Abstract. A parametric mean length is defined as the quantity
\[ R\beta L_u = \frac{R}{R-1} \left[ 1 - \sum p_i^\beta \left( \frac{u_i}{\sum u_i p_i} \right)^{\frac{1}{\beta}} D^{\frac{1}{\beta}} \left( \frac{R}{R-1} \right) \right], \]
where $R > 0 (\neq 1)$, $\sum p_i = 1$. This being the useful mean length of code words weighted by utilities, $u_i$. Lower and upper bounds for $R\beta L_u$ are derived in terms of ‘useful’-R-norm information measure for the incomplete power distribution, $p^\beta$.

1. Introduction

Consider the following model for a random experiment $S$,

\[ S_N = [E; P; U] \]

where $E = (E_1, E_2, \ldots, E_N)$ is a finite system of events happening with respective probabilities $P = (P_1, P_2, \ldots, P_N)$, $p_i \geq 0$, and $\sum p_i = 1$ and credited with utilities $U = (u_1, u_2, \ldots, u_N)$, $u_i > 0$, $i = 1, 2, \ldots, N$. Denote the model by $E$, where

\[ E = \begin{bmatrix} E_1 & E_2 & \ldots & E_N \\ p_1 & p_2 & \ldots & p_N \\ u_1 & u_2 & \ldots & u_N \end{bmatrix}. \quad (1.1) \]

We call (1.1) a Utility Information Scheme (UIS). Belis and Guiasu [3] proposed a measure of information called ‘useful information’ for this scheme, given by

\[ H(U; P) = - \sum u_i p_i \log p_i, \quad (1.2) \]
where $H(U; P)$ reduces to Shannon’s [9] entropy when the utility aspect of the scheme is ignored i.e., when $u_i = 1$ for each $i$. Throughout the paper, $\sum$ will stand for $\sum_{i=1}^{N}$ unless otherwise stated and logarithms are taken to base $D(D > 1)$.

Guiasu and Picard [5] considered the problem of encoding the outcomes in (1.1) by means of a prefix code with codewords $w_1, w_2, \ldots, w_N$ having lengths $n_1, n_2, \ldots, n_N$ and satisfying Kraft’s inequality [4].

$$\sum_{i=1}^{N} D^{-n_i} \leq 1. \quad (1.3)$$

Where $D$ is the size of the code alphabet. The useful mean length $L_u$ of code was defined as

$$L_u = \frac{\sum u_i n_i p_i}{\sum u_i p_i} \quad (1.4)$$

and the authors obtained bounds for it in terms of $H(U; P)$. Longo [7], Gurdial and Pessoa [6], Autar and Khan [1], Singh and Rajeev [8], have studied generalized coding theorems by considering different generalized measures of (1.2) and (1.4) under condition (1.3) of unique decipherability.

In this paper, we study some coding theorems by considering a new function depending on the parameters $R$ and $\beta$ and a utility function. Our motivation for studying this new function is that it generalizes ‘useful’ R-norm information measure already existing in the paper Singh and Rajeev [8], Bockee and Lubbe [2].

2. Coding Theorems

In this section, we define ‘useful’ R-norm information measure as:

$$r_{\beta}H(U; P) = \frac{R}{R-1} \left[ 1 - \left( \frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^\beta} \right)^{\frac{1}{\beta}} \right]. \quad (2.1)$$

where $R > 0 (\neq 1)$, $\beta > 0$, $p_i \geq 0$, $i = 1, 2, \ldots, N$ and $\sum p_i = 1$.

(i) When $\beta = 1$ then (2.1) reduces to ‘useful’ R-norm information measure studied by Singh and Rajeev [8].

i.e. $rH(U; P) = \frac{R}{R-1} \left[ 1 - \left( \frac{\sum u_i p_i^R}{\sum u_i p_i} \right)^{\frac{1}{R}} \right]. \quad (2.2)$

(ii) When $u_1 = 1$ and $\beta = 1$, (2.1) reduces to R-norm entropy as considered by Bockee and Lubbe [2].

i.e. $rH(P) = \frac{R}{R-1} \left[ 1 - \left( \frac{p_1^R}{\sum p_i^R} \right)^{\frac{1}{R}} \right]. \quad (2.3)$

