TAMKANG JOURNAL OF MATHEMATICS
Volume 21, Number 4, Winter 1990

SOME CHARACTERIZATIONS OF UNCONDITIONAL SCHAUDER DECOMPOSITIONS OF BANACH SPACES

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1. Introduction

In analogy with the concept of unconditional Schauder basis in Banach spaces, the study of unconditional Schauder decomposition was initiated in [3,5]. The concept was further studied by several authors [1,4,6,8]. The notion of Markuševič decomposition (M-decomposition) can be found in [2], where it has been called 'biorthogonal decompositions'.

In this paper, we give several characterizations of unconditional Schauder decompositions in terms of M-decompositions. We show that an M-decomposition (G_n) of a Banach space E with the associated sequence of projections (v_n) is an unconditional Schauder decomposition if for every subset S of natural numbers the set $[\cup_{n\in S} v_n^*(E^*)]$ norms $[\cup_{n\in S} G_n]$. We also give a characterization of unconditional Schauder decomposition of a Banach space having an M-decomposition in terms of the multipliers of the elements of E.

2. Preliminaries

Throughtout E will denote a Banach space over the field $\mathbf{K}(\mathbf{R}$ or $\mathbf{C})$, [ ] the closed linear span of the indicated sets and B_E the closed unit ball of E.

A pair of sequences (G_n, v_n), where (G_n) is a sequece of closed linear subspaces of E with $G_n \neq \{0\}$ and (v_n) is a sequence of projections with $v_n(E) = G_n$ for all n, is said to be a *generalized biorthogonal system* if

$$v_i v_j = \delta_{ij} v_i = \delta_{ij} v_j, \ (i, j \in \mathbf{N}).$$

The sequence (G_n) is said to be a *Markuševič decomposition* (*M-decomposition*) of E if $[\cup_{n=1}^{\infty} G_n] = E$ and $v_n(x) = 0$, for all n, imply $x = 0$. The sequence (v_n) is said to be the *associated sequence* of projections (a.s.p.) to the M-decomposition (G_n). For any subset S of $\mathbf{N}$, write

$$W(S) = [\cup_{n\in S} G_n] \text{ and } W^*(S) = [\cup_{n\in S} v_n^*(E^*)].$$

Received August 16, 1989.

The work of the second author is supported by the CSIR(India).

A closed linear subspace V of E^* is said to *norm* E, if there is a constant $C > 0$ such that

$$C\|x\| \leq \sup_{f \in B_V} |f(x)|.$$

The greatest number which satisfies the above inequality is said to be the characteristic of V.

An M-decomposition (G_n) of E with the a.s.p. (v_n) is said to be a *Schauder decomposition*, if for every $x \in E$ the series $\Sigma_{n=1}^{\infty} v_n(x)$ converges to x. In this case $W^*(\mathbf{N})$norms E([8], Theorem 15.7). A schauder decomposition (G_n) with the a.s.p. (v_n) is said to be an *unconditional Schauder decomposition* if the series $\Sigma_{n=1}^{\infty} v_n(x)$ is unconditionally convergent for every x. In the sequel we shall need a result which we give in the form of a lemma.

Lemma 2.1. ([7], *Lemma* 16.1, *p*.458). *Let* (x_n) *be a sequence in* E. *Then the series* $\Sigma_{n=1}^{\infty} x_n$ *is unconditionally convergent if and only if for every sequence* (β_n) *of scalars with* $|\beta_n| \leq 1\,(n = 1, 2, \cdots)$, *the series* $\Sigma_{n=1}^{\infty} \beta_n x_n$ *is convergent; or equivalently*

$$\sup_{|\beta_1|,|\beta_2|,\ldots \leq 1} \; \sup_{1 \leq n < \infty} \|\sum_{i=1}^{n} \beta_i x_i\| < +\infty.$$

The concept of unconditional Schauder decomposition has been studied by many authors. We list below some known characterizations of unconditional Schauder decompositions which we shall be using in the sequel.

