n(\lambda)E(P:\alpha_r:q;\rho_s:z\lambda^2)d\lambda \tag{12}$$

$$= 2^{\rho-2}\frac{\pi^2}{\sin\left(\frac{\rho+n}{2}\right)\sin\left(\frac{\rho-n}{2}\pi\right)}E(\alpha_1,\ldots,\alpha\rho;1-\frac{\rho+n}{2},1-\frac{\rho-n}{2},\rho_1,\ldots,\rho_q:4z)$$

$$+\sum_{n,-n}\frac{\pi^2}{\sin\left(\frac{\rho+n}{2}\pi\right)\sin\pi n}\cdot z^{-\frac{\rho+n}{2}}E\left(\begin{matrix}\alpha_1+\frac{\rho+n}{2},\ldots,\alpha_p+\frac{\rho+n}{2}:4z\\ 1+\frac{\rho+n}{2},1+n,\rho_1+\frac{\rho+n}{2},\ldots,\rho_s+\frac{\rho+n}{2}\end{matrix}\right)$$

where

$$p\geq q+1,\ R(\rho\pm n+2\alpha_r)>0,\ r=1,2,\ldots,p,\ |\text{amp } z|<\pi.$$

([10], p. 82):

$$\int_0^\infty e^{-u}I_n(u)u^{\rho-1}E(p;\alpha_r:q;\rho_s:z/\mu^2)du$$

$$=\frac{\sin(\rho-n)\pi}{(2\sqrt{2})\cos(\rho\pi)}E\left[\begin{matrix}\alpha_1,\ldots,\alpha_p,\frac{\rho+n}{2},\frac{\rho-n}{2},\frac{\rho+n+1}{2}\frac{\rho-n+1}{2}:z\\ \frac{3}{4}+\frac{1}{2}\rho,\frac{1}{4}+\frac{1}{2}\rho,\rho_1,\ldots,\rho_q\end{matrix}\right]$$

$$-\frac{\cos(n\pi)}{(4\sqrt{2}\cdot\pi)\sin\left(\frac{\rho}{2}-\frac{1}{4}\right)\pi}\cdot z^{\frac{\rho}{2}-\frac{1}{4}}$$

$$E\left[\begin{matrix}\alpha_1+\frac{1}{4}-\frac{\rho}{2},\ldots,\alpha_p+\frac{1}{4}-\frac{\rho}{2},\frac{1}{4}+\frac{n}{2},\frac{3}{4}+\frac{n}{2},\frac{1}{4}-\frac{n}{2},\frac{3}{4}-\frac{n}{2}:z\\ \frac{5}{4}-\frac{\rho}{2},\frac{1}{2},\rho_1+\frac{1}{4}-\frac{\rho}{2},\ldots,\rho_q+\frac{1}{4}-\frac{\rho}{2}\end{matrix}\right]$$

$$-\frac{\cos(n\pi)}{(4\sqrt{2}\cdot\pi)\sin\left(\frac{3}{4}-\frac{\rho}{2}\right)\pi}z^{\frac{\rho}{2}-\frac{3}{4}}$$

$$\cdot E\left[\begin{matrix}\alpha_1+\frac{3}{4}-\frac{\rho}{2},\ldots,\alpha_p+\frac{3}{4}-\frac{\rho}{2},\frac{3}{4}+\frac{n}{2},\frac{5}{4}+\frac{n}{2},\frac{5}{4}-\frac{n}{2}:z\\ \frac{7}{4}-\frac{\rho}{2},\frac{3}{2},\rho_1+\frac{3}{4}-\frac{\rho}{2},\ldots,\rho_q+\frac{3}{4}-\frac{\rho}{2}\end{matrix}\right]$$

where

$$R(\rho+n)>0,\ R(2\alpha r-\rho+\frac{1}{2})>0\ r=1,2,3,\ldots,P,\ |\text{ amp } z\,|<\ \pi.$$

([10] p. :

$$\int_0^\infty e^{-u} I_n(u) u^{\rho-1}(P;\alpha_r : q;\rho_s : zu^2)du$$

$$= \frac{\pi}{\sqrt{2}\sin(\rho+n)\pi} E\left[\begin{matrix} \alpha_1,\ldots,\alpha_p,\frac{1}{4}-\frac{\rho}{2},\frac{3}{4}-\frac{\rho}{2} : z \\ 1-\frac{\rho+n}{2},\frac{1}{2}-\frac{\rho+n}{2},\rho_1,\ldots,\rho_q,1+\frac{n-\rho}{2},\frac{1}{2}+\frac{n-\rho}{2}\end{matrix}\right]$$

$$-\frac{\pi z^{-\left(\frac{\rho+n}{2}\right)}}{2\sqrt{2}\cdot\sin\left(\frac{\rho+n}{2}\right)} E\left[\begin{matrix} \alpha_1+\frac{\rho+n}{2},\ldots,\alpha_p+\frac{\rho+n}{2},\frac{1}{4}+\frac{n}{2},\frac{3}{4}+\frac{n}{2} : z \\ 1+\frac{\rho+n}{2},\frac{1}{2},\rho_1+\frac{\rho+n}{2},\ldots,\rho_q+\frac{\rho+n}{2},1+n,\frac{1}{2}+n\end{matrix}\right]$$

$$-\frac{\pi z^{-\left(\frac{\rho+n+1}{2}\right)}}{2\sqrt{2}\cdot\cos\left(\frac{\rho+n}{2}\pi\right)}$$

$$E\left[\begin{matrix} \alpha_1+\frac{\rho+n+1}{2},\ldots,\alpha_p+\frac{\rho+n+1}{2},\frac{3}{4}+\frac{n}{2},\frac{5}{4}+\frac{n}{2} : z \\ 1+\frac{\rho+n+1}{2},\frac{3}{2},\rho_1+\frac{\rho+n+1}{2},\ldots,\rho_q+\frac{\rho+n+1}{2},\frac{3}{2}+n,1+n\end{matrix}\right]$$

where

$$R(n+\rho+2\alpha_r)>0,\ r=1,2,\ldots,p,\ R\left(\frac{1}{2}-\rho\right)>0,\ |\operatorname{amp} z|<\pi.$$

§ 2. Subsidiary theorems.

The theorems to be proved are

$$z^l e^{-\frac{1}{2}z} W_{k_1m}(z) = \frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\frac{1}{2}+m+l,\frac{1}{2}-m+l,1 : ze^{i\pi} \\ 1-k+l\end{matrix}\right] \tag{15}$$

$$z^l e^{-\frac{1}{2}z} W_{k_1m}(z) = (2\pi)^{-\frac{1}{2}(n-1)} n^{k+l+\frac{1}{2}} \tag{16}$$

$$\frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\Delta(n;\frac{1}{2}+m+l),\Delta(n;\frac{1}{2}-m+l) : \left(\frac{z}{n}\right)^n e^{i\pi} \\ \Delta(n;1-k+l)\end{matrix}\right]$$

where n is any positive integer an the symbol Δ has the same meaning as in (7).

$$\left(\frac{z}{2}\right)^l W_{k_1m}(z)W_{-k_1m}(z) = \pi^{-\frac{1}{2}}(2\pi)^{-n} n^{\frac{1}{2}+2l-2k} \tag{17}$$

$$\times \sum_{i,-i} \frac{1}{i} E \left[\begin{matrix} \Delta\left(n; \frac{1}{2} + \frac{1}{2}l\right), \Delta\left(n; 1 + \frac{1}{2}l\right), \Delta\left(n; \frac{1}{2}l + m + \frac{1}{2}\right) \\ \Delta\left(n; \frac{1}{2}l - m + \frac{1}{2}\right) : \left(\frac{z}{4n^2}\right)^n e^{i\pi} \\ \Delta\left(n; 1 + k + \frac{1}{2}l\right), \ \Delta(n; 1 + k - \frac{1}{2}l) \end{matrix} \right]$$

where n is any positive integer, and the symbols Δ and $\Sigma_{i,-i}$ have the previons meanings.

Proof of (16). In (5), take

$$\alpha = \frac{1}{2} + k + m, \ \beta = \frac{1}{2} + k - m,$$

apply (1) and get

$$\begin{aligned} z^k e^{-\frac{1}{2}z} W_{k_1 m}(z) &= \frac{1}{2\pi} \sum_{i,-i} \frac{1}{i} E\left(\frac{1}{2} + k + m, \frac{1}{2} + k - m :: ze^{i\pi}\right) \\ &= \frac{1}{2\pi} \sum_{i,-i} \frac{1}{i} E\left(\frac{1}{2} + k + m, \frac{1}{2} + k - m, 1 : 1 : ze^{i\pi}\right). \end{aligned}$$

Here multiply both rides by z^{l-k} and get

$$\begin{aligned} z^l e^{-\frac{1}{2}z} W_{k_1 m}(z) &= \frac{1}{2\pi} z^{l-k} \sum_{i,-i} \frac{1}{i} E\left(\frac{1}{2} + k + m, \frac{1}{2} + k - m, 1 : 1 : ze^{i\pi}\right) \\ &= \frac{1}{2\pi} \sum_{i,-i} \frac{1}{i} E\left(\frac{1}{2} + m + l, \frac{1}{2} - m + l, \ 1 : 1 - k + l : ze^{i\pi}\right) \ \textit{by } (6) \end{aligned}$$

This is formula (15). Formula (16) follows immediately from (15) by applying (7).

