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ON THE COEFFICIENTS OF ANALYTIC FUNCTIONS OF FAST GROWTH REPRESENTED BY DIRICHLET SERIES

SUNITA RANI AND G.S. SRIVASTAVA

1. Let $f(s) = \Sigma_{n=1}^{\infty} a_n e^{s\lambda_n}$, where $s = \sigma + it$ (σ and t real variables), $0 \leq \lambda_1 < \lambda_2 < \cdots, \lambda_n \to \infty$ as $n \to \infty$ and $\{a_n\}$ is a sequence of nonzero complex numbers, be a Dirichlet Series. It is well known that if

$$\limsup_{n\to\infty} \frac{n}{\lambda_n} = D < \infty, \tag{1.1}$$

then $f(s)$ represents an analytic function in a half plane $Re\, s < \alpha$, where

$$\limsup_{n\to\infty} \frac{\log |a_n|^{-1}}{\lambda_n} = \alpha, \ -\infty < \alpha \leq \infty. \tag{1.2}$$

We denote by D_α the class of all functions $f(s)$, analytic in the half plane $\sigma < \alpha$, $-\infty < \alpha < \infty$. Then for $f \in D_\alpha$, we set

$$M(\sigma) = l.u.b._{-\infty<t<\infty} |f(\sigma+it)|, \qquad m(\sigma) = \max_{n\geq 1}\{|a_n| e^{\sigma\lambda_n}\}$$
$$\text{and } N(\sigma) = \max\{n : m(\sigma) = |a_n| e^{\sigma\lambda_n}\}.$$

Then $M(\sigma)$, $m(\sigma)$ and $N(\sigma)$ are called respectively the *maximum modulus, maximun term and the rank of the maximum term* of $f(s)$.

Nandan [1] defined the order ρ of $f \in D_\alpha$ as

$$\limsup_{\sigma\to\alpha} \frac{\log\log M(\sigma)}{\log[(1-e^{\sigma-\alpha})^{-1}]} = \rho, \qquad 0 \leq \rho \leq \infty. \tag{1.3}$$

If $\rho = \infty$, the analytic function f is said to be *of fast growth*. For such functions, Nautiyal [2] introduced the concept of (β, δ) order and lower (β, δ) order.

Let L^0 be the class of all functions h satisfying the following conditions:

(i) $h(x)$ is defined on $[a, \infty)$, is positive, continuous, strictly increasing and $h(x) \to \infty$ as $x \to \infty$,

(ii) $\lim_{x\to\infty} \frac{h[x(1+\eta(x))]}{h(x)} = 1$ for every function $\eta(x) \to 0$ as $x \to \infty$.

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Let Δ denote the class of all functions h satisfying conditions (i) and (iii) $\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$ for every c, $0 < c < \infty$.

Evidently, the class of function Δ is a proper subset of L^0.

For a function $f \in D_\alpha$, set

$$(1.4)\qquad \lim_{\sigma\to\alpha} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\beta(\log M(\sigma))}{\delta(1/1-\exp(\sigma-\alpha))} = \frac{\rho(\beta,\delta,f)}{(\beta,\delta,f)},$$

where $\beta \in \Delta$ and $\delta \in L^0$. As mentioned above, $\rho(\beta,\delta,f)$ and $\lambda(\beta,\delta,f)$ are called (β,δ) order and (β,δ) lower order of f respectively. Nautiyal obtained the coefficient characterizations of $\rho(\beta,\delta)$ and $\lambda(\beta,\delta)$. We thus have

Theorem A. [2, Theorem 1]. *Let $f(s) \in D_\alpha$ with (β,δ) order $\rho(\beta,\delta,f)$. Assume that*

$$(1.5)\qquad \beta(x/G(x,c)) \simeq \beta(x) \text{ as } x\to\infty,\ 0<c<\infty,$$

where $G(x,c) = \delta^{-1}(c\,\beta(x))$. Then

$$(1.6)\qquad \rho(\beta,\delta,f) = \lim_{n\to\infty} \sup \frac{\beta(\lambda_n)}{\delta[\lambda_n/\log^+(|\,a_n\,|\exp(\alpha\lambda_n))]}$$

where $\log^+ x = \max(0, \log x)$.

