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ON LYAPUNOV TYPE FINITE DIFFERENCE INEQUALITY

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Abstract. Lyapunov type finite difference inequality is established which in the special case yields implicit lower bound on the distance between consecutive zeros of a nontrivial solution of a second order linear finite difference equation.

The classical inequality of Lyapunov [5] states that if $y(t)$ is a nontrival solution of the second order differential equation

$$y'' + p(t)y = 0, \tag{1}$$

where $p(t)$ is real and continuous, and if $y(t)$ has at least two zeros on the interval [a,b], then

$$(b-a)\int_a^b |\, p(t)\,|\, dt > 4. \tag{2}$$

Inequality (2) provides an implicit lower bound on the distance between the zeros of a nontrivial solution of (1) by means of an integral measurement of p. The Lyapunov inequality has received considerable attention since its appearance and a number of papers have been appeared in the literature which deals with the various extensions, generalizations and applications of this inequality, see [1-8] and the references given therein.

The main purpose of this note is to establish a Lyapunov type inequality for the second order linear finite difference equation

$$\Delta(r(n)\Delta x(n)) + p(n)x(n) = 0, \tag{3}$$

for $n \in I$, where $I = \{a, a+1, a+2, \ldots, b\}$, a and $b = a+m$, $(m \geq 2)$ integers, the operator Δ is defined by $\Delta z(n) = z(n+1) - z(n)$ for $n \in I$. It is assumed that $p(n)$ and $r(n)$ for $n \in I$ are real-valued functions and $r(n) > 0$ for $n \in I$. Here our approach is more direct and elementary and the result provides a new estimate on this type of inequality.

Our main result is established in the following theorem.

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Theorem. *Let $x(n)$ be a solution of equation (3) such that $x(a) = x(b) = 0$, $x(n) \neq 0$, for $n \in I^0 = \{a+1, a+2, \ldots, b-1\}$. Let k be a point in I^0 where $|x(n)|$ is maximized. Then*

$$4 \leq \Big(\sum_{n=a}^{b-1} \frac{1}{r(n)}\Big)\Big(\sum_{n=a}^{b-1} | p(n) |\Big). \tag{4}$$

Proof. Let $M = |x(k)|$, $k \in I^0$. It is obvious that

$$x(k) = \sum_{n=a}^{k-1} \Delta x(n), \qquad k \in I, \tag{5}$$

and

$$x(k) = -\sum_{n=k}^{b-1} \Delta x(n), \qquad k \in I. \tag{6}$$

From (5) and (6) we observe that

$$2M \leq \sum_{n=a}^{b-1} | \Delta x(n) | . \tag{7}$$

Now squaring both sides of (7) and using the Schwarz inequality, the following formula of summation by parts

$$\sum_{s=0}^{n-1} u(s)\Delta v(s) = (u(n)v(n) - u(o)v(o)) - \sum_{s=0}^{n-1} v(s+1)\Delta u(s), \tag{8}$$

and the facts that $x(a) = x(b) = 0$ and equation (3) we observe that

$$\begin{aligned} 4M^2 &\leq \Big(\sum_{n=a}^{b-1} r^{-\frac{1}{2}}(n) r^{\frac{1}{2}}(n) \mid \Delta x(n) \mid\Big)^2 \\ &\leq \Big(\sum_{n=a}^{b-1} \frac{1}{r(n)}\Big)\Big(\sum_{n=a}^{b-1} (r(n)\Delta x(n))\Delta x(n)\Big) \\ &= \Big(\sum_{n=a}^{b-1} \frac{1}{r(n)}\Big)\Big(-\sum_{n=a}^{b-1} x(n+1)\Delta(r(n)\Delta x(n))\Big) \\ &= \Big(\sum_{n=a}^{b-1} \frac{1}{r(n)}\Big)\Big(\sum_{n=a}^{b-1} x(n+1)p(n)x(n)\Big) \\ &\leq \Big(\sum_{n=a}^{b-1} \frac{1}{r(n)}\Big) M^2 \Big(\sum_{n=a}^{b-1} | p(n) |\Big). \end{aligned} \tag{9}$$

Dividing both sides of (9) by M^2 we get the desired inequality in (4). This completes the proof of Theorem.

It is interesting to note that in the special case when $r(n) = 1$, the inequality established in (4) reduces to the following inequality

$$4 \leq (b-a) \sum_{n=a}^{b-1} | p(n) | . \tag{10}$$

The inequality (10) yields the implicit lower bound on the distance between consecutive zeros of a nontrivial solution of equation (3).

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