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A FIXED POINT THEOREM FOR SOME NON-SELF-MAPPINGS

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Abstract. A fixed point theorem is proved for continuous mappings from a nonempty closed subset K, of a Banach space X, into X, and which satisfies contractive definition definition (3) and property (a) below.

The Main Theorem.

The following result was established in [5]: Let X be a Banach space, K a nonempty closed subset of X. Let $T : K \to X$ satisfy the following contractive condition on K:

There exists a constant h, $0 < h < 1$ such that, for each $x, y \in K$,

$$(1) \quad d(Tx, Ty) \leq h \max.\{d(x,y)/2, \quad d(x,Tx), d(y,Ty), [d(x,Ty) + d(y,Tx)]/q\},$$

where q is any real number satisfying $q \geq 1 + 2h$. Suppose that T has the additional property:

(a) for each $x \in \partial K$, the boundary of $K, Tx \in K$, then T has a unique fixed point.

In proving his theorem [5], Rhoades constructed two sequences $\{x_n\}$ and $\{x'_n\}$ as follows:

Definition. Let $x_0 \in K$. Define $x'_1 = Tx_0$. If $x'_1 \in K$, set $x_1 = x'_1$. If $x'_1 \notin K$, choose $x_1 \in \partial K$ so that $d(x_0, x_1) + d(x_1, x'_1) = d(x_0, x'_1)$. Set $x'_2 = Tx_1$. If $x_2 \in K$, set $x_2 = x'_2$. If not, choose $x_2 \in \partial K$ so that $d(x_1, x_2) + d(x_2, x'_2) = d(x_1, x'_2)$. Continuing in this manner, we obtain $\{x_n\}$, $\{x'_n\}$ satisfying:

(i) $x'_{n+1} = Tx_n$,

(ii) $x_n = x'_n$ if $x'_n \in K$, and

(iii) $x_n \in \partial K$ and $d(x_{n-1}, x_n) + d(x_n, x'_n) = d(x_{n-1}, x'_n)$ if $x'_n \notin K$.

Let $P = \{x_i \in \{x_n\} : x_i = x'_i\}$ and $Q = \{x_i \in \{x_n\} : x_i \notin x'_i\}$. The sequence $\{x_n\}$ will be referred to as the *general orbit* of T at x_0.

Rhoades [5], proceeded in his proof by showing that for any $x_0 \in K$, the general orbit of T at x_0 is a Cauchy sequence that converges to the unique fixed point of T. On

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the other hand, in [3], the author has shown that if we require T to be continuous and K compact then we may replace condition (1) on T by the following weaker condition:

$$\text{For all} \quad x, y \in K, \; x \neq y,$$

$$(2) \qquad d(Tx,Ty) < \max.\{d(x,y)/2,\; d(x,Ty),\; d(y,Ty), [d(x,Ty)+d(y,Tx)]/q\}$$

where $q \geq 3$ and still conclude that T has a unique fixed point.

In this paper, we prove a fixed point theorem for the mapping T satisfying (a) and the following condition:

Let R^+ denote the set of non-negative real numbers and let $h : R^+\backslash\{0\} \to (0,1)$ be a decreasing function. Suppose that for all $x \neq y$, $x, y \in K$:

$$(3) \; d(Tx,Ty) \leq h(d(x,y))\cdot\max.\{d(x,y)/2, d(x,Tx),\; d(y,Ty), [d(x,Ty)+d(y,Tx)]/q\},$$

where q is any real number satisfying $q \geq 1 + 2h(d(x,y))$.Observe that the above three conditions on T are related as follows : (1) $\Rightarrow$ (3) $\Rightarrow$ (2).Our results show that general orbit for the mapping T satisfying (a) and (3) at any point $x_0 \in K$ is a Cauchy sequence. Moreover, under the additional assumption that T is continuous we may conclude that this Cauchy sequence converges to a unique fixed point of T.

Theorem. *Let X be a Banach space, K a nonempty closed subset of $X, T : K \to X$ a continuous mapping satisfying* (3) *on K. If T has property* (a) *then T has a unique fixed point in K.*

Proof. We will use the following notation: $\tau_n = d(x_n, x_{n+1})$ and $s_n = d(x_n, x_{n+2})$. It is easy to see that $s_n > 0$ for each n. Moreover, following the proof of (Theorem 3.1,[2]) we may assume that $\tau_n > 0$ for each n.

Step I: We first wish to estimate $d(x_n, x_{n+1})$.
Case I. $\quad x_n, x_{n+1} \in P$.