Theorem B. [2, Lemma 6]. *Let $f(s) \in D_\alpha$ with (β,δ) - order $\rho(\beta,\delta,f) > 0$ and lower (β,δ) order $\lambda(\beta,\delta,f)$. Assume that (1.5) is satisfied and*

$$(1.7)\qquad \lim_{n\to\infty} \inf(\lambda_n - \lambda_{n-1}) = \theta > 0,$$

$$(1.8)\qquad \Psi(n) = \frac{\log|\,a_n/a_{n+1}\,|}{\lambda_{n+1}-\lambda_n} \text{ is a non-decreasing function of } n \text{ for all } n > n_0,$$

$$(1.9)\qquad \lim_{\sigma\to\alpha} \inf \frac{\beta(\lambda_{N(\sigma)})}{\delta[1/(1-e^{\sigma-\alpha})]} = \lambda(\beta,\delta,f).$$

Then

$$(1.10)\qquad \lim_{n\to\infty} \inf \frac{\beta(\lambda_{n-1})}{\delta[\lambda_n/\log^+(|\,a_n\,|\exp(\alpha\lambda_n))]} = \lambda(\beta,\delta,f).$$

It is evident that for two analytic functions having same (β,δ) order, we need further classification to compare their growth. The authors introduced in [3] the concept of (β,δ,Y,ρ)-type and (β,δ,Y,ρ) lower type for analytic functions $f(s) \in D_\alpha$. Hence if $0 < \rho < \infty$, then we define

$$(1.11)\qquad \lim_{\sigma\to\alpha} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\beta(\log M(\sigma))}{\delta[\{\gamma(\frac{1}{1-e^{\sigma-\alpha}})\}^{\rho}]} = \frac{T(\beta,\sigma,\gamma,\rho,f)}{t(\beta,\sigma,\gamma,\rho,f)},$$

where $\beta, \gamma \in \Delta$ and $\delta \in L^0$. For simplicity, we shall denote $T(\beta, \delta, \gamma, \rho, f)$ and $t(\beta, \delta, \gamma, \rho, f)$ by T and t respectively. We obtained in [3] the coefficient characterizations of T and t. Hence we have

Theorem C. *Let $f(s) \in D_\alpha$ be of type T and assume that*

$$\beta\left(\frac{x}{G(x,c,\rho)}\right) \sim \beta(x) \text{ as } x \to \infty,\ 0 < c < \infty, \tag{1.12}$$

where $G(x,c,\rho) = \gamma^{-1}\left[\{\delta^{-1}(\frac{\beta(x)}{c})\}^{1/\rho}\right]$.

$$\lim_{n\to\infty} \sup \frac{\beta(\lambda_n)}{\delta[\{\gamma(\frac{\lambda_n}{\log^+ | a_n | + \alpha\lambda_n})\}^\rho]} = T. \tag{1.13}$$

Theorem D. *If $f(s) \in D_\alpha$ and t is defined by (1.11) then*

$$t \geq \lim_{k\to\infty} \inf \frac{\beta(\lambda_{n_{k-1}})}{\delta[\{\gamma(\frac{\lambda_{n_k}}{\log^+ | a_{n_k} | + \alpha\lambda_{n_k}})\}^\rho]}, \tag{1.14}$$

where $\{n_k\}$ is any increasing sequence of positive integers, $n_k \to \infty$ as $k \to \infty$.

In the present paper, we shall obtain the coefficient characterizations of $\rho(\beta, \delta, f)$, $\lambda(\beta, \delta, f)$, as defined by (1.4) and T, t as defined by (1.11), in terms of the ratio of the consecutive coefficients i.e. $|a_n/a_{n+1}|$.

