TAMKANG JOURNAL OF MATHEMATICS
Volume 40, Number 2, 211-216, Summer 2009

SOME MORE RESULTS ON A GENERALIZED 'USEFUL' R-NORM INFORMATION MEASURE

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Abstract. A parametric mean length is defined as the quantity

$$_{R\beta}L_u = \frac{R}{R-1}\left[1 - \sum P_i^{\beta}\left(\frac{u_i}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{R}} D^{-n_i\left(\frac{R-1}{R}\right)}\right],$$

where $R > 0$ ($\neq 1$), $\sum p_i = 1$. This being the useful mean length of code words weighted by utilities, u_i. Lower and upper bounds for $_{R\beta}L_u$ are derived in terms of 'useful'-R-norm information measure for the incomplete power distribution, p^{β}.

1. Introduction

Consider the following model for a random experiment S,

$$S_N = [E; P; U]$$

where $E = (E_1, E_2, \ldots, E_N)$ is a finite system of events happening with respective probabilities $P = (P_1, P_2, \ldots, P_N)$, $p_i \geq 0$, and $\sum p_i = 1$ and credited with utilities $U = (u_1, u_2, \ldots, u_N)$, $u_i > 0$, $i = 1, 2, \ldots, N$. Denote the model by E, where

$$E = \begin{bmatrix} E_1 & E_2 & \ldots & E_N \\ p_1 & p_2 & \ldots & p_N \\ u_1 & u_2 & \ldots & u_N \end{bmatrix}. \tag{1.1}$$

We call (1.1) a Utility Information Scheme (UIS). Belis and Guiasu [3] proposed a measure of information called 'useful information' for this scheme, given by

$$H(U; P) = -\sum u_i p_i \log p_i, \tag{1.2}$$

Received December 18, 2007; revised November 25, 2008.
Key words and phrases. R-norm entropy, 'useful' R-norm information, utilities, power probabilities.

where $H(U;P)$ reduces to Shannon's [9] entropy when the utility aspect of the scheme is ignored i.e., when $u_i = 1$ for each i. Throughout the paper, $\sum$ will stand for $\sum_{i=1}^{N}$ unless otherwise stated and logarithms are taken to base $D(D > 1)$.

Guiasu and Picard [5] considered the problem of encoding the outcomes in (1.1) by means of a prefix code with codewords $w_1, w_2, \ldots, w_N$ having lengths $n_1, n_2, \ldots, n_N$ and satisfying Kraft's inequality [4].

$$\sum_{i=1}^{N} D^{-n_i} \leq 1. \tag{1.3}$$

Where D is the size of the code alphabet. The useful mean length L_u of code was defined as

$$L_u = \frac{\sum u_i n_i p_i}{\sum u_i p_i} \tag{1.4}$$

and the authors obtained bounds for it in terms of $H(U;P)$. Longo [7], Gurdial and Pessoa [6], Autar and Khan [1], Singh and Rajeev [8], have studied generalized coding theorems by considering different generalized measures of (1.2) and (1.4) under condition (1.3) of unique decipherability.

In this paper, we study some coding theorems by considering a new function depending on the parameters R and β and a utility function. Our motivation for studying this new function is that it generalizes 'useful' R-norm information measure already existing in the paper Singh and Rajeev [8], Bockee and Lubbe [2].

2. Coding Theorems

In this section, we define 'useful' R-norm information measure as:

$$_{R\beta}H(U;P) = \frac{R}{R-1}\left[1 - \left(\frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{R}}\right], \tag{2.1}$$

where $R > 0\ (\neq 1)$, $\beta > 0$, $p_i \geq 0$, $i = 1, 2, \ldots, N$ and $\sum p_i = 1$.

(i) When $\beta = 1$ then (2.1) reduces to 'useful' R-norm information measure studied by Singh and Rajeev [8].

$$\text{i.e. } {}_{R}H(U;P) = \frac{R}{R-1}\left[1 - \left(\frac{\sum u_i p_i^{R}}{\sum u_i p_i}\right)^{\frac{1}{R}}\right]. \tag{2.2}$$

(ii) When $u_1 = 1$ and $\beta = 1$, (2.1) reduces to R-norm entropy as considered by Bockee and Lubbe [2].

$$\text{i.e. } {}_{R}H(P) = \frac{R}{R-1}\left[1 - \left(\sum p_i^{R}\right)^{\frac{1}{R}}\right]. \tag{2.3}$$

(iii) When $\beta = 1$ and $R \to 1$, (2.1) reduces to a measure of 'useful' information for the incomplete distribution due to Belis and Guiasu [3].