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq h(d(x_{n-1}, x_n)) \cdot \max.\{d(x_{n-1}, x_n)/2, d(x_{n-1}, Tx_{n-1}), \\ &\quad d(x_n, Tx_n), [d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})]/q\} \\ (A1) \qquad &= h(\tau_{n-1}) \cdot \tau_{n-1}. \end{aligned}$$

Case II. $\quad x_n \in P$, $x_{n+1} \in Q$.

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x'_{n+1}) = d(Tx_{n-1}, Tx_n) \\ (A2) \qquad &\leq h(\tau_{n-1}) \cdot \tau_{n-1}. \end{aligned}$$

Case III. $\quad x_n \in Q$, $x_{n+1} \in P$. Since $x_n \in Q$ and is a convex combination of x_{n-1} and x'_n, it follows that $d(x_n, x_{n+1}) \leq \max.\{d(x_{n-1}, x_{n+1}), d(x'_n, x_{n+1})\}$. If $d(x_{n-1}, x_{n+1}) \leq$

$d(x'_n, x_{n+1})$, then $d(x_n, x_{n+1}) \leq d(x'_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq h(\tau_{n-1}) \cdot \max.\{\tau_{n-1}/2, d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), [d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})]/q\} = h(\tau_{n-1}) \cdot \max.\{d(x_{n-1}, x'_n), [d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)]/q\}$.

So, in the case where $d(x_{n-1}, x'_n)$ is the maximum, we get:

$$\begin{aligned} d(x_n, x_{n+1}) &\leq h(\tau_{n-1}) \cdot d(x_{n-1}, x'_n) \\ &\leq h(\tau_{n-1} \cdot h(\tau_{n-2}) \cdot \tau_{n-2}. \end{aligned} \tag{A3}$$

(by Case II, since $x_n \in Q$ implies that $x_{n-1} \in P$). On the other hand, if $[d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)]/q$ is the maximum, we get:

$$\begin{aligned} d(x_n, x_{n+1}) &\leq h(\tau_{n-1}) \cdot [d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)]/q \\ &\leq h(\tau_{n-1}) \cdot [\tau_{n-1} + \tau_n + d(x_n, x'_n)]/q \\ &= h(\tau_{n-1}) \cdot [d(x_{n-1}, x'_n) + \tau_n]/q \end{aligned}$$

Therefore, $[1 - h(\tau_{n-1})/q] \cdot \tau_n \leq [h(\tau_{n-1})/q] \cdot d(x_{n-1}, x'_n)$
and thus

$$\begin{aligned} \tau_n &\leq \frac{h(\tau_{n-1})}{q - h(\tau_{n-1})} \cdot d(x_{n-1}, x'_n) \\ &\leq h(\tau_{n-1}) \cdot d(x_{n-1}, x'_n). \end{aligned}$$

Again, we conclude that:

$$\tau_n \leq h(\tau_{n-1}) \cdot h(\tau_{n-2}) \cdot \tau_{n-2}. \tag{A4}$$

Finally, we considew the possibility that $d(x'_n, x_{n+1}) < d(x_{n-1}, x_{n+1})$, here we have,

$$\begin{aligned} \tau_n &\leq d(x_{n-1}, x_{n+1}) = d(Tx_{n-2}, Tx_n) \\ &\leq h(s_{n-2}) \cdot \max.\{s_{n-2}/2,\ d(x_{n-2}, Tx_{n-2}), \\ &\qquad d(x_n, Tx_n), [d(x_{n-2}, Tx_n) + d(x_n, Tx_{n-2})]/q\}. \end{aligned} \tag{*}$$

Note that

$$\begin{aligned} s_{n-2}/2 &\leq [d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)]/2 \\ &\leq \max.\{\tau_{n-2}, \tau_{n-1}\} \\ &= \tau_{n-2}. \end{aligned}$$