2. We now prove

Theorem 1. *Let $f(s) \in D_\alpha$ be of (β,δ) order $\rho(\beta, \delta, f) = \rho$. Suppose that conditions (1.5) and (1.8) are satisfied. Further, let*

$$\frac{d \log G(t,c)}{d \log t} \Rightarrow O(1) \text{ as } t \to \infty \text{ for any constant } c,\ 0 < c < \infty. \tag{2.1}$$

Then

$$\lim_{n\to\infty} \sup \frac{\beta(\lambda_n)}{\delta[\frac{\lambda_n - \lambda_{n-1}}{\log^+ | a_n/a_{n-1} | + \alpha(\lambda_n - \lambda_{n-1})}]} = \rho. \tag{2.2}$$

Proof. Let us denote the right hand side of (2.2) by A and let assume that $A < \infty$. Then for $\varepsilon > 0$, there exists integer n_0 such that

$$\beta(\lambda_n) < (A+\varepsilon)\delta[\frac{\lambda_n - \lambda_{n-1}}{\log^+ | a_n/a_{n-1} | + \alpha(\lambda_n - \lambda_{n-1})}],\ n > n_0$$

or, for all $n > n_0$,

$$\log \mid a_n/a_{n-1} \mid < \frac{\lambda_n - \lambda_{n-1}}{G(\lambda_n, 1/(A+\varepsilon))} - \alpha(\lambda_n - \lambda_{n-1})$$

where $G(\lambda_n, 1/(A+\varepsilon)) = \delta^{-1}[\beta(\lambda_n)/(A+\varepsilon)]$. Writing the above inequality for $n = n_0 + 1,\ n_0 + 2, \ldots, k$ and adding all the inequalities thus obtained, we get

$$\sum_{n=n_0+1}^{k} \log \mid a_n/a_{n-1} \mid < \sum_{n=n_0+1}^{k} \frac{\lambda_n - \lambda_{n-1}}{G(\lambda_n, 1/(A+\varepsilon))} - \alpha \sum_{n=n_0+1}^{k} (\lambda_n - \lambda_{n-1})$$

or,

$$\log \mid a_k \mid + \alpha\lambda_k < O(1) + \sum_{n=n_0+1}^{k} \frac{(\lambda_n - \lambda_{n-1})}{G(\lambda_n, 1/(A+\varepsilon))}.$$

To estimate the expression on the right hand side of above inequality, we define a step function $n(t) = \lambda_n$, $\lambda_n < t \leq \lambda_{n+1}$ and let $F(t) = 1/G(t, 1/(A+\varepsilon))$. Now rearranging the summation on right hand side, we have

$$\begin{aligned}
&\sum_{n=n_0+1}^{k} \frac{\lambda_n - \lambda_{n-1}}{G(\lambda_n, 1/(A+\varepsilon))} \\
&= \lambda_k F(\lambda_k) - \sum_{n=n_0+1}^{k-1} \lambda_n \{F(\lambda_{n+1}) - F(\lambda_n)\} - \lambda_{n_0-1} F(\lambda_{n_0}) \\
&= \lambda_k F(\lambda_k) - \int_{\lambda_{n_0}}^{\lambda_{k-1}} n(t)\, dF(t) - O(1).
\end{aligned}$$

Now

$$\begin{aligned}
-\int_{\lambda_{n_0}}^{\lambda_{k-1}} n(t)\, dF(t) &= \int_{\lambda_{n_0}}^{\lambda_{k-1}} \frac{n(t)}{G^2(t)} G'(t)\, dt \\
&= \int_{\lambda_{n_0}}^{\lambda_{k-1}} \frac{n(t)}{tG(t)} \cdot \frac{tG'(t)}{G(t)}\, dt,
\end{aligned}$$

where $G(t) = G(t, 1/(A+\varepsilon))$ and $G'(t) = \frac{dG(t)}{dt}$. Now by definition, $n(t)/t < 1$ and $F(t) = 1/G(t)$ is a decreasing function. Hence

$$\begin{aligned}
-\int_{\lambda_{n_0}}^{\lambda_{k-1}} n(t)\, dF(t) &< \int_{\lambda_{n_0}}^{\lambda_{k-1}} \frac{tG'(t)}{G(t)} \frac{dt}{G(t)} \\
&= [\int_{\lambda_{n_0}}^{\sqrt{\lambda_{k-1}}} + \int_{\sqrt{\lambda_{k-1}}}^{\lambda_{k-1}}] \frac{tG'(t)}{G(t)} \frac{dt}{G(t)}.
\end{aligned}$$