Proof of (17). To prove (17), take in (3)

$$\alpha = \frac{1}{2} + k + m, \ \beta = \frac{1}{2} + k - m,$$

apply (1) and get

$$\left(\frac{z}{2}\right)^{2k} W_{k_1 m}(z) W_{-k_1 m}(z) = \pi^{-\frac{1}{2}} \frac{1}{2\pi} \sum_{i,-i} \frac{1}{i} \tag{18}$$
$$E \left[\begin{matrix} k + \frac{1}{2}, k + 1, \frac{1}{2} + m + k, \frac{1}{2} - m + k : \frac{e^{i\pi} z^2}{4} \\ 1 + 2k \end{matrix} \right]$$

Here multiply by $\left(\frac{z^2}{4}\right)^{\frac{1}{2}l-k}$ and get

$$\begin{aligned}
&\left(\frac{z}{2}\right)^l W_{k_1m}(z)W_{-k_1m}(z)\\
&= \pi^{-\frac{1}{2}}\frac{(z/2)^{l-2k}}{2\pi}\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}k+\frac{1}{2}, k+1, \frac{1}{2}+m+k, \frac{1}{2}-m+k : e^{i\pi}\frac{z^2}{4}\\ 1+2k\end{matrix}\right]\\
&= \pi^{-\frac{1}{2}}(z/2)^{l-2k}\times\frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}k+\frac{1}{2}, k+1, \frac{1}{2}+m+k, \frac{1}{2}-m+k, 1 : e^{i\pi}\frac{z^2}{4}\\ 1, 1+2k\end{matrix}\right]\\
&= \pi^{-\frac{1}{2}}\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}l, 1+\frac{1}{2}l, 1, \frac{1}{2}l+m+\frac{1}{2}, \frac{1}{2}l-m+\frac{1}{2} : e^{i\pi}\frac{z^2}{4}\\ 1+k+\frac{1}{2}l, 1+k-\frac{1}{2}l\end{matrix}\right] \qquad by\ (6).
\end{aligned}$$

Thus we have

$$\left(\frac{z}{2}\right)^l W_{k_1m}(z)W_{-k_1m}(z) = \pi^{-\frac{1}{2}}\frac{1}{2\pi}\sum_{i,-i}\frac{1}{i} \tag{19}$$

$$E\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}l, 1+\frac{1}{2}l, 1, \frac{1}{2}l+m+\frac{1}{2}, \frac{1}{2}l-m+\frac{1}{2} : e^{i\pi}\frac{z^2}{4}\\ 1+k+\frac{1}{2}l, 1+k-\frac{1}{2}l\end{matrix}\right]$$

Again formula (17) follows from (19) by applying (7).

§ 3. Integrals involving one Whittaker functions:

From 8 (with $n=1$) and (1) one gets

$$\begin{aligned}
&\int_0^\infty \exp\left(-\lambda+\frac{z}{2\lambda}\right)\lambda^{\rho+k-1}W_{k_1m}\left(\frac{z}{\lambda}\right)d\lambda\\
&= \frac{z^k}{\Gamma\left(\frac{1}{2}-k+m\right)\Gamma\left(\frac{1}{2}-k-m\right)}E\left(\rho, \frac{1}{2}-k+m, \frac{1}{2}-k-m :: z\right),
\end{aligned} \tag{20}$$

where $R(\rho)>0$.

(15) in combination with (8) (with $n=1$) gives

$$\begin{aligned}
&\int_0^\infty \exp\left(-\lambda-\frac{z}{2\lambda}\right)\lambda^{\rho-l-1}W_{k_1m}\left(\frac{z}{\lambda}\right)dy\\
&= z^{-l}\times\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\frac{1}{2}+m+l, \frac{1}{2}-m+l, 1, \rho : ze^{i\pi}\\ 1-k+l\end{matrix}\right]
\end{aligned} \tag{21}$$

where $R(\rho) > 0$.

Here apply (7) to the right hand side and get

$$\int_0^\infty \exp\Big(-\lambda + \frac{z}{2\lambda}\Big)\lambda^{\rho+l-1} W_{k_1m}\Big(\frac{z}{\lambda}\Big)d\lambda \;=\; z^{-l}(2\pi)^{1-n} n^{l+\rho-k} \tag{22}$$

$$\times \frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\Delta(n;\frac{1}{2}+m+l), \Delta(n;\frac{1}{2}-m+l), \Delta(n;\rho) : \left(\frac{z}{n^2}\right)^n e^{i\pi}\\ \Delta(n;1-k+l)\end{matrix}\right]$$

where n is any positive integer, $R(\rho) > 0$.

When $l = k$, the last formula gives if $R(\rho) > 0$ and n is any positive integer

$$\int_0^\infty \exp\Big(-\lambda - \frac{z}{2\lambda}\Big)\lambda^{\rho-k-1} W_{k_1m}\Big(\frac{z}{\lambda}\Big)d\lambda \;=\; 2^{-k}(2\pi)^{1-n} n\rho \tag{23}$$

$$\frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\Delta(n;\frac{1}{2}+m+k), \Delta(n;\frac{1}{2}-m+k), \Delta(n,\rho) : \left(\frac{z}{n^2}\right)^n e^{i\pi}\\ \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}.\end{matrix}\right]$$

In (9) replace ρ by $l+k$, λ by $\frac{\lambda}{z}$, then replace z by $1/(1-z)$, take

$$\alpha \;=\; \frac{1}{2} - k + m, \qquad \beta \;=\; \frac{1}{2} - k - m,$$

apply (1), so getting

$$\int_0^\infty e^{-\lambda(\frac{1}{2}-z)}\lambda^{l-1} W_{k_1m}(\lambda)d\lambda \;=\; \frac{\pi}{\sin(l+k)\pi\Gamma\Big(\frac{1}{2}-k+m\Big)\Gamma\Big(\frac{1}{2}-k-m\Big)}$$

$$\Big\{\frac{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)}{\Gamma(1-l-k)}(1-z)^{-k-l}F\left[\begin{matrix}\frac{1}{2}-k+m, \frac{1}{2}-k-m; 1-z\\ 1-l-k\end{matrix}\right]$$

$$-\frac{\Gamma(\frac{1}{2}+m+l)\Gamma(\frac{1}{2}-m+l)}{\Gamma(1+l+k)}F\left[\begin{matrix}\frac{1}{2}+m+l, \frac{1}{2}-m+l; 1-z\\ 1+l+k\end{matrix}\right]\Big\} \tag{A}$$

In virtue of the formula (p. 352.):

$$E(P;\alpha_r : q; \rho_s : z) \;=\; \frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\ldots\Gamma(\rho_q)^p} F_q\Big(p;\alpha_r : q; \rho_s; -\frac{1}{z}\Big). \tag{24}$$

The expression between brachets { } in A can be simplified by the formula ([1], p. 349)

$$F(\alpha,\beta;\gamma;z) \;=\; \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z)$$

$$+\frac{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\ \beta,\gamma-\alpha-\beta+1;1-z).$$

Thus A gives GOLDSTEIN'S, formula ([1], p. 396) namely

$$\int_0^\infty e^{-\lambda(\frac{1}{2}-z)}\lambda^{l-1}W_{k_1m}(\lambda)d\lambda$$
$$= \frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}F\left(\begin{matrix} l+m+\frac{1}{2}, l-m+\frac{1}{2}; z \\ l-k+1 \end{matrix}\right), \tag{26}$$

where

$$R(z) < 1,\ R(l \pm m) > -\frac{1}{2}.$$

Also (9) (with $n = 1$) in combination with (15) gives GOLDSTEIN'S formula (26) again.

(1) in combination with (9) gives if $R(\rho \pm n) > 0$

$$\int_0^\infty e^{\frac{z^2}{2\lambda^2}}\lambda^{\rho+2k-1}K_n(2\lambda)W_{k_1m}\left(\frac{z^2}{\lambda^2}\right)d\lambda = \frac{z^{2k}}{4\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)} \tag{27}$$
$$E(\frac{1}{2}-k+m, \frac{1}{2}-k-m, \frac{1}{2}\rho+\frac{1}{2}n, \frac{1}{2}\rho-\frac{1}{2}n : z^2),$$

(15) in combination with (9) gives if $R(\rho \pm n) > 0$

$$\int_0^\infty e^{\frac{z^2}{2\lambda^2}}\lambda^{p-2l-1}K_n(2\lambda)W_{k_1m}\left(\frac{z^2}{\lambda^2}\right)d\lambda \tag{28}$$
$$= \frac{1}{8\pi}\sum_{i,-i}\frac{1}{i}E\left(\frac{1}{2}+m+l, \frac{1}{2}-m+l, 1, \frac{1}{2}\rho+\frac{1}{2}n, \frac{1}{2}\rho-\frac{1}{2}n : 1-k+l : e^{i\pi}z^2\right)$$

When $k = l$, (28) gives if $R(\rho \pm n) > 0$

$$\int_0^\infty e^{-\frac{z^2}{2\lambda^2}}\lambda^{\rho-2k-1}K_n(2\lambda)W_{k_1m}\left(\frac{z^2}{\lambda^2}\right)d\lambda \tag{29}$$
$$= \frac{z^{-2k}}{8\pi}\sum_{i,-i}\frac{1}{i}E(\frac{1}{2}+m+k, \frac{1}{2}-m+k, \frac{1}{2}\rho-\frac{1}{2}n, \frac{1}{2}\rho+\frac{1}{2}n : e^{i\pi}z^2).$$