(iv) When $u_i = 1$ for each i, i.e. when the utility aspect is ignored, $\sum p_i = 1$, $\beta = 1$ and $R \to 1$, the measure (2.1) reduces to Shannon's entropy [9].

$$\text{i.e.}\quad H(P) = -\sum p_i \log p_i. \tag{2.4}$$

(v) When $u_i = 1$ for each i, the measure (2.1) becomes R-norm entropy for the β -power distribution derived from P. We call ${}_{R\beta}H(U;P)$ in (2.1) the generalized 'useful' R-norm information measure for the incomplete power distribution P^β.

Further consider

Definition. The 'useful' mean length ${}_{R\beta}L_u$ with respect to 'useful' R-norm information measure is defined as :

$${}_{R\beta}L_u = \frac{R}{R-1}\left[1 - \sum p_i^\beta \left(\frac{u_i}{\sum u_i p_i^\beta}\right)^{\frac{1}{R}} D^{-n_i\left(\frac{R-1}{R}\right)}\right], \tag{2.5}$$

where $R > 0$ $(\neq 1)$, $\sum p_i = 1$.

(i) For $\beta = 1$ and $R \to 1$, ${}_{R\beta}L_u$ in (2.5) reduces to the useful mean length L_u of the code given in (1.4).

(ii) For $\beta = 1$, $u_i = 1$ for each i and $R \to 1$, ${}_{R\beta}L_u$ becomes the optimal code length defined by Shannon [9].

(iii) For $\beta = 1$, $u_i = 1$ then (2.5) reduced to ${}_RL$ considered by Bockee and Lubbe [2].

$$\text{i.e.}\quad {}_RL = \frac{R}{R-1}\left[1 - \sum p_i D^{-n_i\left(\frac{R-1}{R}\right)}\right], \tag{2.6}$$

the average length of code word considered by Bockee and Lubbe [2].

We establish a result, that in a sense, provides a characterization of ${}_{R\beta}H(U;P)$ under the condition of unique decipherability.

Theorem 2.1. *For all integers $D > 1$*

$${}_{R\beta}L_u \geq {}_{R\beta} H(U;P) \tag{2.7}$$

under the condition (1.3). *Equality holds if and only if*

$$n_i = -\log_D \left(\frac{u_i P_i^{R\beta}}{\sum u_i p_i^{R\beta}}\right). \tag{2.8}$$

Proof. We use Holder's [10] inequality

$$\sum x_i y_i \geq \left(\sum x_i^p\right)^{\frac{1}{p}} \left(\sum y_i^q\right)^{\frac{1}{q}} \tag{2.9}$$

for all $x_i \geq 0$, $y_i \geq 0$, $i = 1, 2, \ldots, N$ when $P < 1\ (\neq 1)$ and $p^{-1} + q^{-1} = 1$, with equality if and only if there exists a positive number c such that

$$x_i^p = c y_i^q. \tag{2.10}$$

Setting

$$x_i = p_i^{\frac{R\beta}{R-1}} \left(\frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{R-1}} D^{-n_i},$$

$$y_i = p_i^{\frac{R\beta}{1-R}} \left(\frac{u_i}{\sum u_i p_i^\beta} \right)^{\frac{1}{1-R}},$$

$p = 1 - \frac{1}{R}$ and $q = 1 - R$ in (2.9) and using (1.3) we obtain the result (2.7) after simplification for $\frac{R}{R-1}$ as $R > 1$.