Theorem 2.2. *Let* (G_n) *be a sequence of subspaces of a Banach space* E *with* $G_n \neq \{0\}\,(n = 1, 2, \cdots)$. *Then, the following statements are equivalent :*

(a) (G_n) *is an unconditional Schauder decomposition of* E

(b) (*Grinbluim* [3]) *There is a constant* $1 \leq M < +\infty$, *such that for any two disjoint finite subsets* A *and* B *of* $\mathbf{N}$, $x_j \in G_j\ (j \in A)$ *and* $y_j \in G_j\ (j \in B)$, *we have*

$$\|\sum_{j \in A} x_j\| \leq M \|\sum_{j \in A} x_j + \sum_{j \in B} y_j\|.$$

(c) (*McArthur* [5]) *Every permutation* $(G_{\sigma(n)})$ *of* (G_n) *is a Schauder decomposition.*

(d) (*Lindenstrauss-Pelczynski* [4]) *There is a constant* $1 \leq K < \infty$ *such that for any* $x_j \in G_j (j = 1, 2, \ldots, n)$ *and* $\varepsilon_j = \pm 1\,(j = 1, 2, \ldots, n)$ *(or equivalently* $|\varepsilon_j| \leq 1$, $j = 1, 2, \ldots, n$)), *we have*

$$\|\sum_{j=1}^{n} \varepsilon_j x_j\| \leq K \|\sum_{j=1}^{n} x_j\|.$$

(e) *(Bachelis [1]) For every subset* S *of* $\mathbf{N}$, *we have*

$$E = W(S) \oplus W(\mathbf{N} \backslash S).$$

3. Characterization theorems.

Theorem 3.1. *Let (G_n) be an M-decomposition of E with the a.s.p. (v_n). Then (G_n) is an unconditional Schauder decomposition if and only if for every subset S of* $\mathbf{N}$, *$W^*(S)$ norms $W(S)$.*

Proof. Let S be an arbitrary subset of $\mathbf{N}$ and T be the quotient mapping of E onto $E/W(\mathbf{N}\backslash S)$. Since $W(S^*)$ norms $W(S)$, there is a constant $K > 0$ such that, for $x \in W(S)$, we have

$$\|Tx\| = \sup_{f \in B_{W(\mathbf{N}\backslash S)}} |f(x)| \geq \sup_{f \in B_{W^*(S)}} |f(x)| \geq K\|x\|.$$

Therefore, $T_1 = T \mid W(S)$ is an isomorphism onto $E/W(\mathbf{N}\backslash S)$, whence $T_1^{-1}T$ is a projection of E onto $W(S)$ along $W(\mathbf{N}\backslash S)$. Hence

$$E = W(S) \oplus W(\mathbf{N}\backslash S).$$

Thus, in view of Theorem 2.1(e), (G_n) is an unconditional Schauder decomposition of E with the a.s.p. (v_n).

Conversely, let (G_n) be an unconditional Schauder decomposition. Let S be an arbitrary subset of $\mathbf{N}$ and P be a continuous linear projection of E onto $W(S)$ along $W(\mathbf{N}\backslash S)$. It is easy to see that

$$P^*(W^*(\mathbf{N})) \subset W^*(S).$$

Since $W^*(\mathbf{N})$ norms E, there is a $K > 0$ such that

$$\sup_{f \in B_{W^*(\mathbf{N})}} |f(x)| \geq K\|x\|, \ (x \in W(S)).$$

Hence

$$\begin{aligned} \sup_{g \in B_{W^*(S)}} |g(x)| &\geq \sup_{f \in B_{W^*(\mathbf{N})}} |(P^*f)(x)| / \|P^*\| \\ &= \sup_{f \in B_{W^*(\mathbf{N})}} |f(x)| / \|P^*\| \\ &\geq K\|x\|/\|P^*\| \ (x \in W(S)). \end{aligned}$$

Thus $W^*(S)$ norms $W(S)$.

Theorem 3.2. *Let (G_n) be an M-decomposition of E with the a.s.p. (v_n). Then, (G_n) is an unconditional Schauder decomposition if and only if there is a constant $C > 0$ such that for every subset S of* $\mathbf{N}$, *we have*

$$\operatorname{dist}(B_{W(S)}, W(\mathbf{N}\backslash S)) > C.$$

Proof. Note that, for any subset S of $\mathbf{N}$, the set $\{\Sigma_{i \in A} x_i : x_i \in G_i, i \in A, A$ is finite a subset of $S\}$ is dense in $W(S)$. Now the proof can be completed by invoking Theorem 2.2 (a) $\Leftrightarrow$ (b).