(iii) When $\beta = 1$ and $R \to 1$, (2.1) reduces to a measure of ‘useful’ information for the incomplete distribution due to Belis and Guiasu [3].
(iv) When $u_i = 1$ for each $i$, i.e. when the utility aspect is ignored, $\sum p_i = 1$, $\beta = 1$ and $R \to 1$, the measure (2.1) reduces to Shannon’s entropy [9].

\[ H(P) = -\sum p_i \log p_i. \] (2.4)

(v) When $u_i = 1$ for each $i$, the measure (2.1) becomes R-norm entropy for the $\beta$-power distribution derived from $P$. We call $R^\beta H(U; P)$ in (2.1) the generalized ‘useful’ R-norm information measure for the incomplete power distribution $P^\beta$.

Further consider

**Definition.** The ‘useful’ mean length $R^\beta L_u$ with respect to ‘useful’ R-norm information measure is defined as:

\[ R^\beta L_u = \frac{R}{R-1} \left[ 1 - \sum p_i^\beta \left( \frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{\beta}} \left( \frac{R-1}{R} \right) \right], \] (2.5)

where $R > 0 \neq 1$, $\sum p_i = 1$.

(i) For $\beta = 1$ and $R \to 1$, $R^\beta L_u$ in (2.5) reduces to the useful mean length $L_u$ of the code given in (1.4).

(ii) For $\beta = 1$, $u_i = 1$ for each $i$ and $R \to 1$, $R^\beta L_u$ becomes the optimal code length defined by Shannon [9].

(iii) For $\beta = 1$, $u_i = 1$ then (2.5) reduced to $RL$ considered by Bockee and Lubbe [2].

\[ i.e. \quad R L = \frac{R}{R-1} \left[ 1 - \sum p_i D^{-n_i \left( \frac{R-1}{R} \right)} \right], \] (2.6)

the average length of code word considered by Bockee and Lubbe [2].

We establish a result, that in a sense, provides a characterization of $R^\beta H(U; P)$ under the condition of unique decipherability.

**Theorem 2.1.** For all integers $D > 1$

\[ R^\beta L_u \geq R^\beta H(U; P) \] (2.7)

under the condition (1.3). Equality holds if and only if

\[ n_i = -\log_D \left( \frac{u_i L_i^{R^\beta}}{\sum u_i L_i^{R^\beta}} \right). \] (2.8)

**Proof.** We use Holder’s [10] inequality

\[ \sum x_i y_i \geq \left( \sum x_i^p \right)^{\frac{1}{p}} \left( \sum y_i^q \right)^{\frac{1}{q}} \] (2.9)
for all $x_i \geq 0, y_i \geq 0, i = 1, 2, \ldots, N$ when $P < 1$ (≠ 1) and $p^{-1} + q^{-1} = 1$, with equality if and only if there exists a positive number $c$ such that

$$x_i^p = cy_i^q.$$  \hspace{1cm} (2.10)

Setting

$$x_i = P_i^{\mu_i} \left( \frac{u_i}{\sum u_i P_i} \right)^{\frac{1}{\alpha}} D^{-n_i},$$

$$y_i = P_i^{\mu_i} \left( \frac{u_i}{\sum u_i P_i} \right)^{\frac{1}{\alpha}} R^{-1},$$

$p = 1 - \frac{1}{R}$ and $q = 1 - R$ in (2.9) and using (1.3) we obtain the result (2.7) after simplification for $\frac{R}{R-1}$ as $R > 1$.