(1) in combination with (10) gives

$$4i\pi\int_0^\infty \exp\left(\frac{z^2}{2\lambda^2}\right)\lambda^{\rho+2k-1}J_n(2\lambda)W_{k_1m}(z^2/\lambda^2)d\lambda = \tag{30}$$
$$\frac{z^{2k}}{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)}i^{\rho-n}E\left(\frac{1}{2}-k+m, \frac{1}{2}-k-m, \frac{1}{2}\rho+\frac{1}{2}n, \frac{1}{2}\rho-\frac{1}{2}n : z^2e^{-i\pi}\right)$$
$$-\frac{z^{2k}}{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)}i^{n-\rho}E\left(\frac{1}{2}-k+m, \frac{1}{2}-k-m, \frac{1}{2}\rho+\frac{1}{2}n, \frac{1}{2}\rho-\frac{1}{2}n : z^2e^{i\pi}\right),$$

where

$$R(\rho+n)>0,\ R(\frac{5}{2}-2k-\rho\pm 2m)>0,\ |\text{ amp } z\,|<\pi$$

(15) in combination with 10) gives

$$4i\pi\int_0^\infty \exp\Big(-\frac{z^2}{2\lambda^2}\Big)\lambda^{\rho-2\rho-1}J_n(2\lambda)W_{k_1m}(z^2/\lambda^2)d\lambda \tag{31}$$

$$= z^{-}\frac{-n}{2\pi}\sum_{i,-i}\frac{1}{i}E\Big(\frac{1}{2}+m+l,\frac{1}{2}-m+l,1,\frac{1}{2}\rho+\frac{1}{2}n,\frac{1}{2}\rho-\frac{1}{2}n:1-k+l:z^2e^{-i\pi}\Big)$$

$$= -z^{-2l}i^{n-\rho}\frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E\Big(\frac{1}{2}+m+l,\frac{1}{2}-m+l,1,\frac{1}{2}\rho+\frac{1}{2}n,\frac{1}{2}\rho-\frac{1}{2}n:1-k+l:z^2e^{i\pi}\Big)$$

where

$$R(\rho+n)>0,\ R(\frac{5}{2}+2l-\rho\pm 2m)>0.$$

When $k=l$, (31) gives if

$$R(\rho+n)>0,\ R(\frac{5}{2}+2k-\rho\pm 2m)>0,\ |\text{ amp } z\,|<\pi$$

$$(4i\pi)z^{2l}\int_0^\infty \exp\Big(-\frac{z^2}{\lambda^2}\Big)\lambda^{\rho-2k-1}J_n(2\lambda)W_{k_1m}(z^2/\lambda^2)d\lambda \tag{32}$$

$$=i^{\rho-n}\times\frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E\Big(\frac{1}{2}+m+l,\frac{1}{2}-m+l,\frac{1}{2}\rho+\frac{1}{2}n,\frac{1}{2}\rho-\frac{1}{2}n:z^2e^{-i\pi}\Big)$$

$$-\,i^{n-\rho}\times\frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E\Big(\frac{1}{2}+m+l,\frac{1}{2}-m+l,\frac{1}{2}\rho+\frac{1}{2}n,\frac{1}{2}\rho-\frac{1}{2}n:z^2e^{i\pi}\Big)$$

(1) in combination with (12) gives

$$\int_0^\infty \exp\Big(\frac{1}{2}z^2\lambda^2\Big)\lambda^{\rho-2k-1}K_n(2\lambda)W_{k_1m}(z^2\lambda^2)d\lambda \tag{33}$$

$$= \frac{\pi^2 2^{\rho-2}z^{2k}\Gamma\Big(\frac{\rho+n}{2}\Big)\Gamma\Big(\frac{\rho-n}{2}\Big)}{1}\,{}_2F_2\left(\begin{matrix}\frac{1}{2}-k+m,\frac{1}{2}-k-m;-1/4z^2\\ 1-\frac{\rho+n}{2},1-\frac{\rho-n}{2}\end{matrix}\right)$$

$$+\frac{z^{2k}}{\Gamma\Big(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)}\sum_{n,-n}\Gamma\Big(\frac{1}{2}-k+m+\frac{\rho+n}{2}\Big)$$

$$\times\,\Gamma\Big(\frac{1}{2}-k-m+\frac{\rho+n}{2}\Big)\Gamma(-n)\Gamma\Big(-\frac{\rho+n}{2}\Big)$$

$$\times\, z^{-(\rho+n)}\,{}_2F_2\left(\begin{matrix}\frac{1}{2}-k+m+\frac{\rho+n}{2},\frac{1}{2}-km+\frac{\rho+n}{2};-1/4z^2\\ 1+\frac{\rho+n}{2},1+n\end{matrix}\right)$$

where

$$R(\rho+n+1-2k\pm 2m)>0,\ |\operatorname{amp} z|<\pi.$$

(15) in combination with (12) gives

$$\int_0^\infty \exp\Big(-\frac{1}{2}z^2\lambda^2\Big)\lambda^{\rho+2l-1}W_{k_1m}(z^2\lambda^2)K_n(\lambda)d\lambda \tag{34}$$

$$=\pi\sum_{n,-n}\left[\frac{\Gamma(-n)\Gamma\Big(\frac{1}{2}+m+l+\frac{1}{2}\rho+\frac{1}{2}n\Big)\Gamma\Big(\frac{1}{2}-m+l+\frac{1}{2}\rho+\frac{1}{2}n\Big)}{\Gamma\Big(1-k+l+\frac{1}{2}\rho+\frac{1}{2}n\Big)z^{\rho+n}} \right.$$
$$\left. {}_2F_2\Big(\frac{1}{2}+m+l+\frac{1}{2}\rho+\frac{1}{2}n,\frac{1}{2}-m+l+\frac{1}{2}\rho+\frac{1}{2}n;1+n,1-k+l+\frac{1}{2}\rho+\frac{1}{2}n;-\frac{1}{4z^2}\Big)\right]$$

where

$$R(\rho\pm n\pm 2m+2\rho+1)>0,\ R(\rho\pm n+2)>0$$

and z is real and positive.

(1) in combination with (13) gives

$$\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)z^{-2k}\int_0^\infty \exp\Big(-\mu+\frac{z^2}{2\mu^2}\Big)\mu^{\rho+2k-1}I_n(\mu)W_{k_1m}\Big(\frac{z^2}{\mu^2}\Big)d\mu \tag{35}$$

$$=\frac{\sin(\rho-n)\pi}{(2\sqrt{2})\cos(\rho\pi)}E\left[\begin{matrix}\frac{1}{2}-k+m,\frac{1}{2}-k+m,\frac{\rho+n}{2},\frac{\rho-n}{2},\frac{\rho+n+1}{2},\frac{\rho-n+1}{2}:z^2\\ \frac{3}{4}+\frac{1}{2}\rho,\frac{1}{4}+\frac{1}{2}\rho\end{matrix}\right]$$

$$-\frac{\cos(n\pi)}{(4\sqrt{2}\cdot\pi)\sin\Big(\frac{\rho}{2}-\frac{1}{4}\Big)\pi}z^{\frac{\rho}{2}-\frac{1}{4}}$$

$$E\left[\begin{matrix}\frac{1}{2}-k+m+\frac{1}{4}-\frac{\rho}{2},\frac{1}{2}-k-m+\frac{1}{4}-\frac{\rho}{2},\frac{1}{4}+\frac{n}{2},\frac{3}{4}+\frac{n}{2},\frac{1}{4}-\frac{n}{2},\frac{3}{4}-\frac{n}{2}:z^2\\ \frac{5}{4}-\frac{\rho}{2},\frac{1}{2}\end{matrix}\right]$$

$$-\frac{\cos(n\pi)}{(4\sqrt{2}\cdot\pi)\sin\Big(\frac{3}{4}-\frac{\rho}{2}\Big)\pi}z^{\frac{\rho}{2}-\frac{3}{4}}$$

$$E\left[\begin{matrix}\frac{1}{2}-k+m+\frac{3}{4}-\frac{\rho}{2},\frac{1}{2}-k-m+\frac{3}{4}-\frac{\rho}{2},\frac{3}{4}+\frac{n}{2},\frac{5}{4}+\frac{n}{2},\frac{3}{4}-\frac{n}{2},\frac{5}{4}-\frac{n}{2}:z^2\\ \frac{7}{4}-\frac{\rho}{2},\frac{3}{2}\end{matrix}\right]$$

where

$$R(\rho+n)>0,\ R(1-2k\pm 2m-\rho)>-\frac{1}{2},\ |\operatorname{amp} z|<\pi,$$

(15) in combination with (13) gives

$$z^{2l}\int_0^\infty \exp\Big(-\mu-\frac{z^2}{2\mu^2}\Big)\mu^{\rho-2l-1}I_n(\mu)W_{k_1,m}\Big(\frac{z^2}{\mu^2}\Big)d\mu$$

$$=\frac{1}{2\pi}\cdot\frac{\sin(\rho-n)\pi}{2\sqrt{2}\cdot\cos(\rho\pi)}$$

$$\times\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\frac{1}{2}+m+l,\frac{1}{2}-m+l,1,\frac{1}{2}\rho+\frac{1}{2}n,\frac{1}{2}\rho+\frac{1}{2}n+\frac{1}{2},\\ \frac{1}{2}\rho-\frac{1}{2}n,\frac{1}{2}\rho-\frac{1}{2}n+\frac{1}{2}:e^{i\pi}z^2\\ \frac{3}{4}+\frac{1}{2}\rho,\frac{1}{4}+\frac{1}{2}\rho,1-k+l\end{matrix}\right]$$

