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NOTE ON DISCRETE HARDY'S INEQUALITY

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1. Introduction.

In [1] Copson established the following Hardy's inequality involving series of positive terms.

Theorem A. *If* $p > 1$, $\lambda_n > 0$, $a_n > 0$, $\Lambda_n = \Sigma_{i=1}^{n}\lambda_i$, $A_n = \Sigma_{i=1}^{n}\lambda_i a_i$ *and* $\Sigma_{n=1}^{\infty}\lambda_n a_n^p$ *converges, then*

$$\sum_{n=1}^{\infty}\lambda_n(A_n/\Lambda_n)^p \leq (p/p^{-1})^p\Sigma_{n=1}^{\infty}\lambda_n a_n^p. \tag{1}$$

The constant is the best possible.

B.G. Pachpatte [3] has recently established the following generalization of the inequality (1)

Theorem B. *Let* $p, \lambda_n, a_n, \Lambda_n$ *and* A_n *be as in Theorem A and let* $H(u)$ *be a real-valued positive convex function defined for* $u > 0$. *If* $\Sigma_{n=1}^{\infty}\lambda_n H^p(a_n)$ *converges, then*

$$\sum_{n=1}^{\infty}\lambda_n H^p(A_n/\Lambda_n) \leq (p/p-1)^p\Sigma_{n=1}^{\infty}\lambda_n H^p(a_n). \tag{2}$$

The constant is the best possible.

In the present note we will establish some new inequalities which generalize the inequalities (1) and (2).

2. Main results.

Theorem 1. *let* $p > 1, \beta_n > 0, \lambda_n > 0, a_n > 0$. $\Sigma_{n=1}^{\infty}\lambda_n a_n^p$ *converge, and further let* $\Lambda_n = \Sigma_{i=1}^{n}\beta_i\lambda_i, A_n = \Sigma_{i=1}^{n}\beta_i\lambda_i a_i$. *If there exists* $\kappa > 0$ *such that*

$$p-1 + \frac{(\beta_{n+1}-\beta_n)\Lambda_n}{\beta_{n+1}\beta_n\lambda_n} \geq \frac{p}{\kappa} \qquad \text{for } n = 1,2,3,\ldots, \tag{3}$$

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then

$$\sum_{n=1}^{\infty} \lambda_n (A_n/\Lambda_n)^p \;\le\; \kappa^p \sum_{n=1}^{\infty} \lambda_n a_n^p. \tag{4}$$

The case $\beta_n = 1$, $n = 1, 2, \ldots$ and $k = p/(p-1)$ shows the constant in (4) to be the best possible.

Proof. Define $\alpha_n = A_n \Lambda_n^{-1}$ for $n = 1, 2, 3, \ldots$ and let $\alpha_0 = \lambda_0 = \beta_0 = 1$. By making use of the elementary inequality,

$$x^p \;+\; (p-1)y^p \;\ge\; pxy^{p-1}$$

where x and y are nonnegative numbers and agree that $\Lambda_0 = 0$. We have, for $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
-p\lambda_{n+1} a_{n+1} \alpha_{n+1}^{p-1} &= -p\beta_{n+1}\lambda_{n+1} a_{n+1} (\alpha_{n+1}^{p-1}/\beta_{n+1}) \\
&= -p(\alpha_{n+1}\Lambda_{n+1} - \alpha_n \Lambda_n)(\alpha_{n+1}^{p-1}/\beta_{n+1}) \\
&= -p(\Lambda_{n+1}/\beta_{n+1})\alpha_{n+1}^p + p(\Lambda_n/\beta_{n+1})\alpha_n \alpha_{n+1}^{p-1} \\
&\le -p(\Lambda_{n+1}/\beta_{n+1})\alpha_{n+1}^p + (\Lambda_n/\beta_{n+1})\alpha_n^p \\
&\quad + (p-1)(\Lambda_n/\beta_{n+1})\alpha_{n+1}^p.
\end{aligned}$$

so that,

$$\begin{aligned}
&(p-1)\lambda_{n+1}\alpha_{n+1}^p + \frac{(\beta_{n+1}-\beta_n)\Lambda_n}{\beta_{n+1}\beta_n\lambda_n}\lambda_n \alpha_n^p - p\lambda_{n+1} a_{n+1} \alpha_{n+1}^{p-1} \\
&\le\; (p-1)\lambda_{n+1}\alpha_{n+1}^p + \frac{(\beta_{n+1}-\beta_n)\Lambda_n}{\beta_{n+1}\beta_n\lambda_n}\alpha_n^p - p(\Lambda_{n+1}/\beta_{n+1})\alpha_{n+1}^p \\
&\quad + (\Lambda_n/\beta_{n+1})\alpha_n^p + (p-1)(\Lambda_n/\beta_{n+1})\alpha_{n+1}^p \\
&= \frac{\Lambda_n \alpha_n^p}{\beta_n} - \frac{\Lambda_{n+1}\alpha_{n+1}^p}{\beta_{n+1}}.
\end{aligned}$$