So, we may conclude either,

$$\tau_n \leq h(s_{n-2}) \cdot \tau_{n-2} \tag{A5}$$

or, in case the maximum of the right hand side of (*) is $[d(x_{n-2}, x_{n+1}) + d(x_{n-1}, x_n)]/q$, it follows that $d(x_{n-1}, x_{n+1}) \leq h(s_{n-2}) \cdot [d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})]/q$,

i.e., $[q - h(s_{n-2})] \cdot d(x_{n-1}, x_{n+1}) \le h(s_{n-2}) \cdot [1 + h(\tau_{n-2})] \cdot d(x_{n-2}, x_{n-1})$ (since $d(x_n, x_{n-1}) \le d(x_{n-1}, x'_n) \le h(\tau_{n-2}) \cdot \tau_{n-2}$).
Therefore, $d(x_{n-1}, x_{n+1}) \le \frac{h(s_{n-2})}{[1+h(s_{n-2})]} \cdot [1 + h(\tau_{n-2})] \cdot \tau_{n-2}$ and thus, $\tau_n \le \frac{h(s_{n-2})}{[1+h(s_{n-2})]} \cdot [1 + h(\tau_{n-2})] \cdot \tau_{n-2}$. Now, if $s_{n-2} \ge \tau_{n-2}$, then $h(s_{n-2}) \le h(\tau_{n-2})$ and consequently, $\frac{h(s_{n-2})}{1+h(s_{n-2})} \le \frac{h(\tau_{n-2})}{1+h(\tau_{n-2})}$. It follows:

$$(A6) \qquad \tau_n \le h(\tau_{n-2}) \cdot \tau_{n-2}.$$

On the other hand, if $s_{n-2} < \tau_{n-2}$, then $h(s_{n-2}) \ge h(\tau_{n-2})$ and thus $1 + h(s_{n-2}) \ge 1 + h(\tau_{n-2})$, or $([1 + h(\tau_{n-2})]/[1 + h(s_{n-2})]) \le 1$ and thus we get,

$$(A7) \qquad \tau_n \le h(s_{n-2}) \cdot \tau_{n-2}.$$

Finally, using the seven conclusions (A1)-(A7), we may conclude that for $n = 2, 3, 4, \ldots$, we have

$$\tau_n < \tau_{n-1}$$

or

$$(A) \qquad \tau_n < \tau_{n-2}.$$

Step II. We will prove that the sequence $\{\tau_n\}_{n=0}^{\infty}$ converges to 0. To do that, we consider two cases. In the first one we assume that $\{x_n\}$ has a subsequence $\{x_{n(k)}\}$ with the property that $x_{n(k)+1}$ and $x_{n(k)+2} \in P$. Here we consider the sequence $\tau_{n(k)}$. By (A) we observe that $\tau_{n(k)} \le \tau_{n(k-1)+1}$ or $\tau_{n(k)} \le \tau_{n(k-1)+2}$. Noting that $x_{n(k-1)+1}$ and $x_{n(k-1)+2} \in P$, it follows that $\tau_{n(k-1)+2} < \tau_{n(k-1)+1} < \tau_{n(k-1)}$. So, we may conclude that for $k \ge 2$, $\tau_{n(k)} < \tau_{n(k-1)}$ and thus $\tau_{n(k)} \to \tau$. We show that $\tau = 0$. Observe that for $k = 1, 2, 3, \ldots$, we have,

$$\tau_{n(k+1)} \le d(x'_{n(k)+1}, x'_{n(k)+2}) < \tau_{n(k)}$$

and thus $\lim_{k\to\infty} d(x'_{n(k)+1}, x'_{n(k)+2}) = \tau$. If $\tau > 0$, then $d(x'_{n(k)+1}, x'_{n(k)+2}) \le h(\tau_{n(k)}) \cdot \tau_{n(k)}$ and as $k \to \infty$ we obtain $\tau \le h(\tau) \cdot \tau < \tau$. Contradiction. Moreover, for j sufficiently large, $\exists k = k(j)$ such that $n(k) \le j \le n(k+1)$ and thus $0 < \tau_j \le \tau_{n(k)}$. Since $\tau_{n(k)} \to 0$, we conclude that $\lim_{j\to\infty} \tau_j = 0$. In the second case, we assume that eventually the sequence $\{x_n\}$ cannot have two consecutive points that are in P, ie., $\exists$ a positive integer N such that for every $n \ge N$, if $x_n \in P$ then $x_{n+1} \in Q$, Assume that $x_n(i) \in Q$ for $i = 1, 2, 3, \ldots$, where $n(i) + 2 = n(i+1)$ and $n(i) - 2 = n(i-1)$, and consider the sequence $\{\tau_{n(i)}\}$. Note that $\tau_{n(i)}$ is convergent, and suppose that $\tau_{n(i)} \to \tau$. By (Observation 2.1, [2]), we may assume that $\exists$ a subsequence of $\{x_{n(i)}\}$ denoted by $x_{n(t)}$ such that either,

(B) for $t = 1, 2, 3, \ldots, \tau_{n(t)} \le d(x_{n(t)+1}, x'_{n(t)})$, or

(C) for $t = 1, 2, 3, \ldots, \tau_{n(t)} \le d(x_{n(t)+1}, x_{n(t)-1})$.