According to assumption (2.1), $\frac{tG'(t)}{G(t)} \to O(1)$ as $t \to \infty$. Hence we have

$$-\int_{\lambda_{n_0}}^{\lambda_{k-1}} n(t)dF(t) < O(1)\frac{[\sqrt{\lambda_{k-1}} - \lambda_{n_0}]}{G(\lambda_{n_0}, \frac{1}{A+\varepsilon})} + \frac{O(1)(\lambda_{k-1} - \sqrt{\lambda_{k-1}})}{G(\sqrt{\lambda_{k-1}}, \frac{1}{A+\varepsilon})}.$$

Since $G(t) \to \infty$ as $t \to \infty$, we finally get $-\int_{\lambda_{n_0}}^{\lambda_{k-1}} n(t)dF(t) < o(\lambda_{k-1})$. Hence we have, since $\{\lambda_k\}$.

$$\log^+ | a_k | + \alpha\lambda_k < O(1) + \lambda_k F(\lambda_k) + o(\lambda_k)$$

or

$$\frac{\log^+ | a_k | + \alpha\lambda_k}{\lambda_k} < \frac{1}{G(\lambda_k, \frac{1}{A+\varepsilon})} + o(1), \qquad k > n_0$$

or, using the definition of $G(\lambda_k, \frac{1}{A+\varepsilon})$, we have

$$\frac{\beta(\lambda_k)}{\delta[\frac{\lambda_k}{\log^+ | a_k | + \alpha\lambda_k}]} < A + \varepsilon, \qquad k > n_0.$$

Now proceeding to limits as $k \to \infty$, we get in view of (1.6),

$$\rho(\beta, \delta, f) \leq A. \tag{2.3}$$

The above inequality obviously holds if $A = \infty$.

To obtain the reverse inequality, we use the condition (1.8). Then, for any $n > n_0$,

$$\begin{aligned} \log | a_{n_0}/a_n | &= \log | \frac{a_{n_0}}{a_{n_0+1}} \cdots \frac{a_{n-1}}{a_n} | \\ &= \sum_{k=n_0}^{n-1} (\lambda_{k+1} - \lambda_k)\Psi(k) \\ &\leq \Psi(n-1) \cdot (\lambda_n - \lambda_{n_0}) \end{aligned}$$

since $\Psi(k)$ is a non decreasing function of k. Hence we have

$$\log^+ | a_n | \geq O(1) + \frac{\lambda_n - \lambda_{n_0}}{\lambda_n - \lambda_{n-1}} \log^+ | a_n/a_{n-1} |$$

or

$$\alpha + \frac{\log^+ | a_n/a_{n-1} |}{\lambda_n - \lambda_{n-1}} \leq \alpha + \frac{\log^+ | a_n |}{\lambda_n} - o(1) \tag{2.4}$$

i.e.
$$\frac{\lambda_n}{\alpha\lambda_n + \log^+ | a_n |} \leq \frac{\lambda_n - \lambda_{n-1}}{\alpha(\lambda_n - \lambda_{n-1}) + \log^+ | a_n/a_{n-1} |}.$$

Since δ is an increasing function, hence we get

$$\lim_{n\to\infty} \sup \frac{\beta(\lambda_n)}{\delta[\frac{(\lambda_n - \lambda_{n-1})}{\log^+ | a_n/a_{n-1} | + \alpha(\lambda_n - \lambda_{n-1})}]} \leq \lim_{n\to\infty} \sup \frac{\beta(\lambda_n)}{\delta[\frac{\lambda_n}{\log^+ | a_n | + \alpha\lambda_n}]}$$

i.e. $A \leq \rho(\beta, \delta, f)$.

Combining the above inequality with (2.3), we get (2.2). This completes proof of Theorem 1.

