TAMKANG JOURNAL OF MATHEMATICS
Volume 21, Number 4, Winter 1990

ALMOST CONVERGENCE AND ALMOST SUMMABILITY

EKREM SAVAS

Summary. The purpose of this paper is to introduce and discuss the spaces of almost summable sequences. Also some matrix transformations have been characterized.

1. Introduction

Let s be the set of all sequences with real or complex terms and let $\ell_{\infty,c}$ and c_0 denote, respectively, the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup_k \ |x_k|$.

Let D be the shift operator on s,i.e.,

$$D((x_k)) \ = \ (x_{k+1}).$$

It may be recalled that Banach limit L is a nonegative linear functional on ℓ_∞ such that L is invariant under the shift operator (i.e., $L(Dx) = L(x)$ for all $x \in \ell_\infty$) and that $L(e) = 1$ where $e = (1,1,1,\ldots)$, [1]. A sequence $x \in \ell_\infty$ is said to be almost convergent if all Banach limits of x coincide, [2].

Let $\hat{c}$ denote the set of all almost convergent sequences.

For any sequence x, we write

$$t_{mn} \ = \ t_{mn}(x) \ = \ \frac{1}{m+1}\sum_{i=1}^{m} x_{n+i}.$$

Lorentz [2] established the following result:

Theorem A. *$x \in \hat{c}$ if and only if $t_{mn}(x)$ tends to a limit as $m \to \infty$, uniformly in n.*

Recently, some new sequence spaces which arose naturally from the concept of almost convergence have been introduced by Nanda [4]. If (p_m) is a bounded sequence of positive real numbers, then we define (see, [4]),

$$\bar{\ell}(p) \ = \ \{x : \sum_m \mid t_{mn} \mid^{p_m} \ \text{converges uniformly in } n\}$$

$$\bar{\bar{\ell}}(p) \ = \ \{x : \sup_n \sum_m \mid t_{mn} \mid^{p_m} < \infty\}.$$

Received June 13, 1989.

(Here and afterwards summations without limits run from 1 to ∞). If $p_m = p$ for all m, we write $\overline{\ell}_p$ and $\overline{\overline{\ell}}_p$ for $\overline{\ell}(p)$ and $\overline{\overline{\ell}}(p)$ respectively. If $p_m = 1$ we write $\overline{\ell}$ and $\overline{\overline{\ell}}$ for $\overline{\ell}_p$ and $\overline{\overline{\ell}}_p$ respectively.

Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \Sigma_k a_{nk}x_k$ converges for each n. Let X and Y be any two nonempty subsets of s. If $(x_k) \in X$ implies that $Ax = (A_n(x)) \in Y$, we say that A defines a matrix transformation from X into Y and we denote it by $A : X \to Y$. By (X, Y) we mean the class of matrices A such that $A : X \to Y$.

2. Almost summability.

We define

$$\begin{aligned}(A,p) &= \{x : \sum_n | A_n(x) |^{p_n} < \infty\}\\ (A,p)_\infty &= \{x : \sup_n | A_n(x) |^{p_n} < \infty\}\\ (\hat{A},p) &= \{x : \sum_m | t_{mn}(Ax) |^{p_m} \text{ converges uniformly in } n\}\\ (\hat{\hat{A}},p) &= \{x : \sup_n \sum_m | t_{mn}(Ax) |^{p_m} < \infty\}\end{aligned}$$

where

$$t_{mn}(Ax) = \sum_k a(n,k,m)x_k$$

such that

$$a(n,k,m) = \frac{1}{m+1}\sum_{i=0}^{m} a_{n+i,k}.$$

If $p_n = p$ for all n then we write $(A)_p$ and $(A)_p^\infty$ for (A,p) and $(A,p)_\infty$ respectively. If $p = 1$ we omit the suffix p and write (A). Note that (A) denotes the set of all absolutely summable sequences. Similarly if $p_m = p$ for all m we write $(\hat{A})_p$ and $(\hat{\hat{A}}_p)$ for $(\hat{A},p)$ and $(\hat{\hat{A}},p)$ respectively.

We have

Theorem 1. $(\hat{A},p) \subset (\hat{\hat{A}},p)$.