$$-\frac{\cos(n\pi)}{2\pi(4\sqrt{2}\cdot\pi)\sin(\frac{3}{4}-\frac{\rho}{2})\pi}z^{\rho-\frac{1}{2}}$$

$$\times\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\frac{1}{2}+m+l+\frac{1}{4}-\frac{\rho}{2},\frac{1}{2}-m+l+\frac{1}{4}-\frac{\rho}{2},\frac{5}{4}+\frac{\rho}{2},\\ \frac{1}{4}+\frac{n}{2},\frac{3}{4}+\frac{n}{2},\frac{1}{4}-\frac{n}{2},\frac{3}{4}-\frac{n}{2}:z^2e^{i\pi}\\ \frac{5}{4}-\frac{\rho}{2},\frac{1}{2},1-k+l+\frac{1}{4}-\frac{\rho}{2}\end{matrix}\right]$$

$$-\frac{\cos(n\pi)}{2\pi(4\sqrt{2}\cdot\pi)\sin(\frac{3}{4}-\frac{\rho}{2})\pi}z^{\rho-\frac{3}{2}}$$

$$+\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\frac{1}{2}+m+l+\frac{3}{4}-\frac{\rho}{2},\frac{5}{4}-m+l-\frac{\rho}{2},\frac{7}{4}+\frac{\rho}{2},\\ \frac{3}{4}+\frac{n}{2},\frac{5}{4}+\frac{n}{2},\frac{3}{4}-\frac{n}{2},\frac{5}{4}-\frac{n}{2}:z^2e^{i\pi}\\ \frac{7}{4}-\frac{\rho}{2},\frac{3}{2},\frac{7}{4}-k+l-\frac{\rho}{2}\end{matrix}\right]$$

where

$$R(\rho+n)>0,\ R(1+2l\pm 2m-\rho)>-\frac{1}{2},\ R\Big(\frac{5}{2}-\rho\Big)>0,\ |\operatorname{amp} z|<\pi.$$

(1) in combination with (14) gives

$$\Gamma\Big(\frac{1}{2}-k+m\Big)\Gamma\Big(\frac{1}{2}-k-m\Big)z^{-2k}\int_0^\infty \exp(-\mu+\frac{1}{2}z^2\mu^2)\mu^{\rho-2k-1}I_n(\mu)W_{k_1m}(z^2\mu^2)d\mu$$

$$=\frac{\pi\Gamma\Big(\frac{1}{2}-k+m\Big)\Gamma\Big(\frac{1}{2}-k-m\Big)\Gamma\Big(\frac{1}{4}-\frac{\rho}{2}\Big)\Gamma\Big(\frac{3}{4}-\frac{\rho}{2}\Big)}{(\sqrt{2})\sin(\rho+n)\pi\Gamma\Big(1+\frac{\rho+n}{2}\Big)\Gamma\Big(\frac{1}{2}\Big)\Gamma(1+n)\Gamma\Big(\frac{1}{2}+n\Big)}\tag{37}$$

$$ {}_4F_4\left(\begin{matrix}\frac{1}{2}-k+m,\frac{1}{2}-k-m,\frac{1}{4}-\frac{\rho}{2},\frac{3}{4}-\frac{\rho}{2};-1/z^2\\ 1+\frac{\rho+n}{2},\frac{1}{2},1+n,\frac{1}{2}+n\end{matrix}\right) $$

$$ +\frac{\pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}-k+m+\frac{\rho+n}{2}\right)\Gamma\left(\frac{1}{2}-k-m+\frac{\rho+n}{2}\right)\Gamma\left(\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)\Gamma\left(-\frac{\rho+n}{2}\right)}{2\sqrt{2}\Gamma(1+n)\Gamma\left(\frac{1}{2}+n\right)z^{\rho+n}} $$

$$ \times\ {}_4F_4\left(\begin{matrix}\frac{1}{2}-k+m+\frac{\rho+n}{2},\frac{1}{2}-k-m+\frac{\rho+n}{2},\frac{1}{4}+\frac{n}{2},\frac{3}{4}+\frac{n}{2};-1/z^2\\ 1+\frac{\rho+n}{2},\frac{1}{2},1+n,\frac{1}{2}+n\end{matrix}\right) $$

$$ +\frac{\Gamma\left(1-k+m+\frac{\rho+n}{2}\right)\Gamma\left(1-k-m+\frac{\rho+n}{2}\right),\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)\Gamma\left(\frac{5}{4}+\frac{n}{2}\right)\Gamma\left(-\frac{\rho+n+1}{2}\right)}{2\sqrt{2}\Gamma(3/2)\Gamma\left(\frac{3}{2}+n\right)\Gamma(1+n)z^{\rho+n+1}} $$

$$ \times\ {}_4F_4\left(\begin{matrix}1-k+m+\frac{\rho+n}{2},1-k-m+\frac{\rho+n}{2},\frac{3}{4}+\frac{n}{2},\frac{5}{4}+\frac{n}{2};-1/z^2\\ 1+\frac{\rho+n+1}{2},\frac{3}{2},\frac{3}{2}+n,1+n\end{matrix}\right) $$

where

$$ R(n+\rho+1-2k\pm 2m)>0,\ |\operatorname{amp} z|<\pi,\ R\left(\frac{1}{2}-\rho\right)>0. $$

(15) in combintation with (14) gives

$$ z^{2l}\int_0^\infty \exp\left(-\mu-\frac{1}{2}z^2\mu^2\right)\mu^{\rho+2l-1}I_n(\mu)W_{k_1m}(z^2\mu^2)d\mu \tag{38} $$

$$ =\frac{1}{2\sqrt{2}}\frac{\Gamma\left(\frac{1}{2}+m+l+\frac{\rho+n}{2}\right)\Gamma\left(\frac{1}{2}-m+l+\frac{\rho+n}{2}\right)\Gamma\left(\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-k+l+\frac{\rho+n}{2}\right)\Gamma(1+n)\Gamma\left(\frac{1}{2}+n\right)}z^{-(\rho+n)} $$

$$ \times\ {}_4F_4\left(\begin{matrix}\frac{1}{2}+m+l+\frac{\rho+n}{2},\frac{1}{2}-m+l+\frac{\rho+n}{2},\frac{1}{4}+\frac{n}{2},\frac{3}{4}+\frac{n}{2};1/z^2\\ \frac{1}{2},1-k+l+\frac{\rho+n}{2},1+n,\frac{1}{2}+n\end{matrix}\right) $$

$$ +\frac{1}{2\sqrt{2}}\frac{\Gamma\left(1+m+l+\frac{\rho+n}{2}\right)\Gamma\left(1-m+l+\frac{\rho+n}{2}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)\Gamma\left(\frac{5}{4}+\frac{n}{2}\right)z^{-(\rho+n+1)}}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(1-k+\rho+\frac{\rho+n+1}{2}\right)\Gamma\left(\frac{3}{2}+n\right)\Gamma(1+n)} $$

$$ \times\ {}_4F_4\left(\begin{matrix}1+m+l+\frac{\rho+n}{2},1-m+l+\frac{\rho+n}{2},\frac{3}{4}+\frac{n}{2},\frac{5}{4}+\frac{n}{2};1/z^2\\ \frac{3}{2},\frac{3}{2}-k+\rho+\frac{\rho+n}{2},\frac{3}{2}+n,1+n\end{matrix}\right) $$

where

$$R(n+p+1\pm 2m+2l)>0,\ R\Big(\frac{1}{2}-p\Big)>0,\ and\ \mid \text{amp } z \mid<\pi.$$

It may be noted that the formula (22), (29), (31), (32), (36) and (38) can be generalized by applying (7).

§ 4. Integerals involving the product of two Whittaker functions:

(4) in combination with (8) ($n=2$) gives if $R(\rho)>0$

$$\int_0^\infty e^{-\lambda}\lambda^{\rho+2k-1}W_{k_1m}\Big(2i\frac{z}{\lambda}\Big)W_{k_1m}\Big(-2i\frac{z}{\lambda}\Big)d\lambda = \frac{2^{\rho-1}z^{2k}}{\pi\Gamma\Big(\frac{1}{2}-k+m\Big)\Gamma\Big(\frac{1}{2}-k-m\Big)}$$
$$\times E\Big[\frac{1}{2}-k+m,\frac{1}{2}-k-m,\frac{1}{2}-k,1-k,\frac{1}{2}\rho,\frac{1}{2}\rho+\frac{1}{2}:1-2k:z^2/4\Big] \tag{39}$$