Next we prove

Theorem 2. *Let $f(s) \in D_\alpha$ be of lower (β, δ) order $\lambda(\beta, \delta, f) = \lambda$. If $f(s)$ satisfies (1.9) then*

$$(2.5) \quad \lambda = \max_{\{n_m\}} \lim_{m\to\infty} \inf \frac{\beta(n_{m-1})}{\delta[(\lambda_{n_m} - \lambda_{n_{m-1}})/\log^+ | a_{n_m}/a_{n_{m-1}} | + \alpha(\lambda_{n_m} - \lambda_{n_{m-1}})]}$$

where maximum on the right hand side of (2.5) is taken over all increasing sequences of natural numbers $\{n_m\}$ such that $n_m \to \infty$ as $m \to \infty$.

Proof. Let the limit inferior on the right hand side of (2.5) be denoted by B. Clearly $0 \leq B \leq \infty$. First let $0 < B < \infty$. Then for any $\varepsilon > 0$ and all integers $k > N$, we have

$$\beta(\lambda_{n_{k-1}}) > (B - \varepsilon)\delta[\frac{\lambda_{n_k} - \lambda_{n_{k-1}}}{\log^+ | a_{n_k}/a_{n_{k-1}} | + \alpha(\lambda_{n_k} - \lambda_{n_{k-1}})}]$$

or

$$\log | a_{n_k}/a_{n_{k-1}} | > \frac{\lambda_{n_k} - \lambda_{n_{k-1}}}{G(\lambda_{n_{k-1}}, 1/(B - \varepsilon))} - \alpha(\lambda_{n_k} - \lambda_{n_{k-1}})$$

where $G(\lambda_{n_{k-1}}, 1/(B - \varepsilon)) = \delta^{-1}[\beta(\lambda_{n_{k-1}})/(B - \varepsilon)]$.

Writing above inequality for $k = N, N + 1, \cdots, m$ and adding all the inequalities thus obtained, we get

$$\sum_{k=N}^{m} \log | a_{n_k}/a_{n_{k-1}} | > \sum_{k=N}^{m} \frac{\lambda_{n_k} - \lambda_{n_{k-1}}}{G(\lambda_{n_{k-1}}, 1/(B - \varepsilon))} - \alpha(\lambda_{n_m} - \lambda_{n_{N-1}})$$

or,

$$\log | a_{n_m} | + \alpha\lambda_{n_m} > O(1) + \sum_{k=N}^{m} \frac{\lambda_{n_k} - \lambda_{n_{k-1}}}{G(\lambda_{n_{k-1}}, 1/(B - \varepsilon))}.$$

Since $1/G(t)$ is a decreasing function therefore

$$\log^+ | a_{n_m} | + \alpha\lambda_{n_m} > O(1) + \frac{\lambda_{n_m} - \lambda_{n_{N-1}}}{G(\lambda_{n_{m-1}}, 1/(B - \varepsilon))}$$

or

$$G(\lambda_{n_{m-1}}, 1/(B-\varepsilon)) > \frac{\lambda_{n_m}}{\log^+ | a_{n_m} | + \alpha\lambda_{n_m}} + o(1), \qquad m > N.$$

On using the definition of $G(\lambda_{n_{m-1}}, 1/(B-\varepsilon))$, and proceeding to limits, we have

$$\begin{aligned} B &\leq \liminf_{m\to\infty} \frac{\beta(\lambda_{n_{m-1}})}{\delta[\lambda_{n_m}/(\log^+ | a_{n_m} | + \alpha\lambda_{n_m})]} \\ &\leq \lambda(\beta,\delta,f) \qquad \text{(from Lemma 6, [2]).} \end{aligned}$$

Since $\{n_m\}$ was any arbitrary sequence of positive integers, we get

(2.6)

$$\lambda(\beta,\delta,f) \geq \max_{\{n_m\}} \liminf_{m\to\infty} \frac{\beta(\lambda_{n_{m-1}})}{\delta[(\lambda_{n_m} - \lambda_{n_{m-1}})/\{\log^+ | a_{n_m}/a_{n_{m-1}} | + \alpha(\lambda_{n_m} - \lambda_{n_{m-1}})\}]}.$$

