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ON QUASI $\star$-BARRELLED SPACES

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Abstract. In this paper, a new class of locally convex spaces, called quasi $\star$-barrelled spaces is introduced. These spaces are characterized by : A locally convex space E is Quasi $\star$-barrelled if every bornivorous $\star$-barrel in E is a neighbourhood of O in E. This class of spaces is a generalization of quasi-barrelled spaces and $\star$-barrelled spaces (K.Anjaneyulu; Gayal : Jour. Math. Phy. Sci. Madras, 1984). Some properties of quasi $\star$-barrelled spaces are studied. Lastly one example each of

(i) a quasi $\star$-barrelled space which is not quasi-barrelled.
(ii) a quasi $\star$-barrelled space which is not $\star$-barrelled.
is given.

Introduction

Let E be a locally convex space, not necessarily Hausdorff, and E' its dual space. Consider the following collections of subsets of E':

$$\begin{aligned}
\mathcal{A} &= \{A : A \text{ is relatively } \beta(E';E)\text{-compact}\} \\
\mathcal{B} &= \{A : A \text{ is relatively } \sigma(E',E)\text{-compact}\} \\
\mathcal{C} &= \{A : A \text{ is } \sigma(E',E)\text{-bounded}\} \\
\mathcal{D} &= \{A : A \text{ is } \beta(E',E)\text{-bounded}\} \\
\mathcal{P} &= \{A : A \text{ is equicontinuous}\}
\end{aligned}$$

It is well known [2] that (i) E is barrelled if and only if the Collections $\mathcal{C}$ and $\mathcal{P}$ coincide (ii) E is quasi-barrelled space if and only if the collections $\mathcal{D}$ and $\mathcal{P}$ coincide (iii) if the space E is quasi-complete and quasi-barrelled then it is a barrelled space. Several authors have considered other types of spaces such as quasi M-barrelled [5] σ-barrelled [6] $\star$-barrelled [1]. Neverth-less, it seems to be of interest to consider those locally convex spaces E, here called quasi $\star$-barrelled, for which the collections $\mathcal{A}$ and $\mathcal{P}$ coincide. Section 1 deals with the definition and a characterization of quasi $\star$-barrelled spaces. The class of quasi $\star$-barrelled spaces properly include $\star$-barrelled spaces. Properties of quasi $\star$-barrelled spaces are studied in section 2. In section 3, some counter examples

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are given. The notations and definitions used here, and in what follows, are those of [2], unless explicitly stated to the countrary.

1. Quasi $\star$-barrelled spaces:

Definition. Let E be a locally convex space and E' its dual. A subset of E is said to be a bornivrous $\star$-barrel if it is the polar of a relatively compact subset of E' for the topology $\beta(E', E)$. The locally convex space E is said to be quasi $\star$-barrelled if every bornivorous $\star$-barrel in E is a neighbourhood of 0.

Clearly, every bornivorous $\star$-barrel in E is bornivorous barrel, it follows that every quasi-barrelled space is quasi $\star$-barrelled.

Proposition 1. *A locally convex space E is quasi $\star$-barrelled space if and only if every subset of E' which is relatively $\beta(E', E)$-compact is equicontinuous.*

Proof. Suppose that E is a quasi $\star$-barrelled space. Let M be a relatively compact subset of E' for the topology $\beta(E', E)$. Then its polar $M^0 = V$ (Say) is a borni-vorous $\star$-barrel in E and hence a neighbourhood of O. Since

$$M \subset M^{00} = (M^0)^0 = V^0$$

It follows that M is equicontinuous. Conversely, suppose that the condition holds and let B be a bornivorous $\star$-barrel in E. Then $B = M^0$ for some relatively compact subset M of E' for the topology $\beta(E', E)$. By assumption, M is equicontinuous and hence there exists a neighbourhood V of O in E such that

$$M \subset V^0$$

But then

$$B = M^0 \supset V^{00} \supset V$$

and so B is a neighbourhood of O in E. Thus E is quasi $\star$-barrelled space.

Proposition 2. *Every $\star$-barrelled space is quasi $\star$-barrelled.*

Proof. Let E be a quasi $\star$-barrelled space and E' its dual. Let V be a relatively compact subset of E' for the topology $\beta(E', E)$. Then $\overline{V}$ is compact for the topology $\beta(E', E)$ and so is compact for the topology $\sigma(E', E)$. It follows that V is relatively compact for the topology $\sigma(E', E)$. Since E is a $\star$-barrelled space, V is equicontinuous. Therefore, E is a quasi $\star$-barrelled space.

Proposition 3. *Let E be a $\star$-barrelled locally convex space. Then (i) E is quasi $\star$-barrelled if and only if it is quasi-M barrelled. (ii) E is barrelled if and only if it is quasibarrelled.*

Proof. (i) is obvious by [1].

(ii) If E is also a barrelled space, then it is alwarys quasibarrelled. Conversely, assume that E is quasi-barrelled. Since every $\star$-barrelled space is sequentially barrelled [1] E is sequentially barrelled, hence E is barrelled by proposition 4.1 [7].

2. Further Properties

Proposition 4. *Let F be a vector space, $(E_i)_{i\in I}$ a family of quasi $\star$-barrelled Hausdroff spaces and for each $i \in I$ let f_i be a linear mapping from E into F such that $\cup f_i(E_i)$ spans F. Suppose that F equipped with finest locally convex topology for which all the mappings f_i are continuous is a Hausdroff space. Then F is a quasi $\star$-barrelled space.*

Proof. See [1].

Corollary 1. *Let E be a quasi $\star$-barrelled space and M a closed subspace of E. Then the quotient space E/M is quasi $\star$-barrelled.*

Corollary 2. *The locally convex direct sum of a family $(E_i)_{i\in I}$ of quasi $\star$-barrelled Hausdroff spaces is a quasi $\star$-barrelled space.*

Proposition 5. *Any separable (DF)-Space E is quasi $\star$-barrelled.*

Proof. It follows [4, corollary 4(a)] that E is quasi-barrelled. Hence E is quasi $\star$-barrelled.

3. Examples.

(i) A quasi $\star$-barrelled space need not be quasi-barrelled. Let F be a non reflexive Banach space and $E = (F', \mathcal{T}(F', F))$, the Mackey dual space of F. Then $E' = F$. It is proved [4, p.195] that E is neither quasi-barrelled nor barrelled. We show that it is a quasi $\star$-barrelled space. Let V be a relatively compact set in E' for the topology $\beta(E', E)$. Then $\overline{V}$ is compact for the topology $\beta(E', E)$. Since $(E', \beta(E', E)) = F$, a non reflexive Banach space, it is complete for the topology $\beta(E', E)$. Therefore, the closed absolutely convex hull, W of $\overline{V}$ is compact for the topology $\beta(E', E)$ and hence compact for the topology $\sigma(E', E)$. Since $(E, \mathcal{T}(E, E'))$ is a Mackey space, W is equicontinuous.

Since
$$V \subset \overline{V} \subset W$$
it follows that E is quasi $\star$-barrelled.

(ii) A quasi $\star$-barrelled space which is not $\star$-barrelled. Let ϕ: The vector space of all sequences (real or complex) having only finitely many nonzero components, equipped with supremum norm topology. It is a normed vector space and hence quasi barrelled and so quasi-$\star$-barrelled space. But it is not $\star$-barrelled for if it were a $\star$-barrelled space, then being quasi-barrelled, it would be barrelled by proposition 3 which is not true by Iyahen [3].

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