By adding the inequalities for $n = 0, 1, 2, 3, \ldots, N-1$, we have

$$\begin{aligned}
&\sum_{n=0}^{N-1}(p-1)\lambda_{n+1}\alpha_{n+1}^p \;+\; \sum_{n=0}^{N-1}\frac{(\beta_{n+1}-\beta_n)\Lambda_n}{\beta_{n+1}\beta_n\lambda_n}\lambda_n\alpha_n^p - p\sum_{n=0}^{N-1}\lambda_{n+1}a_{n+1}\alpha_{n+1}^{p-1} \\
&\le\; (-\Lambda_N/\beta_N)\alpha_N^p \;\le\; 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{n=1}^{N}(p-1)\lambda_n\alpha_n^p \;+\; \sum_{n=1}^{N-1}\frac{(\beta_{n+1}-\beta_n)\Lambda_n}{\beta_{n+1}\beta_n\lambda_n}\lambda_n\alpha_n^p \\
&\le\; p\sum_{n=1}^{N}\lambda_n a_n \alpha_n^{p-1}.
\end{aligned}$$

Using (3) and since $(p-1)\lambda_N \alpha_N^p \geq 0$, we have

$$\sum_{n=1}^{N-1} \lambda_n \alpha_n^p \leq \kappa \sum_{n=1}^{N} \lambda_n a_n \alpha_n^{p-1}. \tag{5}$$

By let N tend to infinity in (5) and using Holder inequality with indices p and $p/p-1$, we have

$$\sum_{n=1}^{\infty} \lambda_n \alpha_n^p \leq \kappa \sum_{n=1}^{\infty} \lambda_n a_n \alpha_n^{p-1} = \kappa \sum_{n=1}^{\infty} \lambda_n^{1/p} a_n \lambda_n^{(p-1/p)} \alpha_n^{p-1}$$

$$\leq \kappa (\sum_{n=1}^{\infty} \lambda_n a_n^p)^{\frac{1}{p}} (\sum_{n=1}^{\infty} \lambda_n \alpha_n^p)^{\frac{p-1}{p}}.$$

Dividing the above inequality by the last factor on the right and raising the result to the pth power, we obtain

$$\sum_{n=1}^{\infty} \lambda_n \alpha_n^p \leq \kappa^p \sum_{n=1}^{\infty} \lambda_n a_n^p,$$

this is the desired inequality (4).

Remark 1. Theorem 1 redues to Theorem A when $\kappa = p/(p-1)$ and $\beta_n = 1$, for $n = 1, 2, 3, \ldots$.

Theorem 2. *Let H be a real-valued positive convex function defined on $(0, \infty)$, and let $p, \beta_n, \lambda_n, a_n, \Lambda_n, A_n$ and κ be as in Theorem 1. If $\Sigma_{n=1}^{\infty} \lambda_n H^p(a_n)$ converges, then*

$$\sum_{n=1}^{\infty} \lambda_n H^p(A_n/\Lambda_n) \leq \kappa^p \sum_{n=1}^{\infty} \lambda_n H^p(a_n). \tag{6}$$

Proof. Since H is a convex function, by Jensen's inequality, we have

$$H(A_n/\Lambda_n) \leq F_n/\Lambda_n,$$

where

$$F_n = \sum_{i=1}^{n} \lambda_i \beta_i H(a_i).$$

Thus

$$\sum_{n=1}^{\infty} \lambda_n H^p(A_n/\Lambda_n) \leq \sum_{n=1}^{\infty} \lambda_n (F_n/\Lambda_n)^p. \tag{7}$$

Replace a_n by $H(a_n)$ in (4), we have

$$\sum_{n=1}^{\infty} \lambda_n (F_n/\Lambda_n)^p \leq \kappa^p \sum_{n=1}^{\infty} \lambda_n H^p(a_n). \tag{8}$$

The desired inequality then follows from (7) and (8).

Remark 2. The inequality (4) is the spacial case of the inequality (6) when $H(u) = u$, and Theorem 2 reduces to Theorem B when $\kappa = p/(p-1)$ and $\beta_n = 1$, for $n = 1, 2, 3, \ldots$. This case shows the constant in (6) to be the best possible.

The following is a discrete analogue of Theorem 3 in [2].

Theorem 3. *Let $p, \beta_n, \lambda_n, a_n, \Lambda_n, A_n$ and κ be as in Theorem 1, and let $\varphi > 0$ be defined on $(0, \infty)$ so that $\varphi'' \geq 0$ and*

$$\varphi\varphi'' \geq (1 - 1/p)(\varphi')^2. \tag{9}$$

If $\Sigma_{n=1}^{\infty}\lambda_n\varphi(a_n)$ converges, then

$$\sum_{n=1}^{\infty} \lambda_n\varphi(A_n/\Lambda_n) \leq \kappa^p \sum_{n=1}^{\infty} \lambda_n\varphi(a_n). \tag{10}$$

Proof. Let $\psi(u) = \varphi^{1/p}(u), u > 0$. Then, by (9), $\psi'' \geq 0$. Hence ψ is convex on $(0, \infty)$. Thus, by Theorem 2, we have

$$\sum_{n=1}^{\infty} \lambda_n\psi^p(A_n/\Lambda_n) \leq \kappa^p \sum_{n=1}^{\infty} \lambda_n\psi^p(a_n),$$

and therefore

$$\sum_{n=1}^{\infty} \lambda_n\varphi(A_n/\Lambda_n) \leq \kappa^p \sum_{n=1}^{\infty} \lambda_n\varphi(a_n).$$

This completes the proof.

Remark 3. Theorem 3 reduces to Theorem 2 and Theorem 1 when $\varphi(u) = H^p(u)$ and $\varphi(u) = u^p$, respectively. Also we note that the inequality (2) and (1) are the special cases of the inequality (10) when $\varphi(u) = H^p(u), \kappa = p/(p-1)$ and $\beta_n = 1$, for $n = 1, 2, \ldots$, respectively.

References

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