(4) in combination with (9) (with $n=2$) gives

$$\int_0^\infty e^{-\lambda}\lambda^{\rho-2k-1}W_{k_1m}(2iz\lambda)W_{k_1,m}(-2iz\lambda)d\lambda \tag{40}$$

$$=\frac{2\rho\pi^{\frac{3}{2}}\operatorname{cosec}(\rho\pi)\cdot z^{2k}}{2^{-2k}\Gamma\Big(1-\frac{\rho}{2}\Big)\Gamma\Big(\frac{1}{2}-\frac{\rho}{2}\Big)}\ {}_4F_3\left(\begin{matrix}\frac{1}{2}-k+m,\frac{1}{2}-k-m,\frac{1}{2}-k,1-k;-\frac{1}{4z^2}\\ 1-\frac{1}{2}\rho,\frac{1}{2}-\frac{1}{2}\rho,1-2k\end{matrix}\right)$$
$$-\frac{\pi^{\frac{1}{2}}z^{2k-\rho}\Gamma\Big(\frac{1}{2}-k+m+\frac{\rho}{2}\Big)\Gamma\Big(\frac{1}{2}-k-m+\frac{\rho}{2}\Big)\Gamma\Big(1-k+\frac{\rho}{2}\Big)\Gamma\Big(\frac{1}{2}-k+\frac{\rho}{2}\Big)}{2\Gamma\Big(\frac{1}{2}-k+m\Big)\Gamma\Big(\frac{1}{2}-k-m\Big)\Gamma\Big(1+\frac{\rho}{2}\Big)\Gamma\Big(1-2k+\frac{\rho}{2}\Big)\sin\Big(\frac{\rho}{2}\Big)\pi}$$
$$\times\ {}_4F_3\Big(\frac{1}{2}-k+m+\frac{\rho}{2},\frac{1}{2}-k-m+\frac{\rho}{2},1-k+\frac{\rho}{2},\frac{1}{2}-k+\frac{\rho}{2};\frac{1}{2},1+\frac{\rho}{2},1-2k+\frac{\rho}{2};-\frac{1}{4z^2}\Big)$$
$$+\frac{2^{-2}\pi z^{2k-\rho-1}\Gamma\Big(\frac{1}{2}-k+m+\frac{\rho}{2}+\frac{1}{2}\Big)\Gamma\Big(1-k-m+\frac{\rho}{2}\Big)\Gamma\Big(\frac{3}{2}-k+\frac{\rho}{2}\Big)\Gamma\Big(1-k+\frac{\rho}{2}\Big)}{\sin\Big(\frac{\rho+1}{2}\Big)\pi\Gamma\Big(\frac{1}{2}-k+m\Big)\Gamma\Big(\frac{1}{2}-k-m\Big)\Gamma\Big(\frac{3}{2}+\frac{\rho}{2}\Big)\Gamma\Big(\frac{3}{2}\Big)\Gamma\Big(\frac{3}{2}-2k+\frac{\rho}{2}\Big)}$$
$$\times\ {}_4F_3\Big(1-k+m+\frac{\rho}{2},1-k-m+\frac{\rho}{2},\frac{3}{2}-k+\frac{\rho}{2},1-k+\frac{\rho}{2};\frac{3}{2},\frac{3}{2}+\frac{\rho}{2},\frac{3}{2}-2k+\frac{\rho}{2};-\frac{1}{4z^2}\Big),$$

where

$$R(\rho+1-2k\pm 2m)>0,\ \mid \text{amp } z\mid<\pi.$$

(4) in combination with (10) gives if $R(\rho\pm n)>0$

$$4\int_0^\infty \lambda^{\rho-2k-1}K_n(2\lambda)W_{k_1m}(2iz/\lambda)W_{k_1m}(-2iz/\lambda)d\lambda \tag{41}$$

$$= \frac{z^{2k}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-k+m\right)\Gamma\left(\frac{1}{2}-k-m\right)}$$

$$E\left[\begin{matrix}\frac{1}{2}-k+m, \frac{1}{2}-k-m, \frac{1}{2}-k, 1-k, \frac{\rho+n}{2}, \frac{\rho-n}{2} : z^2 \\ 1-2k\end{matrix}\right]$$

(4) in combination with (11) gives

$$4i\pi \int_0^\infty \lambda^{\rho+2k-1} J_n(2\lambda) W_{k_1 m}(2iz/\lambda) W_{k_1 m}(-2iz/\lambda)\,d\lambda \tag{42}$$

$$= \frac{z^{2k}\cdot i^{n-\rho}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-k+m\right)\Gamma\left(\frac{1}{2}-k-m\right)}$$

$$E\left[\begin{matrix}\frac{1}{2}-k+m, \frac{1}{2}-k-m, \frac{1}{2}-k, 1-k, \frac{\rho+n}{2}, \frac{\rho-n}{2} : z^2 e^{i\pi} \\ 1-2k\end{matrix}\right]$$

$$- \frac{z^{2k} i^{n-\rho}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-k+m\right)\Gamma\left(\frac{1}{2}-k-m\right)}$$

$$E\left[\begin{matrix}\frac{1}{2}-k+m, \frac{1}{2}-k-m, \frac{1}{2}-k, 1-k, \frac{\rho+n}{2}, \frac{\rho-n}{2} : z^2 e^{-i\pi} \\ 1-2k\end{matrix}\right]$$

where

$$R(\rho+n)>0,\ R\left(\frac{3}{2}-\rho+1-2k\pm 2m\right)>0\ R\left(\frac{5}{2}-\rho-2k\right)>0.$$

(4) in combination with (12) gives

$$z^{-2k}\sqrt{\pi}\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)\int_0^\infty \lambda^{\rho-2k-1} K_n(\lambda) W_{k_1 m}(2iz\lambda) W_{k_1 m}(-2iz/\lambda)\,d\lambda$$

$$= \frac{2^{\rho-2}\Gamma\left(\frac{\rho+n}{2}\right)\Gamma\left(\frac{\rho-n}{2}\right)\Gamma\left(\frac{1}{2}-k+m\right)\Gamma\left(\frac{1}{2}-k-m\right)\Gamma\left(\frac{1}{2}\right)}{2^{-2k}} \tag{44}$$

$${}_4F_3\left(\begin{matrix}\frac{1}{2}-k+m, \frac{1}{2}-k-m, \frac{1}{2}-k, 1-k; -\frac{1}{4z^2} \\ 1-\frac{\rho+n}{2}, 1-\frac{\rho-n}{2}, 1-2k\end{matrix}\right)$$

$$+ \sum_{n,-n} \frac{\Gamma(-n)\Gamma\left(\frac{1}{2}-k+m+\frac{\rho+n}{2}\right)\Gamma\left(\frac{1}{2}-k-m+\frac{\rho+n}{2}\right)\Gamma\left(\frac{1}{2}-k+\frac{\rho+n}{2}\right)\Gamma\left(1-k+\frac{\rho+n}{2}\right)}{\Gamma\left(1-2k+\frac{\rho+n}{2}\right) z^{\rho+n}}$$

$$
{}_4F_3\left(\begin{matrix}\frac{1}{2}-k+m+\frac{\rho+n}{2},\frac{1}{2}-k-m+\frac{\rho+n}{2},\frac{1}{2}-k+\frac{\rho+n}{2},1-k+\frac{\rho+n}{2};-\frac{1}{4z^2}\\ 1+\frac{\rho+n}{2},1+n,1-2k+\frac{\rho+n}{2}\end{matrix}\right)
$$

where

$$R(\rho\pm n+1-2k+2m)>0,\ R(\rho\pm n+1-2k)>0,\ |\operatorname{amp} z|<\pi.$$

(4) in combination with (13) gives

$$
\int_0^\infty e^{-\mu}\mu^{\rho+2k-1}I_n(\mu)W_{k_1m}\left(\frac{2iz}{\mu}\right)W_{k_1m}\left(-\frac{2iz}{\mu}\right)d\mu \tag{45}
$$

$$
=\frac{z^{2k}\sin(\rho-n)\pi}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-k+m\right)\Gamma\left(\frac{1}{2}-k-m\right)\cdot\left(2\sqrt{2}\right)\cos(\rho\pi)}
$$

$$
\times E\left[\begin{matrix}\frac{1}{2}-k+m,\frac{1}{2}-k-m,\frac{1}{2}-k,1-k,\frac{\rho+n}{2},\frac{\rho-n}{2},\frac{\rho+n+1}{2}\\ \frac{\rho-n+1}{2}:z^2\\ \frac{3}{4}+\frac{1}{2}\rho,\frac{1}{4}+\frac{1}{2}\rho,1-2k\end{matrix}\right]
$$

$$
-\frac{z^{2k}\cos(n\pi)z^{\frac{\rho}{2}-\frac{1}{4}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-k+m\right)\Gamma\left(\frac{1}{2}-k-m\right)(4\sqrt{2}\pi)\sin(\frac{\rho}{2}-\frac{1}{4})\pi}
$$

$$
\times E\left[\begin{matrix}\frac{3}{4}-k+m-\frac{\rho}{2},\frac{3}{4}-k-m-\frac{\rho}{2},\frac{3}{4}-k-\frac{\rho}{2},\frac{5}{4}-k-\frac{\rho}{2},\\ \frac{1}{4}+\frac{n}{2},\frac{3}{4}+\frac{n}{2},\frac{1}{4}-\frac{n}{2},\frac{3}{4}-\frac{n}{2}:z^2\\ \frac{5}{4}-\frac{\rho}{2},\frac{1}{2},1-2k+\frac{1}{4}-\frac{\rho}{2}\end{matrix}\right]
$$

$$
-\frac{z^{2k}\cos(n\pi)z^{\frac{\rho}{2}-\frac{3}{4}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-k+m\right)\Gamma\left(\frac{1}{2}-k-m\right)(4\sqrt{2}\pi)\sin(\frac{\rho}{2}-\frac{1}{4})\pi}
$$

$$
\times E\left[\begin{matrix}\frac{5}{4}-k+m+\frac{\rho}{2},\frac{5}{4}-k-m-\frac{\rho}{2},\frac{5}{4}-k-\frac{\rho}{2},\frac{7}{4}-k-\frac{\rho}{2},\\ \frac{3}{4}+\frac{n}{2},\frac{5}{4}+\frac{n}{2},\frac{3}{4}-\frac{n}{2},\frac{5}{4}-\frac{n}{2}:z^2\\ \frac{7}{4}-\frac{\rho}{2},\frac{3}{2},\frac{7}{4}-2k-\frac{\rho}{2}\end{matrix}\right]
$$

where

$$R(\rho+n)>0,\ R\Big(1-2k\pm 2m-\rho+\frac{1}{2}\Big)>0,\ R\Big(\frac{3}{2}-2k\Big)>0,\ |\operatorname{amp} z|<\pi$$