To prove the reverse inequality, let the range of the rank $N(\sigma)$ be the sequence $\{n_k\}$. Also, let $\Psi(n)$ denote the jump points of $N(\sigma)$. Then

$$\begin{aligned} N(\sigma) &= n_k \text{ for } \Psi(n_k) \leq \sigma < \Psi(n_{k+1}), \qquad k = 1, 2, \cdots, \text{ where} \\ \Psi(n_k) &= \frac{\log | a_{n_{k-1}}/a_{n_k} |}{\lambda_{n_k} - \lambda_{n_{k-1}}}. \end{aligned}$$

Now, under the assumption (1.9) we have

$$\begin{aligned} \lambda(\beta,\delta,f) &= \liminf_{\sigma\to\alpha} \frac{\beta(\lambda_{N(\sigma)})}{\delta\{(1-e^{\sigma-\alpha})^{-1}\}} \\ &= \liminf_{k\to\infty} \frac{\beta(\lambda_{n_k})}{\delta[\{1-\exp(\Psi(n_{k+1})-\alpha)\}^{-1}]}. \end{aligned}$$

It can be easily seen that $1 - e^{\sigma-\alpha} \simeq \alpha - \sigma$ as $\sigma \to \alpha$. Hence in view of property (ii) of δ, we get

$$\begin{aligned} \lambda(\beta,\delta,f) &= \liminf_{k\to\infty} \frac{\beta(\lambda_{n_{k-1}})}{\delta[\{\alpha - \Psi(n_k)\}^{-1}]} \\ &= \liminf_{k\to\infty} \frac{\beta(\lambda_{n_{k-1}})}{\delta[(\lambda_{n_k} - \lambda_{n_{k-1}})/\{\log^+ | a_{n_k}/a_{n_{k-1}} | + \alpha(\lambda_{n_k} - \lambda_{n_{k-1}})\}]}. \end{aligned}$$

Hence

$$\lambda(\beta,\delta,f) \leq \max_{\{n_m\}} \liminf_{m\to\infty} \frac{\beta(\lambda_{n_{m-1}})}{\delta[\lambda_{n_m} - \lambda_{n_{m-1}})/\{\log^+ | a_{n_m} | + \alpha(\lambda_{n_m} - \lambda_{n_{m-1}})\}]}.$$

Combining the above inequality with (2.6) we get (2.5). This completes the proof of Theorem 2.

3. In this section, we shall obtain coefficient characterization of the type T and lower type t as defined by (1.11). We prove

Theorem 3. *Let $f(s) \in D_\alpha$ be of $(\beta, \delta, \gamma, f)$-type T and lower $(\beta, \delta, \gamma, f)$ type t. Suppose that the condition (1.8) and (1.12) are satisfied. Then*

$$T = \lim_{n\to\infty} \sup \frac{\beta(\lambda_n)}{\delta[\{\gamma(\frac{\lambda_n - \lambda_{n-1}}{\log^+ | a_n/a_{n-1} | + \alpha(\lambda_n - \lambda_{n-1})})\}^\rho]}. \tag{3.1}$$

Proof. Let the expression on the right hand side of (3.1) be denoted by Q. Clearly $0 \leq Q \leq \infty$. First let $0 < Q < \infty$. Then for $\varepsilon > 0$ we have for all sufficiently large $n > N(\varepsilon)$,

$$\beta(\lambda_n) < (Q+\varepsilon)\delta[\{\gamma(\frac{\lambda_n - \lambda_{n-1}}{\log^+ | a_n/a_{n-1} | + \alpha(\lambda_n - \lambda_{n-1})})\}^\rho]$$

or, for all $n > N$,

$$\log | a_n/a_{n-1} | < \frac{\lambda_n - \lambda_{n-1}}{G(\lambda_n, 1/(Q+\varepsilon), \rho)} - \alpha(\lambda_n - \lambda_{n-1}),$$

where $G(\lambda_n, 1/Q+\varepsilon, \rho) = \gamma^{-1}[\{\delta^{-1}(\frac{\beta(\lambda_n)}{Q+\varepsilon})\}^{1/\rho}]$.