Theorem 3.3. *Let (G_n) be an M-decomposition of E with the a.s.p. (v_n). Then (G_n) is an unconditional Schauder decomposition if and only if for every $x \in E$, there exist a sequence of scalars (γ_j) with $\gamma_j \to 0$ and a $z \in E$ with*

$$\sup_{|\beta_1|,|\beta_2|,\ldots,\leq 1}\ \sup_{1\leq n<\infty} \|\sum_{i=1}^{n} \beta_i v_i(z)\| \ < \ +\infty,$$

such that

$$v_j(x) \ = \ \gamma_j v_j(z), \ (j = 1, 2, \cdots).$$

Proof. Let (G_n) be an unconditional Schauder decomposition of E with the a.s.p. (v_n). Then, for any $x \in E$, we have

$$\lim_{k\to\infty} u_k(x) \ = \ \lim_{k\to\infty} \sum_{i=1}^{k} v_i(x) \ = \ x.$$

Therefore, there exists a sequence (m_n) of positive integers such that

$$\|x - u_k(x)\| \ \leq \ 4^{-1-n}, \ (k \geq m_n, \ n = 1, 2, \cdots).$$

Put

$$y_n \ = \ \sum_{i=m_{n-1}+1}^{m_n} v_i(x), \ (n = 1, 2, \cdots).$$

Then, $\|y_n\| \leq 2 \cdot 4^{-n}$ so that

$$\sum_{n=1}^{\infty} 2^{n-1}\|y_n\| \ \leq \ \sum_{n=1}^{\infty} 2^{-n}.$$

Thus, the series $\Sigma_{n=1}^{\infty} 2^{n-1} y_n$ converges. Again, putting

$$z = \sum_{n=1}^{\infty} 2^{n-1} y_n \text{ and } \gamma_j = 2^{1-n}, \ (m_{n-1} + 1 \leq j \leq m_n; \, n = 1, 2, \cdots),$$

we have $\gamma_j \to 0$ and $v_j(x) = \gamma_j v_j(z)$, $(j = 1, 2, \cdots)$. Finally, since (G_n) is an unconditional Schauder decomposition, by Lemma 2.1, we have

$$\sup_{|\beta_1|,|\beta_2|,\ldots\leq 1}\ \sup_{1\leq n<\infty} \|\sum_{i=1}^{n} \beta_i v_i(z)\| \ < \ +\infty.$$

Conversely, under hypothesis, for any sequence (β_i) of scalars with $|\beta_i| \leq 1 (i = 1, 2, \cdots)$, we have

$$\begin{aligned} \|\sum_{i=p}^{q} \beta_i v_i(z)\| \ &= \ \|\sum_{i=p}^{q} \gamma_i (\sum_{j=1}^{i} \beta_j v_j(z) - \sum_{j=1}^{i-1} \beta_j v_j(z))\| \\ &\leq \ (|\gamma_p| + \sum_{i=p}^{q-1} |\gamma_i - \gamma_{i+1}| + |\gamma_q|) \\ &\quad \times \sup_{|\beta_1|,|\beta_2|,\ldots\leq 1}\ \sup_{1\leq n<\infty} \|\sum_{i=1}^{n} \beta_i v_i(z)\|. \end{aligned}$$

Therefore, $(\Sigma_{i=1}^{n}\beta_i v_i(x))$ is a Cauchy sequence and hence converges. Hence, by Lemma 2.1, the series $\Sigma_{i=1}^{\infty} v_i(x)$ is unconditionally convergent in E. Now, in view of

$$v_j(x - \lim_{n\to\infty}\sum_{i=1}^{n} v_i(x)) \;=\; 0,\; (j = 1, 2, \cdots)$$

and that (v_n) is total on E, it follows that for every $x \in E$, $\Sigma_{i=1}^{\infty} v_i(x)$ is unconditionally convergent to x.

Remark. The condition that

$$\sup_{|\beta_1|,|\beta_2|,\ldots\leq 1}\;\sup_{1\leq n<\infty} \|\sum_{i=1}^{n}\beta_i v_i(z)\| \;<\; +\infty.$$

in the Theorem 3.3 cannot be relaxed. Indeed, if (G_n) is a Schauder decomposition of E with the a.s.p. (v_n) which is not an unconditional Schauder decomposition, then proceeding just as in the proof of the necessary part of Theorem 3.3, for each $x \in E$, we have a sequence (γ_j) of scalars with $\gamma_j \to 0$ and a $z \in E$ satisfying

$$v_j(x) \;=\; \gamma_j v_j(z),\; (j = 1, 2, \cdots).$$

But, since (G_n) is not an unconditional Schauder decomposition, for some $x \in E$, the series $\Sigma_{i=1}^{\infty} v_i(x)$ is not unconditionally convergent, whence by Lemma 2.1

$$\sup_{|\beta_1|,|\beta_2|,\ldots\leq 1}\;\sup_{1\leq n<\infty} \|\sum_{i=1}^{n}\beta_i v_i(x)\| \;=\; \infty.$$

But, in view of (3.1), and since $\gamma_j \to 0$, we have

$$\sup_{|\beta_1|,|\beta_2|,\ldots\leq 1}\;\sup_{1\leq n<\infty} \|\sum_{i=1}^{n}\beta_i v_i(z)\| \;=\; \infty.$$

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