(4) in combination with (14) gives

$$\int_0^\infty e^{-\mu}\mu^{\rho-2k-1}I_n(\mu)W_{k_1m}(-2iz\mu)W_{k_1m}(-2iz\mu)d\mu$$

$$=\frac{\sqrt{\pi}z^{2k}\cdot\sqrt{\pi}\Gamma\Big(\frac{1}{4}-\frac{\rho}{2}\Big)\Gamma\Big(\frac{3}{4}-\frac{\rho}{2}\Big)}{\sqrt{2}\sin(\rho+n)\pi 2^{-2k}\Gamma\Big(1-\frac{\rho+n}{2}\Big)\Gamma\Big(\frac{1}{2}-\frac{\rho+n}{2}\Big)\Gamma\Big(1+\frac{n-\rho}{2}\Big)\Gamma\Big(\frac{1}{2}+\frac{n-\rho}{2}\Big)}$$

$$\times\frac{1}{\Gamma(1-k)}\,{}_8F_5\left(\begin{matrix}\frac{1}{2}-k+m,\frac{1}{2}-k-m,\frac{1}{4}-\frac{\rho}{2},\frac{3}{4}-\frac{\rho}{2},1-k,\frac{1}{2}-k;-1/z^2\\ 1-\frac{\rho+n}{2},\frac{1}{2}-\frac{\rho+n}{2},1+\frac{n-\rho}{2},\frac{1}{2}+\frac{n-\rho}{2},1-2k\end{matrix}\right)$$

$$-\frac{z^{2k}\sqrt{\pi}\Gamma\Big(\frac{1}{2}-k+m+\frac{\rho+n}{2}\Big)\Gamma\Big(\frac{1}{2}-k-m+\frac{\rho+n}{2}\Big)\Gamma\Big(1-k+\frac{\rho+n}{2}\Big)}{z^{\rho+n}2\sqrt{2}\cdot\sin\Big(\frac{\rho+n}{2}\pi\Big)\Gamma\Big(\frac{1}{2}-k+m\Big)\Gamma\Big(\frac{1}{2}-k-m\Big)\Gamma\Big(1+\frac{\rho+n}{2}\Big)}$$

$$\times\frac{\Gamma\Big(\frac{1}{2}-k+\frac{\rho+n}{2}\Big)\Gamma\Big(\frac{1}{4}+\frac{n}{2}\Big)\Gamma\Big(\frac{3}{4}+\frac{n}{2}\Big)}{\Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(1-2k+\frac{\rho+2}{2}\Big)\Gamma(1+n)\Gamma\Big(\frac{1}{2}+n\Big)}$$

$$\times\ {}_8F_5\left(\begin{matrix}\frac{1}{2}-k+m+\frac{\rho+n}{2},\frac{1}{2}-k-m+\frac{\rho+n}{2},1-k+\frac{\rho+n}{2},\\ \frac{1}{2}-k+\frac{\rho+n}{2},\frac{1}{4}+\frac{n}{2},\frac{3}{4}+\frac{n}{2};-\frac{1}{z^2}\\ 1+\frac{\rho+n}{2},\frac{1}{2},1-2k+\frac{\rho+n}{2},1+n,\frac{1}{2}+n\end{matrix}\right)$$

$$-\frac{\sqrt{\pi}z^{2k}\Gamma\Big(1-k+m+\frac{\rho+n}{2}\Big)\Gamma\Big(1-k-m+\frac{\rho+n}{2}\Big)}{z^{\rho+n+1}(2\sqrt{2})\cdot\cos\Big(\frac{\rho+n}{2}\Big)\pi\Gamma\Big(\frac{1}{2}-k+m\Big)\Gamma\Big(\frac{1}{2}-k-m\Big)}$$

$$\times\frac{\Gamma\Big(1-k+\frac{\rho+n}{2}\Big)\Gamma\Big(\frac{3}{2}-k+\frac{\rho+n}{2}\Big)\Gamma\Big(\frac{3}{4}+\frac{n}{2}\Big)\Gamma\Big(\frac{5}{4}+\frac{n}{2}\Big)}{\Gamma\Big(\frac{3}{2}+\frac{\rho+n}{2}\Big)\Gamma\Big(\frac{3}{2}\Big)\Gamma\Big(\frac{3}{2}-2k+\frac{\rho+n}{2}\Big)\Gamma\Big(\frac{3}{2}+n\Big)\Gamma(1+n)}$$

$$\times\ {}_8F_5\left(\begin{matrix}1-k+m+\frac{\rho+n}{2},1-k-m+\frac{\rho+n}{2},1-k+\frac{\rho+n}{2},\\ \frac{3}{2}-k+\frac{\rho+n}{2},\frac{3}{4}+\frac{n}{2},\frac{5}{4}+\frac{n}{2};\frac{1}{z^2}\\ \frac{3}{2}+\frac{\rho+n}{2},\frac{3}{2},\frac{3}{2}-2k+\frac{\rho+n}{2},\frac{3}{2}+n,1+n\end{matrix}\right)$$

where

$$R(n+\rho+1-2k\pm 2m)>0,\ R(n+\rho+1-2k)>0,$$
$$R\Big(\frac{1}{2}-\rho\Big)>0,\ |\operatorname{amp} z|<\pi.$$

(19) in combination with 8 (with $n=2$) gives

$$\int_0^\infty e^{-\lambda}\lambda^{\rho-l-1}W_{k_1m}(2z/\lambda)W_{-k_1m}(2z/\lambda)d\lambda\ =\ z^{-l}2^{\rho-2}\pi^{-2}\sum_{i,-i}\frac{1}{i} \tag{47}$$
$$E\left[\begin{matrix}\frac{1}{2}+\frac{l}{2},1+\frac{l}{2},1,\frac{l}{2}+m+\frac{1}{2},\frac{l}{2}-m+\frac{1}{2},\frac{\rho}{2},\frac{\rho+1}{2}:e^{i\pi}\frac{z^2}{4}\\ 1+k+\frac{1}{2}l,1+k-\frac{1}{2}l\end{matrix}\right]$$

where $R(\rho)>0$.

Here apply (7) and get,

$$\int_0^\infty e^{-\lambda}\lambda^{\rho-l-1}W_{k_1m}\Big(\frac{2z}{\lambda}\Big)W_{-k_1m}\Big(\frac{2z}{\lambda}\Big)d\lambda\ =\ 2^{-2l}2^{2\rho+2l-2k-4}\pi^{-4}z^{-1} \tag{48}$$
$$\times\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\Delta\Big(n;\frac{1+l}{k}\Big),\Delta\Big(n;1+\frac{l}{2}\Big),\Delta(n;1),\Delta\Big(n;\frac{l+1}{2}+m\Big),\\ \Delta\Big(n;\frac{l-1}{2}+m\Big),\Delta\Big(n;\frac{\rho}{2}\Big),\Delta\Big(n;\frac{\rho+1}{2}\Big):\Big(\frac{z^2}{n^4}\Big)^n e^{i\pi}\\ \Delta\Big(n;1+k+\frac{1}{2}l\Big),\Delta\Big(n;1+k-\frac{1}{2}l\Big)\end{matrix}\right]$$

where n is any positive integer and $R(\rho)>0$.