Writing the above inequality for $n = N+1, N+2, \cdots, k$ and adding we get

$$\sum_{n=N+1}^{k} \log | a_n/a_{n-1} | < \sum_{n=N+1}^{k} \frac{(\lambda_n - \lambda_{n-1})}{G(\lambda_n, 1/Q+\varepsilon, \rho)} - \alpha(\lambda_k - \lambda_N)$$

or,

$$\log | a_k | + \alpha\lambda_k < O(1) + \sum_{n=N+1}^{k} \frac{\lambda_n - \lambda_{n-1}}{G(\lambda_n, \frac{1}{Q+\varepsilon}, \rho)}.$$

Let us write $F(t) = \frac{1}{G(\lambda_t, 1/Q+\varepsilon, \rho)}$ and $n(t) = \lambda_n$ for $\lambda_n < t \leq \lambda_{n+1}$. Then we have $\frac{n(t)}{t} < 1$. We consider

$$\begin{aligned}\sum_{n=N+1}^{k} \frac{\lambda_n - \lambda_{n-1}}{G(\lambda_n, 1/Q+\varepsilon, \rho)} &= \lambda_k F(\lambda_k) - \lambda_N F(\lambda_{N+1}) - \sum_{n=N+1}^{k-1} \lambda_n [F(\lambda_{n+1}) - F(\lambda_n)] \\ &= \lambda_k F(\lambda_k) - \lambda_N F(\lambda_{N+1}) - \int_{\lambda_{N+1}}^{\lambda_{k-1}} n(t) dF(t).\end{aligned}$$

As in the proof of Theorem 1, we can easily show that

$$-\int_{\lambda_{N+1}}^{\lambda_{k-1}} n(t) dF(t) = o(\lambda_{k-1}).$$

Hence we have for all large k,

$$\log^+ \mid a_k \mid + \alpha\lambda_k < \frac{\lambda_k}{G(\lambda_k, 1/Q+\varepsilon, \rho)}(1+o(1)), \qquad k > N,$$

or, using the definition of $G(\lambda_k, 1/Q+\varepsilon, \rho)$, we have

$$\frac{\beta(\lambda_k)}{\delta[\{\gamma(\lambda_k/\log^+ \mid a_k \mid +\alpha\lambda_k)\}^\rho]} < Q+\varepsilon, \qquad k > N.$$

Now proceeding to limits as $k \to \infty$, we get in view of (1.13),

$$T = T(\beta, \delta, \gamma, \rho, f) \leq Q. \tag{3.2}$$

The above inequality obviously holds if $Q = \infty$. To prove the reverse inequality, we have from (2.4),

$$\alpha + |\frac{\log^+ \mid a_n/a_{n-1} \mid}{\lambda_n - \lambda_{n-1}} \leq \alpha + \frac{\log^+ \mid a_n \mid}{\lambda_n} - o(1)$$

or,

$$\frac{\alpha(\lambda_n - \lambda_{n-1}) + \log^+ \mid a_n/a_{n-1} \mid}{\lambda_n - \lambda_{n-1}} \leq \frac{\log^+ \mid a_n \mid +\alpha\lambda_n}{\lambda_n}$$

or,

$$\delta[\{\gamma(\frac{\lambda_n - \lambda_{n-1}}{\alpha(\lambda_n - \lambda_{n-1}) + \log^+ \mid a_n/a_{n-1} \mid})\}^\rho] \geq \delta[\{\gamma(\frac{\lambda_n}{\log^+ \mid a_n \mid +\alpha\lambda_n})\}^\rho].$$

Therefore

$$\frac{\beta(\lambda_n)}{\delta[\{\gamma(\frac{\lambda_n}{\log^+ \mid a_n \mid +\alpha\lambda_n})\}^\rho]} \geq \frac{\beta(\lambda_n)}{\delta[\{\gamma(\frac{(\lambda_n - \lambda_{n-1})}{\log^+ \mid a_n/a_{n-1} \mid +\alpha(\lambda_n - \lambda_{n-1}})\}^\rho]}$$

which gives, on taking limits

$$T \geq Q \tag{3.3}$$

Again this inequality holds if $Q = 0$. Now combining (3.2) with (3.3), we get (3.1). This completes proof of Theorem 3.