If Case (B) occurs, then by (A3) and (A4) we have:

(D) $\tau_{n(t)} < d(x_{n(t)-1}, x'_{n(t)}) < \tau_{n(t)-2}$, it follows that $\lim \tau_{n(t)} = \lim d(x'_{n(t)-1}, x'_{n(t)}) = \tau = 0$, and $\lim \tau_n = 0$.

Finally, we consider the possibility that (C) occurs. Then by (A5), (A6) and (A7), we may assume that for $t = 1, 2, 3, \ldots$, we have,

$$(E1) \qquad \tau_{n(t)} \le h(\tau_{n(t)-2}) \cdot \tau_{n(t)-2},$$

or

$$(E2) \qquad \tau_{n(t)} \le h(s_{n(t)-2}) \cdot \tau_{n(t)-2}.$$

In the case (E1) occurs, as $t \to \infty$, we get $\tau \le h(\tau) \cdot \tau < \tau$, which is absurd, and thus we conclude that $\tau = 0$. On the other hand if (E2) occurs, without loss of generality, and since $\{s_{n(t)-2}\}$ is bounded, we may assume that $s_{n(t)-2} \to \rho$. If $\rho > 0$, then as $t \to \infty$, we get $\tau \le h(\rho/2) \cdot \tau < \tau$. Contradiction. To show that $\tau = 0$, we note that:

$$\tau_{n(t)-2} - d(x_{n(t)-2}, x_{n(t)}) \le d(x_{n(t)-1}, x_{n(t)}) < d(x_{n(t)-1}, x'_{n(t)}) < \tau_{n(t)-2}.$$

Hence $\lim d(x'_{n(t)-1}, x'_{n(t)}) = \tau$ and we may conclude as we did in the previous two cases that $\lim_{n\to\infty} \tau_n = 0$. So we have:

$$(F) \qquad \lim_{n\to\infty} \tau_n = 0.$$

Step III. We prove that $\{x_n\}$ is a Cauchy sequence. For if it is not Cauchy, then by well-ordering principle there exists $\epsilon > 0$ and two subsequences $\{p(n)\}, \{l(n)\}$ such that for every $n = 0, 1, 2, 3, \ldots$, we find that $p(n) > l(n) > n, d(x_{p(n)}, x_{l(n)}) \ge \epsilon$ and $d(x_{p(n)-1}, x_{l(n)}) < \epsilon$. Put $g_n = d(x_{p(n)}, x_{l(n)})$. For each $n \ge 0$, we have:

$$\begin{aligned} \varepsilon \le g_n &\le d(x_{p(n)-1}, x_{p(n)}) + d(x_{p(n)-1}, x_{\ell(n)}) \\ &< \tau_{p(n)-1} + \varepsilon. \end{aligned}$$

Since $\tau_n \to 0$, it follows that $g_n \to \varepsilon$. It has been shown in details in Assad [1] that (F) allows us to conclude that:

$$\begin{aligned} &\lim d(x_{p(n)+1}, x_{\ell(n)-1}) = \lim d(x_{p(n)-1}, x_{\ell(n)+1}) \\ &= \lim d(x_{p(n)+1}, x_{\ell(n)+1}) = \lim d(x_{p(n)+1}, x_{\ell(n)}) \\ &= \lim d(x_{p(n)}, x_{\ell(n)-1}) = \lim d(x_{p(n)-1}, x_{\ell(n)-1}) \\ &= \lim d(x_{p(n)}, x_{\ell(n)+1}) = \lim d(x_{p(n)-1}, x_{\ell(n)}) = \varepsilon. \end{aligned}$$

Next, we consider the following four possibilites :

$$(G1) \qquad x_{p(n)+1}, \in P \text{ and } x_{\ell(n)+1} \in P, \text{ then}$$

$$\begin{aligned} d(x_{p(n)+1}, x_{\ell(n)+1}) &= d(Tx_{p(n)}, Tx_{\ell(n)}) \\ &\leq h(g_n) \cdot \max.\{g_n/2, \tau_{p(n)}, \tau_{\ell(n)}, \\ &\qquad [d(x_{p(n)}, Tx_{\ell(n)}) + d(x_{\ell(n)}, Tx_{p(n)})]/q\}. \end{aligned}$$