Proof. Let $x \in (\hat{A},p)$. Then there is an integer M such that

$$\sum_{m \geq M} | \sum_k a(n,k,m)x_k |^{p_m} \leq 1. \tag{2.1}$$

Hence it is enough to show that, for fixed m, $\Sigma_k a(n,k,m)x_k$ is bounded. It follows from (2.1) that

$$| \sum_k a(n,k,m)x_k | \leq 1 \text{ for } m \geq M \text{ and for all } n.$$

But if $m \geq 1$

$$(m+1)\sum_k a(n,k,m)x_k - m\sum_k a(n,k,m-1)x_k = \sum_k a_{n+m,k} \tag{2.2}$$

Hence for any fixed $m \geq M+1$, $\Sigma_k a_{n+m,k}x_k$ is bounded. Therefore $\Sigma_k a(n,k,m)x_k$ is bounded for all m, n and this completes the proof.

Theorem 2. $(\widehat{A})_p \subset (A)_\infty$.

Proof. Note that

$$\sup_{m,n} \mid \sum_k a(n,k,m)x_k \mid \leq \sup_n (\sum_m \mid \sum_k a(n,k,m)x_k \mid^p)^{1/p} \tag{2.3}$$

$$\sup_{n,m} \mid \sum_k a(n,k,m)x_k \mid \geq \sup_n \mid \sum_k a(n,k,0)x_k \mid = \sup_n \mid \sum_k a_{nk}x_k \mid \tag{2.4}$$

$$\begin{aligned}\sup_{n,m} \mid \sum_k a(n,k,m)x_k \mid &= \sup_{n,m} \mid \sum_k \frac{1}{m+1}\sum_{i=0}^m a_{n+i,k}x_k \mid \\ &\leq \sup_{n,i} \mid \sum_k a_{n+i,k}x_k \mid \sup_m \frac{\sum_{i=1}^m 1}{m+1} \\ &\leq \sup_{n,i} \mid \sum_k a_{n+i,k}x_k \mid \\ &= \sup_n \mid \sum_k a_{nk}x_k \mid .\end{aligned}$$

Now the result follows from (2.3), (2.4) and (2.5).

If X is a linear space over the field C then a paranorm on X is a function $g : X \to R$ which satisfies the following axioms for $x, y \in X$,

$$\begin{aligned} g(0) &= 0 \\ g(x) &= g(-x) \\ g(x+y) &\leq g(x)+g(y) \\ \lambda \longrightarrow \lambda_0,\ x &\longrightarrow x_0 \textit{ imply } \lambda x \longrightarrow \lambda_0 x_0 \end{aligned}$$

where $\lambda, \lambda_0 \in C$ and $x, x_0 \in X$; in other words, $\mid \lambda - \lambda_0 \mid \to 0$, $g(x - x_0) \to 0$ imply $g(\lambda x - \lambda_0 x_0) \to 0$. A paranormed space is a linear space X with a paranorm g and is written as (X, g).

Theorem 3. *(A,p) is linear topological space paranormed by*

$$f(x) = (\sum_n \mid \sum_k a_{nk}x_k \mid^{p_n})^{1/M}$$

where $M = \max(1, \sup p_n)$. $(\hat{A}, p)$ *is paranormed by*

$$g(x) = \sup(\sum_m | \sum_k a(n,k,m)x_k |^{p_m})^{1/M} \tag{2.6}$$

$(\hat{A}, p)$ *is paranormed by (2.6) if* $\inf p_m > 0$. *Also if* $\inf p_n > 0$ *then* $(A,p)_\infty$ *is paranormed by*

$$h(x) = \sup_n | \sum_k a_{nk} x_k |^{p_n/M} .$$

Proof. Because of Theorem 1 (2.6) is meaningful for $x \in (\hat{A}, p)$. We consider only $(\hat{A}, p)$. It can be proved by "standart" arguments that g is a paranorm on $(\hat{A}, p)$. As one step in the proof, we shall only show that for fixed x, $\lambda x \to 0$ as $\lambda \to 0$. If $x \in (\hat{A}, p)$, then given $\varepsilon > 0$ there is an M such that, for all n

$$\sum_{m \geq M} | \sum_k a(n,k,m)x_k |^{p_m} < \varepsilon. \tag{2.7}$$

So if $0 < \lambda \leq 1$, then

$$\sum_{m \geq M} | \sum_k a(n,k,m)\lambda x_k |^{p_m} \leq \sum_{m \geq M} | \sum_k a(n,k,m)x_k |^{p_m} < \varepsilon,$$

and since, for fixed M,

$$\sum_{m=0}^{M-1} | \sum_k a(n,k,m)\lambda x_k |^{p_m} \to 0$$

as $\lambda \to 0$, this completes the proof.

