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ON SOME PROJECTION METHODS FOR APPROXIMATING FIXED POINTS OF NONLINEAR EQUATIONS IN BANACH SPACE

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Abstract. We use a Newton-like method to approximate a fixed point of a nonlinear operator equation in a Banach space. Our iterates are computed at each step by solving a linear algebraic system of finite order.

I. Introduction

Consider the problem of approximating a fixed point x^* of the operator equation

$$x = T(x) \tag{1}$$

where $T(x)$ is a nonlinear operator defined on a subset D of a Banach space E with values in a Banach space $\hat{E}$.

We study the convergence of the Newton-like methods

$$x_{n+1} = T(x_n) - PT'(x_n)(x_n - x_{n+1}), \qquad n \geq 0 \tag{2}$$

and

$$y_{n+1} = T(y_n) - PT'(x_0)(y_n - y_{n+1}), \qquad x_0 = y_0, \quad n \geq 0 \tag{3}$$

to x^*, where $T'(x_n)$ is the Fréchet derivative of T evaluated at x_n and P is a linear projection operator projecting E on its subspace E_p. If E_p is a finite-dimensional space with $\dim(E_p)=N$, then the iterates (2) and (3) can be computed at each step by solving a system of linear algebraic equations of order at most N. The case when $P = I$, the identity operator on E, has been examined by many authors, under different assumptions [1], [3], [4], [5], [7]. The iterates, however, can rarely be computed in infinite dimensional spaces, since it may be very difficult or impossible to find the inverses of the linear operators $I - T'(x_n)$, $n \geq 0$. The case when T is a continuous linear operator has been examined in [5], [6]. We assume that T is a nonlinear operator. Our conditions are easier to verify than the ones in [5], even in the linear case.

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In this paper, we provide sufficient conditions for the convergence of iterations (2) and (3) to a locally unique fixed point x^* of equation (1).

Finally, we illustrate our results with an example.

II. Convergence theorems.

We can now formulate our main theorem concerning iteration (2).

Theorem 1. *Let $T : D \subset E \to \hat{E}$ and assume*

(a) *the inverse of the linear operator $I - PT'(x_0)$ exists and*

$$\|(I - PT'(x_0))^{-1}(x_0 - T(x_0))\| \leq \eta; \tag{4}$$

(b) *the following inequalities are true for all*

$$x, y \in U(x_0, r) = \{x \in E \,\big|\, \|x - x_0\| < r\} :$$

$$\|(I - PT'(x_0))^{-1}(PT'(x) - PT'(y))\| \leq M\|x - y\|^\lambda \tag{5}$$

and

$$\|(I - PT'(x_0))^{-1}(QT(x) - QT(y))\| \leq q\|x - y\|^\lambda, \; Q = I - P, \; \lambda \in [0, 1). \tag{6}$$

(c) *The conditions*

$$(\eta d)^\lambda < 1, \tag{7}$$

$$Mr^\lambda < 1, \tag{8}$$

$$\eta + \frac{ed^{-1}}{1 - e} \leq r \tag{9}$$

are satisfied, where

$$e = (d\eta)^\lambda, \; d^{\lambda - 1} = c$$

and

$$c(r) = c = \frac{1}{1 - Mr^\lambda}\Big(\frac{2Mr}{1 + \lambda} + q\Big).$$

(d) *The ball $\overline{U}(x_0, r) \subset D$. Then, equation (1) has a fixed point x^* in $\overline{U}(x_0, r)$ where r is chosen to be the minimum number $r > 0$ satisfying (8)-(9). Moreover, the following estimates are true*

$$\|x_n - x^*\| \leq d^{-1}\frac{e^n}{1 - e}, \; n \geq 0 \tag{10}$$

and

$$\|x_{n+1} - x_n\| \leq c\|x_n - x_{n-1}\|^\lambda, \; n \geq 1. \tag{11}$$

Furthermore, if

$$(dr)^{\lambda} < 1 \tag{12}$$

then x^* is the unique fixed point of equation (1) in $\overline{U}(x_0, r)$.