$(G2)$ $x_{p(n)+1} \in P$ and $x_{\ell(n)+1} \in Q$, then $x_{\ell(n)} \in P$ and

$$\begin{aligned} d(x_{p(n)+1}, x_{\ell(n)}) &= d(Tx_{p(n)}, Tx_{\ell(n)-1}) \\ &\leq h(d(x_{p(n)}, x_{\ell(n)-1})) \cdot \max.\{d(x_{p(n)}, x_{\ell(n)-1})/2, \tau_{p(n)}, \tau_{\ell(n)-1}, \\ &\qquad [g_n + d(x_{\ell(n)-1}, x_{p(n)+1})]/1 + 2h(d(x_{p(n)}, x_{\ell(n)-1}))\}. \end{aligned}$$

$(G3)$ $x_{p(n)+1} \in Q$ and $x_{\ell(n)+1} \in P$, then $x_{p(n)} \in P$ and

$$\begin{aligned} d(x_{p(n)}, x_{\ell(n)+1}) &= d(Tx_{p(n)-1}, Tx_{\ell(n)}) \\ &\leq h(d(x_{p(n)-1}, x_{\ell(n)})) \cdot \max.\{d(x_{p(n)-1}, x_{\ell(n)})/2, \tau_{p(n)-1}, \tau_{\ell(n)}, \\ &\qquad [g_n + d(x_{p(n)-1}, x_{\ell(n)+1})]/1 + 2h(d(x_{p(n)-1}, x_{\ell(n)}))\}. \end{aligned}$$

$(G4)$ $x_{p(n)+1} \in Q$ and $x_{\ell(n)+1} \in Q$, then $x_{p(n)}$ and $x_{\ell(n)} \in P$ and,

$$\begin{aligned} d(x_{p(n)}, x_{\ell(n)}) &= d(Tx_{p(n)-1}, Tx_{\ell(n)-1}) \\ &\leq h(d(x_{p(n)-1}, x_{\ell(n)-1})) \cdot \max.\{d(x_{p(n)-1}, x_{\ell(n)-1})/2, \tau_{p(n)-1}, \tau_{\ell(n)-1}, \\ &\qquad [d(x_{p(n)-1}, x_{\ell(n)}) + d(x_{\ell(n)-1}, x_{p(n)})]/1 + 2h(d(x_{p(n)-1}, x_{\ell(n)-1}))\}. \end{aligned}$$

Each of these four Cases: (G1), (G2), (G3) and (G4) leads to the conclusion that $\varepsilon \leq \frac{2h(\varepsilon/2)}{1+2h(\varepsilon/2)} \cdot \varepsilon < \varepsilon$ as $n \to \infty$, which is absurd. Therefore, we conclude that the sequence $\{x_n\}$ is a Cauchy sequence, and by completeness of X, we conclude that the sequence converges to a point in K. Let $\lim_{n\to\infty} x_n = z$.

Finally, we will show that z is the unique fixed point of T. Choose a subsequence $\{x_{b(n)}\}_{n=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ such that $x_{b(n)+1} \in P$ for all $n = 0, 1, 2, \cdots$ Observe that $\lim_{n\to\infty} x_{b(n)+1} = \lim_{n\to\infty} x_n = z$ and by continuity of T we also have $\lim_{n\to\infty} x_{b(n)+1} = \lim_{n\to\infty} Tx_{b(n)} = Tz$. Therefore we obtain that $z = Tz$. If T has two distinct fixed points $x, y \in K$, then

$$\begin{aligned} d(x,y) &= d(Tx,Ty) \\ &\leq h(d(x,y)) \cdot \max.\{d(x,y)/2, d(x,Tx), d(y,Ty), \\ &\qquad [d(x,Ty) + d(y,Tx)]/1 + 2h(d(x,y))\}, \end{aligned}$$

and thus $d(x,y) \leq \frac{2h(d(x,y))}{1+2h(d(x,y))} \cdot d(x,y) < d(x,y)$, a contradiction. Thus the proof is completed.

The following result follows immediately from the Theorem.

Corollary. *Let X be a Banach space, K a nonempty closed subset of $X, T : K \to X$ a continuous mapping satisfying the condition on K,*
(H) *for all $x \neq y, x, y \in K$:*

$$d(Tx,Ty) \leq h(d(x,y)) \cdot \max.\{d(x,Tx), d(y,Ty)\}.$$

If T has property (a), *then T has a unique fixed point.*

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