Lastly we prove

Theorem 4. *Let $f(s) \in D_\alpha$ of be of lower $(\beta, \delta, \gamma, \rho)$ type $t(\beta, \delta, \gamma, \rho, f)$ and satisfy condition (1.12). Then for any increasing sequence $\{n_k\}$ of positive integers, we have*

$$t \geq \liminf_{k\to\infty} \frac{\beta(\lambda_{n_{k-1}})}{\delta[\{\gamma(\frac{\lambda_{n_k} - \lambda_{n_{k-1}}}{\log^+ \mid a_{n_k}/a_{n_{k-1}} \mid +\alpha(\lambda_{n_k} - \lambda_{n_{k-1}})})\}^\rho]}. \tag{3.4}$$

Proof. Let us denote the right hand side of (3.4) by q. Clearly $0 \leq q \leq \infty$. First let $0 < q$. Then for all $k > N$, $\varepsilon > 0$, we have

$$G(\lambda_{n_{k-1}}, \frac{1}{q-\varepsilon}, \rho) \; > \; \frac{\lambda_{n_k} - \lambda_{n_{k-1}}}{\log^+ \mid a_{n_k}/a_{n_{k-1}} \mid + \alpha(\lambda_{n_k} - \lambda_{n_{k-1}})}$$

where

$$G(\lambda_{n_{k-1}}, \frac{1}{q-\varepsilon}, \rho) \; = \; \gamma^{-1}\left[\{\delta^{-1}(\frac{\beta(\lambda_{n_{k-1}})}{q-\varepsilon})\}^{1/\rho}\right]$$

or,

$$log^+ \mid a_{n_k}/a_{n_{k-1}} \mid > \frac{\lambda_{n_k} - \lambda_{n_{k-1}}}{G(\lambda_{n_{k-1}}, 1/q-\varepsilon, \rho)} \; - \; \alpha(\lambda_{n_k} - \lambda_{n_{k-1}}).$$

Writing the above inequality for $m = N, N+1, \cdots, k$ and adding all these inequalities, we get

$$\log^+ \mid a_{n_k} \mid + \alpha\lambda_{n_k} \; > \; \sum_{m=N}^{k} \frac{\lambda_{n_m} - \lambda_{n_{m-1}}}{G(\lambda_{n_{m-1}}, 1/q-\varepsilon, \rho)} \; + \; O(1).$$

As in the proof of Theorem 2. we have

$$\log^+ \mid a_{n_k} \mid + \alpha\lambda_{n_k} \; > \; \frac{\lambda_{n_k} - \lambda_{n_{N-1}}}{G(\lambda_{n_{k-1}}, 1/q-\varepsilon, \rho)} \; + \; O(1)$$

or,

$$G(\lambda_{n_{k-1}}, 1/q-\varepsilon, \rho) \; > \; \frac{\lambda_{n_k}}{\log^+ \mid a_{n_k} \mid + \alpha\lambda_{n_k}} \; + \; o(1).$$

Hence proceding to limits, we get on using (1.14),

$$q \; \leq \; \liminf_{k\to\infty} \frac{\beta(\lambda_{n_{k-1}})}{\delta\left[\{\gamma(\lambda_{n_k}/\log^+ \mid a_{n_k} \mid + \alpha\lambda_{n_k})\}^{\rho}\right]}$$

or,

$$t \; \geq \; \liminf_{k\to\infty} \frac{\beta(\lambda_{n_{k-1}})}{\delta\left[\{\gamma(\lambda_{n_k} - \lambda_{n_{k-1}}/(\log^+ \mid a_{n_k}/a_{n_{k-1}} \mid + \alpha(\lambda_{n_k} - \lambda_{n_{k-1}}))\}^{\rho}\right]}.$$

This proves Theorem 4.

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Department of mathematics, University of Roorkee, ROORKEE, 247667, India.