Proof. From (2) and (3) we get the identity

$$(I - PT'(x_n))(x_{n+1} - x_n) = T(x_n) - T(x_{n-1}) - PT'(x_{n-1})(x_n - x_{n-1}),\ n \geq 1. \tag{13}$$

By the Banach lemma on invertible operators, (5) and (8), it follows that $I - PT'(x)$ is invertible for all $x \in U(x_0, r)$ and

$$||(I - PT'(x))^{-1}(I - PT'(x_0))|| \leq \frac{1}{1 - M||x - x_0||^{\lambda}} \leq \frac{1}{1 - Mr^{\lambda}}. \tag{14}$$

Let us assume that $x_0, x_1, \ldots, x_n \in U(x_0, r)$, then from (4)-(6), (13) and (14) we get

$$\begin{aligned}
&||x_{n+1} - x_n|| \\
&\leq ||(I - PT'(x_n))^{-1}(I - PT'(x_0))|| \Big[||(I - PT'(x_0))^{-1}(PT(x_n) - PT(x_{n-1}) \\
&\quad - PT'(x_{n-1})(x_n - x_{n-1}))|| + ||(I - PT'(x_0))^{-1}(QT(x_n) - QT(x_{n-1}))||\Big] \\
&\leq \frac{1}{1 - Mr^{\lambda}} \Big[||(I - PT'(x_0))^{-1} \int_0^1 PT'(x_{n-1} + t(x_n - x_{n-1})) - PT'(x_{n-1}))(x_n - x_{n-1})dt|| \\
&\quad + q||x_n - x_{n-1}||^{\lambda}\Big] \\
&\leq \frac{1}{1 - Mr^{\lambda}} \Big[\frac{M}{1 + \lambda}||x_n - x_{n-1}|| + q\Big] ||x_n - x_{n-1}||^{\lambda} \\
&\leq c||x_n - x_{n-1}||^{\lambda}, \text{ which shows (11).}
\end{aligned} \tag{15}$$

From (11), we get

$$\begin{aligned}
||x_0 - x_{n+1}|| &\leq ||x_1 - x_0|| + ||x_2 - x_1|| + \cdots + ||x_n - x_{n+1}|| \\
&\leq \eta + c\eta^{\lambda} + c^{1+\lambda}\eta^{\lambda^2} + \cdots + c^{1+\lambda+\cdots+\lambda^{n-1}}\eta^{\lambda^n} \\
&\leq \eta + d^{-1}[(d\eta)^{\lambda} + (d\eta)^{\lambda^2} + \cdots + (d\eta)^{\lambda^n}] \\
&\leq \eta + d^{-1}[(d\eta)^{\lambda} + (d\eta)^{2\lambda} + \cdots + (d\eta)^{n\lambda}] \\
&\leq \eta + d^{-1}e(1 + e + e^2 + \cdots + e^{n-1}) \\
&\leq \eta + d^{-1}e\frac{1 - e^n}{1 - e} \leq \eta + d^{-1}e\frac{1}{1 - e} \leq r \qquad \text{(by9)}.
\end{aligned}$$

Hence, $x_{n+1} \in U(x_0, r)$. For $p \geq 1$,

$$\begin{aligned}
||x_n - x_{n+1p}|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - x_{n+2}|| + \cdots + ||x_{n+p-1} - x_{n+p}|| \\
&\leq d^{-1}(d\eta)^{\lambda^n} + d^{-1}(d\eta)^{\lambda^{n+1}} + \cdots + d^{-1}(d\eta)^{\lambda^{n+p}} \\
&\leq d^{-1}e^n[1 + e + \cdots + e^{p-1}] = d^{-1}e^n\frac{1 - e^p}{1 - e}.
\end{aligned} \tag{16}$$