Theorem 4. *Let* $0 < p_m \leq q_m$, *then* $(\hat{A}, p) \subset (\hat{A}, q)$.

Proof. Let $x \in (A, p)$. then there is an integer M such that (2.1) holds. Hence for $m \geq M$ $| \Sigma_k a(n,k,m)x_k | \leq 1$. So that

$$| \sum_k a(n,k,m)x_k |^{q_m} \leq | \sum_k a(n,k,m)x_k |^{p_m}$$

and this completes the proof.

3. Some matrix transformations

In this section we characterize some matrix transformations

Theorem 5. *Let $b_{nk} > 0$. If*

$$\sup_n \sum_k | a_{nk} | (b_{nk})^{1/p} < \infty \tag{3.1}$$

and

$$\sup_k \sum_m | a(n,k,m) | (b_{nk})^{-1/q} < \infty \tag{3.2}$$

where $p^{-1} + q^{-1} = 1$, then $A \in (\ell_p, \bar{\bar{\ell}}_p)$.

Proof. We have by Hölder's inequality

$$\begin{aligned} | t_{mn}(Ax) |^p \leq & \Big(\sum_k | a(n,k,m) | (b_{nk})^{1/p}\Big)^{p-1} \\ & \sum_k | a(n,k,m) | (b_{nk})^{-1/q} | x_k |^p . \end{aligned}$$

Hence

$$\begin{aligned} \sum_m | t_{mn}(Ax) |^p \leq & \Big(\sum_k | a(n,k,m) | (b_{nk})^{1/p}\Big)^{p-1} \\ & \sum_k | x_k |^p \sum_m | a(n,k,m) | (b_{nk})^{-1/q}. \end{aligned} \tag{3.3}$$

Also it follows from Lemma in [4] that (3.1) is equivalent to

$$\sup_n \sum_k | a(n,k,m) | (b_{nk})^{1/p} < \infty. \tag{3.4}$$

Now the result follows from (3,2), (3.3) and (3.4).

Theorem 6. *Let $1 \leq p < \infty$. Then $A \in (\ell_\infty, \bar{\ell}_p)$ if and only if*

$$\sum_m (\sum_k | a(n,k,m) |)^p < \infty, \text{ uniformly in } n, \tag{3.5}$$

Proof. Sufficiency. Suppose that (3.5) holds and that $x \in \ell_\infty$. then

$$\begin{aligned} \sum_m | t_{mn}(Ax) |^p & \leq \sum_m (\sum_k | a(n,k,m) x_k |)^p \\ & \leq \|x\|_\infty^p \sum_m (\sum_k | a(n,k,m) |)^p. \end{aligned}$$

Therefore $\Sigma_m \mid t_{mn}(Ax) \mid^p$ converges uniformly in n and so $A \in (\ell_\infty, \bar{\ell}_p)$.

Necessity. Suppose that $A \in (\ell_\infty, \bar{\ell}_p)$ and that $x \in \ell_\infty$, Therefore

$$q_n(x) = (\sum_m | t_{mn}(Ax) |^p)^{1/p}$$

exists uniformly in n. Now (q_n) is a sequence of continuous seminorms on ℓ_∞ such that $\sup_n q_n(x) < \infty$. Therefore by Banach-Steinhaus theorem ([3], p.114) there exists a constant K such that

$$q_n(x) \leq K \cdot \|x\| \qquad (\forall n, \ \forall x \in \ell_\infty). \tag{3.6}$$

Putting $x = \text{sgn } a(n, k, m)$ in (3.6) we observe that (3.5) holds. this completes the proof.

References.

[1] Banach, S., *Theorie des Operations linearies* (Warszawa), (1932).

[2] Lorentz, G.G., "A contribution to the theory of divergent sequences", *Acta Math.* 80, 167-190, (1948).

[3] Maddox, I.J., *Elements of Functional Analysis*, cambridge University Press, (1970).

[4] Nanda, S., "Some paranormed sequence spaces", *Rendiconti di Matematica* (1), Vol. 4, serie VII, 11-19, (1984).

[5] Nanda, S., "Absolute almost convergence and absolute almost summability", *Rendiconti di Mathematica dell sue appliczioni* Perie VII-Vol.5 , No.1-2, (1985).

Department of Mathematics, Firat University, ELAZIĞ/TURKEY.