(19) in combination with (9) gives

$$\int_0^\infty e^{-\lambda}\lambda^{\rho+l-1}W_{k_1m}(2z\lambda)W_{-k_1,m}(2z\lambda)d\lambda \tag{49}$$
$$=\frac{\Gamma\Big(\frac{1}{2}+\frac{1}{2}l+\frac{1}{2}\rho\Big)\Gamma\Big(1+\frac{1}{2}l+\frac{1}{2}\rho\Big)\Gamma\Big(\frac{1}{2}+m+\frac{1}{2}l+\frac{1}{2}\rho\Big)\Gamma\Big(\frac{1}{2}-m+\frac{1}{2}l+\frac{1}{2}\rho\Big)}{2z^{\rho+1}\Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(1+k+\frac{1}{2}l+\frac{1}{2}\rho\Big)\Gamma\Big(1+k-\frac{1}{2}l+\frac{1}{2}\rho\Big)}$$
$$\times\ {}_4F_3\left(\begin{matrix}\frac{1}{2}+\frac{1}{2}l+\frac{1}{2}\rho,1+\frac{1}{2}l+\rho,\frac{1}{2}+m+\frac{1}{2}l+\frac{1}{2}\rho,\frac{1}{2}-m+\frac{1}{2}l+\frac{1}{2}\rho;1/4z^2\\ \frac{1}{2},1+k+\frac{1}{2}l+\rho,1+k-\frac{1}{2}l+\frac{1}{2}\rho\end{matrix}\right)$$

$$-\frac{\Gamma\left(1+\frac{1}{2}l+\frac{1}{2}\rho\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}l+\frac{1}{2}\rho\right)\Gamma\left(1+\frac{1}{2}l+m+\frac{1}{2}\rho\right)\Gamma\left(1+\frac{1}{2}l-m+\frac{1}{2}\rho\right)}{2^2 z^{\rho+l+1}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+k+\frac{1}{2}l+\frac{1}{2}\rho\right)\Gamma\left(\frac{3}{2}+k-\frac{1}{2}l+\frac{1}{2}\rho\right)}$$

$$\times\ {}_4F_3\left(\begin{matrix}1+\frac{1}{2}l+\frac{1}{2}\rho,\frac{3}{2}+\frac{1}{2}l+\frac{1}{2}\rho,1+\frac{1}{2}l+m+\frac{1}{2}\rho,1-m+\frac{1}{2}l+\frac{1}{2}\rho;1/4z^2\\ \frac{3}{2},\frac{3}{2}+k+\frac{1}{2}l+\frac{1}{2}\rho,\frac{3}{2}+k-\frac{1}{2}l+\frac{1}{2}\rho\end{matrix}\right)$$

where

$$R(1+l+\rho)>0,\ R(2+\rho)>0,\ R(l\pm 2m+1)>0 \text{ and } z$$

is real and positive.

(19) in combination with (10) gives if $R(\rho\pm n)>0$.

$$4z^l\int_0^\infty \lambda^{\rho-l-1}W_{k_1m}\left(\frac{2z}{\lambda}\right)W_{-k_1m}\left(\frac{2\lambda}{\lambda}\right)K_n(2\lambda)d\lambda \tag{50}$$

$$=\pi^{-\frac{1}{2}}\times\frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\frac{1}{2}+\frac{\rho}{2},1+\frac{l}{2},1,\frac{l}{2}+m+\frac{1}{2},\frac{l}{2}-m+\frac{1}{2},\frac{\rho+n}{2},\frac{\rho-n}{2}e^{i\pi}z^2\\ 1+k+\frac{1}{2}l,1+k+l\end{matrix}\right]$$

(19) in combination with (11) gives

$$4i\pi z^l\int_0^\infty \lambda^{\rho-2l-1}J_n(2\lambda)W_{-k_1m}\left(\frac{2z}{\lambda}\right)W_{k_1m}\left(\frac{2z}{\lambda}\right)d\lambda \tag{51}$$

$$=\ \pi^{-3/2}\sin\left(\frac{\rho-n}{2}\right)\pi$$

$$\times E\left[\begin{matrix}\frac{1}{2}+\frac{l}{2},1+\frac{l}{2},1,\frac{l}{2}+m+\frac{1}{2},\frac{l}{2}-m+\frac{1}{2},\frac{\rho+n}{2},\frac{\rho-n}{2}:z^2\\ 1+k+\frac{1}{2}l,1+k-\frac{1}{2}l\end{matrix}\right]$$

$$+\frac{1}{2\pi^{3/2}}i^{\rho-n-1}$$

$$\times E\left[\begin{matrix}\frac{1}{2}+\frac{l}{2},1+\frac{1}{2}l,1,\frac{l}{2}+m+\frac{1}{2},\frac{l}{2}-m+\frac{1}{2},\frac{\rho+n}{2},\frac{\rho-n}{2}:z^2e^{-2i\pi}\\ 1+k+\frac{1}{2}l,1+k-\frac{1}{2}l\end{matrix}\right]$$

$$-\frac{1}{2\pi^{3/2}}i^{n-\rho-1}$$

$$\times E\left[\begin{matrix}\frac{1}{2}+\frac{l}{2},1+\frac{l}{2},1,\frac{l}{2}+m+\frac{1}{2},\frac{l}{2}-m+\frac{1}{2},\frac{\rho+n}{2},\frac{\rho-n}{2}:z^2e^{2i\pi}\\ 1+k+\frac{1}{2}l,1+k-\frac{1}{2}l\end{matrix}\right]$$

where

$$R(\rho+n)>0,\ R(\frac{3}{2}-\rho+1+l)>0,\ R(\frac{7}{2}-\rho)>0,$$

$$R(\frac{3}{2}-\rho+1\pm 2m+l)>0 \text{ and } z$$

is real and positive.

(19) in combination with (12) gives

$$\int_0^\infty \lambda^{\rho+l-1}K_n(\lambda)W_{k_1m}(2z\lambda)W_{-k_1m}(2z\lambda)d\lambda \ = \tag{52}$$

$$\sum_{n,-n}\frac{\Gamma\Big(\frac{l+\rho+n+1}{2}\Big)\Gamma\Big(1+\frac{\rho+l+n}{2}\Big)\Gamma\Big(m+\frac{l+\rho+n+1}{2}\Big)\Gamma\Big(-m+\frac{l+\rho+n+1}{2}\Big)\Gamma(-n)}{z^{\rho+n+l}\Gamma\Big(1+k+\frac{l+\rho+n}{2}\Big)\Gamma\Big(1+k+\frac{\rho+n-l}{2}\Big)\Gamma\Big(\frac{1}{2}\Big)}$$

$$\times\ {}_4F_3\left(\begin{matrix}\frac{1}{2}+\frac{\rho}{2}+\frac{\rho}{2}+\frac{n}{2},1+\frac{l}{2}+\frac{\rho}{2}+\frac{n}{2},m+\frac{l+\rho+n+1}{2},-m+\frac{l+\rho+n+1}{2};1/4z^2\\ 1+n,1+k+\frac{l+\rho+n}{2},1+k+\frac{\rho+n-l}{2}\end{matrix}\right)$$

where z is real and positive and $|4z^2|>1$.

(13) in combination with (19) gives

$$\int_0^\infty e^{-\mu}\mu^{\rho-l-1}I_n(\mu)W_{k_1m}\Big(\frac{2z}{\mu}\Big)W_{-k_1m}\Big(\frac{2z}{\mu}\Big)d\mu \ = \ \frac{\sin(\rho-n)\pi z^{-l}}{\sqrt{\pi}(2\sqrt{2})\cos(\rho\pi)(2\pi)} \tag{53}$$

$$\times\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\frac{l+1}{2},1+\frac{l}{2},1,\frac{l+1}{2}+m,\frac{l+1}{2}-m,\frac{\rho+n}{2},\frac{\rho+n+1}{2},\\ \frac{\rho-n}{2},\frac{\rho-n+1}{2}:e^{i\pi}z^2\\ \frac{3}{4}+\frac{l}{2},\frac{1}{4}+\frac{l}{2},1+k+\frac{1}{2}l,1+k-\frac{1}{2}l\end{matrix}\right]$$

$$-\ \frac{\pi^{-\frac{1}{2}}\cos(n\pi)z^{\rho-\frac{1}{2}-l}}{(\sqrt{2}\cdot\pi)\sin\Big(\frac{\rho}{2}-\frac{1}{4}\Big)\pi(2\pi)}$$

$$\times\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\frac{3}{4}+\frac{l}{2}-\frac{\rho}{2},\frac{5}{4}+\frac{l}{2}-\frac{\rho}{2},\frac{3}{4}+m+\frac{l-\rho}{2},\frac{3}{4}-m+\frac{l-\rho}{2},\\ \frac{1}{4}+\frac{n}{2},\frac{3}{4}+\frac{n}{2},\frac{1}{4}-\frac{n}{2},\frac{3}{4}-\frac{n}{2}:e^{i\pi}z^2\\ \frac{1}{2},\frac{5}{4}+k+\frac{l-\rho}{2},\frac{5}{4}+k-\frac{l+\rho}{2}\end{matrix}\right]$$

$$-\ \frac{\pi^{\frac{1}{2}}\cos(n\pi)z^{\rho-\frac{3}{2}-l}}{(4\sqrt{2}\cdot\pi)\sin\Big(\frac{3}{4}-\frac{\rho}{2}\Big)\pi(2\pi)}$$

$$\times \sum_{i,-i} \frac{1}{i} E \left[\begin{matrix} \frac{5}{4}+\frac{l-\rho}{2}, \frac{7}{4}+\frac{l-\rho}{2}, \frac{5}{4}+m+\frac{l-\rho}{2}, \frac{5}{4}-m+\frac{l-\rho}{2}, \\ \frac{3}{4}+\frac{n}{2}, \frac{5}{4}+\frac{n}{2}, \frac{3}{4}-\frac{n}{2}, \frac{5}{4}-\frac{n}{2} : e^{i\pi} z^2 \\ \frac{3}{2}, \frac{7}{4}+k+\frac{l-\rho}{2}, \frac{7}{4}+k-\frac{l+\rho}{2} \end{matrix} \right]$$

where

$$R(\rho+n)>0,\ R(\frac{3}{2}-\rho+l)>0,\ R(\frac{3}{2}\pm 2m+l-\rho)>0,\ R(\frac{5}{2}-\rho)>0$$

and $|\text{amp } z| < \pi$.
(14) in combination with (19) gives

$$\int_0^\infty e^{-\mu}\mu^{\rho+l-1} I_n(\mu) W_{k_1 m}(2z\lambda) W_{-k_1 m}(2z\lambda) d\lambda = \tag{54}$$