It now follows from that the sequence $\{x_n\}$ is a Cauchy sequence in a Banach space and as such it converges to some $x^* \in \overline{U}(x_0, r)$. By letting $p \to \infty$ in (16) we obtain (10), whereas by letting $n \to \infty$ in (2) we get $x^* = T(x^*)$. To show uniqueness let us assume that z^* is any fixed point of T in $U(x_0, r)$ and use the identity

$$(I - PT'(x_n))(x_{n+1} - z^*) \ = \ T(x_n) - T(z^*) - PT'(x_n)(x_n - z^*)$$

to get

$$\|x_{n+1} - z^*\| \ \leq \ c\|x_n - z^*\|^\lambda \ \leq \ \cdots \ \leq \ d^{-1}(dr)^{\lambda^n} \ \leq \ d^{-1}(dr)^{\lambda^n} \ \to \ 0$$

as $n \to \infty$ from (12). Hence $x^* = \lim_{n\to\infty} x_n = z^*$.

That completes the proof of the theorem.

Note that for $\lambda = 1$ the proof of the previous theorem can be repeated, but (7) becomes $c < 1$, (9) becomes $\frac{\eta}{1-c} \leq r$, $e = c$, (10) becomes $\|x_n - x^*\| \leq \dfrac{e^n}{1-e}\eta$ and (12) becomes $c < 1$.

the proof of the following theorem concerning iteration (3) is omitted as similar to the proof of theorem 1.

Theorem 2. *Let $T : D \subset E \to \hat{E}$ and assume*
(a) *the following inequalities are true:*

$$\|(I - PT'(x_0))^{-1}(x_0 - T(x_0))\| \ \leq \eta,$$

$$\|(I - PT'(x_0))^{-1}(PT'(x) - PT'(y)\| \ \leq \ M\|x - y\|^\lambda$$

and

$$\|(I - PT'(x_0))^{-1}(QT(x) - QT(y)\| \ \leq \ q\|x - y\|^\lambda,$$

$$Q \ = \ I - P, \ \lambda \in [0, 1), \text{ for all } x, y \in U(x_0, R).$$

(b) *The conditions*

$$(\eta d_1)^\lambda \ < \ 1,$$

$$\eta \ + \ \frac{e_1 d_1^{-1}}{1 - e_1} \ \leq \ R$$

are satisfied, where

$$e_1 \ = \ (d_1\eta)^\lambda, \ d_1^{\lambda-1} \ = \ c_1$$

and

$$c_1(r) \ = \ c_1 \ = \ 2^{1-\lambda}MR \ + \ q.$$

(c) *The ball $\overline{U}(x_0, R) \subset D$.*

Then equation (1) has a fixed point $x^\star$ in $\overline{U}(x_0, R)$ where R is chosen to be the minimum number $R > 0$ satisfying conditions (b). Moreover, the following estimates are true

$$\|y_n - x^\star\| \ \le \ d_1^{-1}\frac{e_1^n}{1-e_1},\ n \ge 0$$

and

$$\|y_{n+1} - y_n\| \ \le \ c_1\|y_n - y_{n-1}\|^\lambda,\ n \ge 1.$$

Furthermore if

$$(d_1 R)^\lambda \ \le \ 1$$

then $x^\star$ is the unique fixed point of equation (1) in $\overline{U}(x_0, R)$.

Note that a remark similar to the one made after Theorem 1 for the case $\lambda = 1$ can now easily follow for Theorem 2.

We now complete this paper with an application.

III. Applications.

Let us consider the following system in $E = \hat{E} = R^k$,

$$v_i \ = \ f_i(v_1, \ldots, v_k),\ i = 1, 2, \cdots, k. \tag{17}$$

Set

$$T(v) \ \ \{f_i(v_1, \ldots, v_k)\},\ i = 1, 2, \cdots, k;$$

$$T'(W)v \ = \ \left\{\sum_{j=1}^{k} f'_{ij}(w_1, \ldots, w_k)v_j\right\},\ i = 1, 2, \cdots, k;$$

$$PT'(w)v \ = \ \begin{cases} \sum_{j=1}^{k} f'_{ij}(w_1, \cdots, w_k)v_j, & i = 1, 2, \cdots, N \\ 0, & i = N+1, \cdots, k, \end{cases}$$

where the symbol f'_{ij} denotes $\partial f_i / \partial v_j$.