**Theorem 2.2.** For every code with lengths $\{n_i\}, i = 1, 2, \ldots, N$, $R_{33} L_u$ can be made to satisfy,

$$R_{33} L_u < R_{33} H(U; P)D \left( \frac{1-n}{R} \right) + \frac{R}{R-1} \left[ 1 - D \left( \frac{1-n}{R} \right) \right].$$ \hspace{1cm} (2.11)

**Proof.** Let $n_i$ be the positive integer satisfying, the inequality

$$-\log D \left( \frac{u_i D^{R_3}}{\sum u_i D^{R_3}} \right) \leq n_i < -\log D \left( \frac{u_i D^{R_3}}{\sum u_i D^{R_3}} \right) + 1.$$ \hspace{1cm} (2.12)

Consider the intervals

$$\delta_i = \left[ -\log D \left( \frac{u_i D^{R_3}}{\sum u_i D^{R_3}} \right), 1 - \log D \left( \frac{u_i D^{R_3}}{\sum u_i D^{R_3}} \right) \right]$$ \hspace{1cm} (2.13)

of length 1. In every $\delta_i$, there lies exactly one positive number $n_i$ such that

$$0 < -\log D \left( \frac{u_i D^{R_3}}{\sum u_i D^{R_3}} \right) \leq n_i < -\log D \left( \frac{u_i D^{R_3}}{\sum u_i D^{R_3}} \right) + 1.$$ \hspace{1cm} (2.14)

It can be shown that the sequence $\{n_i\}, i = 1, 2, \ldots, N$ thus defined, satisfies (1.3). From (2.14) we have

$$n_i < -\log D \left( \frac{u_i D^{R_3}}{\sum u_i D^{R_3}} \right) + 1$$

$$\Rightarrow D^{-n_i} > \left( \frac{u_i D^{R_3}}{\sum u_i D^{R_3}} \right)^{-1}$$

$$\Rightarrow D^{-n_i} \left( \frac{R-1}{R} \right) > \left( \frac{u_i D^{R_3}}{\sum u_i D^{R_3}} \right) \frac{R-1}{R}.$$ \hspace{1cm} (2.15)
Multiplying both sides of (2.15) by \( p_i^\beta \left( \frac{u_i}{\sum u_i p_i^\beta} \right)^\frac{1}{\beta} \), summing over \( i = 1, 2, \ldots, N \) and simplification for \( R > 1 \), gives (2.11).

**Theorem 2.3.** For every code with lengths \( \{n_i\} \), \( i = 1, 2, \ldots, N \), of Theorem 2.1, \( R^\beta L_u \) can be made to satisfy,

\[
R^\beta L_u \geq R^\beta H(U; P) > R^\beta H(U; P)D + \frac{R}{R - 1}(1 - D).
\]

(2.16)

**Proof.** Suppose

\[
\pi_i = -\log_D \left( \frac{u_i p_i^R}{\sum u_i p_i^R} \right).
\]

Clearly \( \pi_i \) and \( \pi_i + 1 \) satisfy ‘equality’ in Holder’s inequality (2.9). Moreover, \( \pi_i \) satisfies Kraft’s inequality (1.3).

Suppose \( n_i \) is the unique integer between \( \pi_i \) and \( \pi_i + 1 \), then obviously \( n_i \), satisfied (1.3).

Since \( R > 0 \) (\( \neq 1 \)), we have

\[
\sum p_i^\beta \left( \frac{u_i}{\sum u_i p_i^\beta} \right)^\frac{1}{\beta} D^{-n_i \frac{(n_i - 1)}{n}} \leq \sum p_i^\beta \left( \frac{u_i}{\sum u_i p_i^\beta} \right)^\frac{1}{\beta} D^{-\pi_i \frac{(n - 1)}{n}} \leq D \left( \sum p_i^\beta \left( \frac{u_i}{\sum u_i p_i^\beta} \right)^\frac{1}{\beta} D^{-\pi_i \frac{(n - 1)}{n}} \right),
\]

(2.18)

Since,

\[
\sum p_i^\beta \left( \frac{u_i}{\sum u_i p_i^\beta} \right)^\frac{1}{\beta} D^{-\pi_i \frac{(n - 1)}{n}} = \left( \sum u_i p_i^R \right)^\frac{1}{\beta}.
\]

Hence, (2.18) becomes

\[
\sum p_i^\beta \left( \frac{u_i}{\sum u_i p_i^\beta} \right)^\frac{1}{\beta} D^{-n_i \frac{(n_i - 1)}{n}} \leq \left( \sum u_i p_i^R \right)^\frac{1}{\beta} < D \left( \sum u_i p_i^R \right)^\frac{1}{\beta},
\]

which gives the result (2.16).

**References**


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