$$\frac{z^{-\rho-n-l}\Gamma\Big(\frac{l+\rho+n+1}{2}\Big)\Gamma\Big(\frac{\rho+l+n+2}{2}\Big)\Gamma\Big(m+\frac{l+\rho+n+1}{2}\Big)}{2\sqrt{2}\Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(1+k+\frac{\rho+l+n}{2}\Big)}$$

$$\times \frac{\Gamma\Big(-m+\frac{l+\rho+n+1}{2}\Big)\Gamma\Big(\frac{1}{4}+\frac{n}{2}\Big)\Gamma\Big(\frac{3}{4}+\frac{n}{2}\Big)}{\Gamma\Big(1+k+\frac{\rho+n+l}{2}\Big)\Gamma(1+n)\Gamma\Big(\frac{1}{2}+n\Big)}$$

$$\times {}_6F_5 \left(\begin{matrix} \frac{l+\rho+n+1}{2}, \frac{l+\rho+n+2}{2}, m+\frac{l+\rho+n+1}{2}, -m+\frac{l+\rho+n+1}{2}, \\ \frac{1}{4}+\frac{n}{2}, \frac{3}{4}+\frac{n}{2}, \frac{1}{4z^2} \\ \frac{1}{2}, 1+k+\frac{\rho+l+n}{2}, 1+k+\frac{\rho+n-l}{2}, 1+n, \frac{1}{2}+n \end{matrix} \right)$$

$$+ \frac{z^{-\rho-n-l}\Gamma\Big(1+\frac{l+\rho+n}{2}\Big)\Gamma\Big(\frac{3}{2}+\frac{l+\rho+n}{2}\Big)\Gamma\Big(1+m+\frac{l+\rho+n}{2}\Big)}{2\sqrt{2}\Gamma\Big(\frac{3}{2}\Big)\Gamma\Big(\frac{3}{2}+k+\frac{\rho+l+n}{2}\Big)}$$

$$\times \frac{\Gamma\Big(1-m+\frac{l+\rho+n}{2}\Big)\Gamma\Big(\frac{3}{4}+\frac{n}{2}\Big)\Gamma\Big(\frac{5}{4}+\frac{n}{2}\Big)}{\Gamma\Big(\frac{3}{2}+k+\frac{\rho+n-l}{2}\Big)\Gamma\Big(\frac{3}{2}+n\Big)\Gamma(1+n)}$$

$$\times {}_6F_5 \left(\begin{matrix} 1+\frac{l+\rho+n}{2}, \frac{3}{2}+\frac{l+\rho+n}{2}, 1+m+\frac{l+\rho+n}{2}, 1-m+\frac{\rho+l+n}{2}, \\ \frac{3}{4}+\frac{n}{2}, \frac{5}{4}+\frac{n}{2}; \frac{1}{4z^2} \\ \frac{3}{2}, \frac{3}{2}+k+\frac{l+\rho+n}{2}, \frac{3}{2}+k+\frac{\rho+n-l}{2}, \frac{3}{2}+n, 1+n \end{matrix} \right)$$

where

$$R(n+\rho+1+l)>0,\ R(n+\rho+1\pm 2m+l)>0,\ R(n+\rho+2)>0$$

and z is real and positive with $|4z^2|>1$.

It may be noted that the formulae (50), (51), (53) can be generalized by applying (7)

§ 5. Integrals involving the product of four Whittaker functions.

(4) in combination with (2) gives

$$\int_0^\infty t^{\rho-1}W_{k_1m}(t)W_{-k_1m}(t)W_{\mu,\nu}\Big(\frac{2iz}{t}\Big)W_{\mu,\nu}\Big(\frac{-2iz}{t}\Big)dt \tag{55}$$

$$= \frac{2^{\rho-2\mu-1}z^{2\mu}}{\pi\Gamma\Big(\frac{1}{2}-\mu+\nu\Big)\Gamma\Big(\frac{1}{2}-\mu-\nu\Big)}$$

$$E\left[\begin{matrix}\frac{1}{2}-\mu+\nu,\frac{1}{2}-\mu-\nu,\frac{1}{2}-\mu,1-\mu,\frac{1}{2}+\frac{1}{2}\rho-\mu+m,\\ \frac{1}{2}+\frac{1}{2}\rho-\mu-m,\frac{\rho+1}{2}-\mu,1+\frac{\rho}{2}-\mu:\frac{z^2}{4}\\ 1-2\mu,1+\frac{1}{2}\rho-\mu-k,1+\frac{1}{2}\rho-\mu+k\end{matrix}\right]$$

where

$$R(\rho-2\mu+2m)>-1,\ R(\rho)>-1 \text{ and } |\operatorname{amp} z|<\pi.$$

(19) in combination with (2) gives

$$\int_0^\infty t^{\rho-1}W_{k_1m}(t)W_{-k_1m}(t)W_{\mu,\nu}(2zt)W_{-\mu,\nu}(2zt)dt = \frac{2^{\rho+l-1}}{2\pi^2 z^l} \tag{56}$$

$$\times\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\frac{l+1}{2},1+\frac{l}{2},1,\frac{l+1}{2}+\nu,\frac{l+1}{2}-\nu,\frac{l+\rho+1}{2}+m,\\ \frac{l+\rho+1}{2}+m,\frac{l+\rho+1}{2},1+\frac{l+\rho+1}{2},1+\frac{l+\rho}{2}:e^{i\pi}\frac{z^2}{4}\\ 1+\mu+\frac{l}{2},1+\mu-\frac{l}{2},1+\frac{1}{2}l+\frac{1}{2}\rho-k,1+\frac{1}{2}l+\frac{1}{2}\rho+\frac{1}{2}k\end{matrix}\right].$$

where

$$R(\rho\pm 2m)>-1,\ R(\rho)>-1,\ |\operatorname{amp} z|<\pi,$$

It may be noted that the parameter l does not appear on the left side of (56). Also (56) can be generalized by (7).

§ 6. Integrals involving the product of an exponential function and 3 three Whittaker functions:

(1) in combination with (2) gives

$$\int_0^\infty t^{\rho-1} e^{\frac{z^2}{2t^2}} W_{k_1 m}(t) W_{-k_1 m}(t) W_{\mu,\nu}\left(\frac{z^2}{t^2}\right) dt$$

$$= \frac{2^{\rho-2\mu-1} z^{2k}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\frac{1}{2}-\mu-\nu\right)}$$

$$\times E\left[\begin{matrix} \frac{1}{2}-\mu+\nu, \frac{1}{2}-\mu-\nu, \frac{\rho+1}{2}-\mu+m, \frac{\rho+1}{2}-\mu-m, \\ 1+\frac{1}{2}\rho-\mu, \frac{1}{2}+\frac{1}{2}\rho-\mu : \frac{z^2}{4} \\ 1+\frac{1}{2}\rho-\mu+k, 1+\frac{1}{2}\rho-\mu-k \end{matrix}\right]$$

where

$$R(\rho \pm 2m) > 1,\ R(\rho) > -1,\ |\operatorname{amp} z| < \frac{\pi}{2}.$$

(15) in combination with (2) gives

$$\int_0^\infty t^{\rho-1} \exp\left(-\frac{z^2}{2t^2}\right) W_{k_1 m}(t) W_{-k_1 m}(t) W_{\mu,\nu}\left(\frac{z^2}{t^2}\right) dt = \frac{z^{-2l} 2^{\rho+2l-2}}{\sqrt{\pi}} \tag{60}$$

$$\frac{1}{2\pi} \sum_{i,-i} \frac{1}{i} E\left[\begin{matrix} \frac{1}{2}+\nu+l, \frac{1}{2}-\nu+l, 1, \frac{\rho+1}{2}+l+m, \frac{\rho+1}{2}+l-m, \\ \frac{\rho+1}{2}+l, 1+\frac{\rho}{2}+l : e^{i\pi}\frac{z^2}{4} \\ 1-\mu+l, 1+l+\frac{1}{2}\rho-k, 1+\frac{1}{2}\rho+l+k \end{matrix}\right]$$

where

$$R(\rho \pm 2m) > -1,\ R(\rho) > 1,\ |\operatorname{amp} z| < \frac{\pi}{2}.$$

It may be noted that l does not appear on the left side of (60). Also (60) can be generalized by (7).

When $l = \mu$, the formula becomes

$$\int_0^\infty \exp\left(\frac{-z^2}{2t^2}\right) t^{\rho-1} W_{k_1 m}(t) W_{-k_1 m}(t) W_{\mu,\nu}\left(\frac{z^2}{t^2}\right) dt = \frac{z^{-2\mu}}{\sqrt{\pi}} \tag{61}$$

$$\frac{2^{2\rho+2\mu-1}}{\sqrt{\pi}} \sum_{i,-i} \frac{1}{i} E\left[\begin{matrix} \frac{1}{2}+\nu+\mu, \frac{1}{2}-\nu+\mu, \frac{\rho+1}{2}+\mu+m, \frac{\rho+1}{2}+\nu-m, \\ \frac{\rho+1}{2}+\mu, 1+\frac{\rho}{2}+\mu : e^{i\pi}\frac{z^2}{4} \\ 1+\mu+\frac{1}{2}\rho-k, 1+\mu+k+\frac{1}{2}\rho \end{matrix}\right]$$

wher

$$R(\rho \pm 2m) > 1, \; R(\rho) > 1, \; |\, \text{amp } z \,| < \frac{\pi}{2}.$$

Also (61) can generalized by (7).

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