Iterations (2) and (3) can be written as

$$\begin{aligned} v_{i,n+1} \ &= \ f_i(v_{1,n}, \ldots, v_{k,n}) + \sum_{j=1}^{k} f'_{ij}(v_{1,n}, \ldots, v_{k,n})(v_{j,n+1} - v_{j,n}),\ i = 1, \cdots, N \quad (18) \\ v_{i,n+1} \ &= \ f_i(v_{1,n}, \cdots, v_{k,n}),\ i = N+1, \cdots, k \end{aligned}$$

and

$$\begin{aligned} \overline{v}_{i,n+1} \ &= \ f_i(\overline{v}_{1,n}, \ldots, \overline{v}_{k,n}) + \sum_{j=1}^{k} f'_{ij}(\overline{v}_{1,0}, \ldots, \overline{v}_{n,0})(\overline{v}_{j,n+1} - \overline{v}_{j,n}),\ i = 1, \cdots, N \quad (19) \\ \overline{v}_{i,n+1} \ &= \ f_i(\overline{v}_{1,n}, \cdots, \overline{v}_{k,n}),\ i = N+1, \cdots, k, \end{aligned}$$

respectively.

If the determinants $D(x_n)$ and D_0 of (18) and (19) respectively, are nonzero, then we have

$$v_{i,n+1} = \frac{\sum_{m=1}^{N} D_{im}(v_n)\overline{f}_m(v_n)}{D(v_n)}, \ i = 1,2,\cdots,N,$$
$$v_{i,n+1} = f_i(v_n), i = N+1,\cdots,k$$

for system (18) and

$$\overline{v}_{i,n+1} = \frac{\sum_{m=1}^{N} D_{im}(\overline{v}_0)\overline{f}_m(v_0)}{D_0)}, \ i = 1,2,\cdots,N,$$
$$\overline{v}_{i,n+1} = f_i(\overline{v}_n), i = N+1,\cdots,k$$

for system (19).

Here

$$\overline{f}_m(v_n) = f_m(v_n) - \sum_{i=1}^{k} f'_{mj}(v_n)v_{j,n} + \sum_{i=N+1}^{k} f'_{mj}(v_n)f_j(v_n),$$
$$\overline{f}_m(v_0) = f_m(v_0) - \sum_{j=1}^{k} f'_{mj}(v_0)v_{j,n} + \sum_{i=N+1}^{k} f'_{mj}(v_0)f_j(v_0),$$

$m = 1,2,\cdots,k$, where $D_{im}(v_n)$, $D_{im}(v_0)$ are the cofactors of the elements at the itersection of the m-th row and i-th column of the determinants $D(x_n)$ and D_0, respectively.

We assume that the following conditions are satisfied on some region under consideration.

$$| f_i(v_i,\ldots,v_k) - f_i(w_1,\ldots,w_k) | \leq \sum_{j=1}^{k} t_{ij} | v_j - w_j |^\lambda, \ i = N+1,\cdots,k, \ \lambda \in [0,1]$$

$$| f'_{ij}(v_i,\ldots,v_k) - f'_{ij}(w_1,\ldots,w_k) | \leq \sum_{s=1}^{k} b_{ijs} | v_s - w_s |^\lambda, \ i = 1,\cdots,N, \ j = 1,\cdots,k,$$

$$| D_{im}(v) | \leq a_{im'} \ | D(v) | \leq a,$$
$$| f'_{ij}(v) | \leq h_{ij}, \ i = 1,\ldots,N, \ j = 1,2,\cdots,k.$$

For any $v \in E$, set $\|v\| = \sup_{1\leq i\leq k} | v_i |$, then the constants q and M appearing in the Theorem 1-2 can be computed by

$$q \leq \sup_{i=N+1,\cdots,k} \sum_{j=1}^{k} t_{ij} \text{ and } M \leq \sup_{i=1,2,\cdots,N} \sum_{j,j=1}^{k} c_{ijs}.$$

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