Theorem 2.2. *For every code with lengths* $\{n_i\}$, $i = 1, 2, \ldots, N$, ${}_{R\beta}L_u$ *can be made to satisfy,*

$${}_{R\beta}L_u <_{R\beta} H(U;P) D^{\left(\frac{1-R}{R}\right)} + \frac{R}{R-1}\left[1 - D^{\left(\frac{1-R}{R}\right)}\right]. \tag{2.11}$$

Proof. Let n_i be the positive integer satisfying, the inequality

$$-\log_D \left(\frac{u_i p_i^{R\beta}}{\sum u_i p_i^{R\beta}} \right) \leq n_i < -\log_D \left(\frac{u_i p_i^{R\beta}}{\sum u_i p_i^{R\beta}} \right) + 1. \tag{2.12}$$

Consider the intervals

$$\delta_i = \left[-\log_D \left(\frac{u_i P_i^{R\beta}}{\sum u_i P_i^{R\beta}} \right), -\log_D \left(\frac{u_i P_i^{R\beta}}{\sum u_i P_i^{R\beta}} \right) + 1 \right] \tag{2.13}$$

of length 1. In every δ_i, there lies exactly one positive number n_i such that

$$0 < -\log_D \left(\frac{u_i P_i^{R\beta}}{\sum u_i p_i^{R\beta}} \right) \leq n_i < -\log_D \left(\frac{u_i P_i^{R\beta}}{\sum u_i p_i^{R\beta}} \right) + 1. \tag{2.14}$$

It can be shown that the sequence $\{n_i\}$, $i = 1, 2, \ldots, N$ thus defined, satisfies (1.3). From (2.14) we have

$$n_i < -\log_D \left(\frac{u_i P_i^{R\beta}}{\sum u_i P_i^{R\beta}} \right) + 1$$

$$\Rightarrow D^{-n_i} > \left(\frac{u_i P_i^{R\beta}}{\sum u_i P_i^{R\beta}} \right) D^{-1}$$

$$\Rightarrow D^{-n_i\left(\frac{R-1}{R}\right)} > \left(\frac{u_i P_i^{R\beta}}{\sum u_i p_i^{R\beta}} \right)^{\frac{R-1}{R}} D^{\frac{1-R}{R}}. \tag{2.15}$$

Multiplying both sides of (2.15) by $p_i^\beta\left(\frac{u_i}{\sum u_i p_i^\beta}\right)^{\frac{1}{R}}$, summing over $i = 1, 2, \ldots, N$ and simplification for $\frac{R}{R-1} > 0$ as $R > 1$, gives (2.11).

Theorem 2.3. *For every code with lengths* $\{n_i\}$, $i = 1, 2, \ldots, N$, *of Theorem* 2.1, ${}_{R\beta}L_u$ *can be made to satisfy,*

$$ {}_{R\beta}L_u \geq {}_{R\beta}H(U;P) > {}_{R\beta}H(U;P)D + \frac{R}{R-1}(1-D). \tag{2.16}$$

Proof. Suppose

$$\overline{n}_i = -\log_D\left(\frac{u_i P_i^{R\beta}}{\sum u_i p_i^{R\beta}}\right). \tag{2.17}$$

Clearly $\overline{n}_i$ and $\overline{n}_i + 1$ satisfy 'equality' in Holder's inequality (2.9). Moreover, $\overline{n}_i$ satisfies Kraft's inequality (1.3).

Suppose n_i is the unique integer between $\overline{n}_i$ and $\overline{n}_i + 1$, then obviously n_i, satisfied (1.3).

Since $R > 0$ $(\neq 1)$, we have

$$\begin{aligned}\sum p_i^\beta\left(\frac{u_i}{\sum u_i p_i^\beta}\right)^{\frac{1}{R}} D^{-n_i\frac{(R-1)}{R}} &\leq \sum p_i^\beta\left(\frac{u_i}{\sum u_i p_i^\beta}\right)^{\frac{1}{R}} D^{-\overline{n}_i\frac{(R-1)}{R}} \\ &< D\left(\sum p_i^\beta\left(\frac{u_i}{\sum u_i p_i^\beta}\right)^{\frac{1}{R}} D^{-\overline{n}_i\frac{(R-1)}{R}}\right). \end{aligned}\tag{2.18}$$

Since,

$$\sum p_i^\beta\left(\frac{u_i}{\sum u_i p_i^\beta}\right)^{\frac{1}{R}} D^{-\overline{n}_i\frac{(R-1)}{R}} = \left(\frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^\beta}\right)^{\frac{1}{R}}.$$

Hence, (2.18) becomes

$$\sum p_i^\beta\left(\frac{u_i}{\sum u_i p_i^\beta}\right)^{\frac{1}{R}} D^{-n_i\frac{(R-1)}{R}} \leq \left(\frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^\beta}\right)^{\frac{1}{R}} < D\left(\frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^\beta}\right)^{\frac{1}{R}},$$

which gives the result (2.16).

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