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NORMS ON CARTESIAN PRODUCT OF LINEAR SPACES

CHI-KWONG LI AND NAM-KIU TSING

Abstract. Let X_i $(i = 1, \cdots, n)$ be real or complex linear spaces, each equipped with a norm $\|\cdot\|_i$. Standard ways of constructing norms $\|\cdot\|$ on the Cartesian product $X = X_1 \times \cdots \times X_n$ are to define

$$\|(x_1, \cdots, x_n)\| = \varphi(\|x_1\|_1, \cdots, \|x_n\|_n)$$

via some functions φ on $\mathbf{R}^n$. Common examples of φ in standard texbooks are norms on $\mathbf{R}^n$. This may mislead peoples to think that any norm φ on $\mathbf{R}^n$ can induce a norm on the product space X in the above way. In this note we show that this is actually false and characterize the functions φ that can give rise to norms on X in the above manner. It turns out that a necessary and sufficient condition on φ is :

for any $a_1, \cdots, a_n, b_1, \cdots, b_n \geq 0$,

(I) $\varphi(a_1, \cdots, a_n) > 0$ if $(a_1, \cdots, a_n) \neq (0, \cdots, 0)$;

(II) $\varphi(\alpha(a_1, \cdots, a_n)) = \alpha\varphi(a_1, \cdots, a_n)$ if $\alpha \geq 0$;

(III) $\varphi(c_1, \cdots, c_n) \leq \varphi(a_1, \cdots, a_n) + \varphi(b_1, \cdots, b_n)$ if $(c_1, \cdots, c_n) = (a_1, \cdots, a_n) + (b_1, \cdots, b_n)$;

(IV) $\varphi(a_1, \cdots, a_n) \leq \varphi(b_1, \cdots, b_n)$ if $a_i \leq b_i$ for all i.

Several interesting consequences of the result are discussed.

Results and proofs

Let X be a finite or infinite dimensional linear space over $\mathbf{F}$, where $\mathbf{F}$ is the field of all real or complex numbers. A *norm* on X is a function $\|\cdot\| : X \to \mathbf{R}$ that satisfies

(N1) $\|x\| > 0$ for all nonzero $x \in X$;

(N2) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $\alpha \in F$, $x \in X$;

(N3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

A simple example is to take $X = \mathbf{F}$ and $\|x\| = | x |$ for all x in X. As the concept of the norm is of crucial importance in the study of mathematical analysis, it is introduced in the beginning chapters of most analysis textbooks.

Let $X_i (i = 1, \cdots, n)$ be linear spaces over the same field, each equipped with a norm $\|\cdot\|_i$. Standard ways of constructing norms $\|\cdot\|$ on the Cartesian product $X =$

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$X_1 \times \cdots \times X_n$ are to define

$$\|(x_1, \cdots, x_n\| = \varphi(\|x_1\|_1, \cdots, \|x_n\|_n)$$

via some functions φ on $\mathbf{R}^n$. Common examples of φ in standard textbooks of functional analysis (for example see [2,p.49], [4,p.41] or [5, p.142]) are

$$\varphi_p(a_1, \cdots, a_n) = (a_1^p + \cdots + a_n^p)^{1/p} \qquad (1 \le p \le \infty). \tag{1}$$

The corresponding norms $\|\cdot\|_{(p)}$ on X will then be

$$\|x\|_{(p)} = (\|x_1\|_1^p + \cdots + \|x_n\|_n^p)^{1/p} \qquad (1 \le p \le \infty). \tag{2}$$

In general, we say that a function $\varphi : \mathbf{R}^n \to \mathbf{R}$ *induces* a norm $\|\cdot\|$ on the Cartesian product $X = X_1 \times \cdots \times X_n$ if the function $\|\cdot\| : X \to \mathbf{R}$ defined by

$$\|(x_1, \cdots, x_n)\| = \varphi(\|x_1\|_1, \cdots, \|x_n\|_n)$$

is a norm on X. A closed look of the problem shows that the domain of definition of φ can be confined to

$$\mathbf{R}_+^n = \{(a_1, \cdots, a_n) \in \mathbf{R}^n : a_1, \cdots, a_n \ge 0\}.$$

In view of the functions φ_p $(1 \le p \le \infty)$ in the above examples, one may be tempted to conjecture that *any* norm $\varphi : \mathbf{R}^n \to \mathbf{R}$ can induce a norm on X. In fact, the subtlety of the situations is so easily overlooked that some authors even state this conjecture, which is not true in general, as a fact (see [3, p.39] for example). To see that the conjecture is not true, one may consider the following counter-example.

Define $\varphi : \mathbf{R}^2 \to \mathbf{R}$ by

$$\varphi(\alpha_1, \alpha_2) = \max\{|\alpha_1 + \alpha_2|, 2|\alpha_1 - \alpha_2|\},$$

which is a norm. Take $X_1 = X_2 = \mathbf{R}$ and $\|\cdot\|_1 = \|\cdot\|_2 = |\cdot|$. Then the function induced by φ is $\|\cdot\| : \mathbf{R}^2 \to \mathbf{R}$ where

$$\begin{aligned}\|(\alpha_1, \alpha_2)\| &= \varphi(|\alpha_1|, |\alpha_2|) \\ &= \max\{||\alpha_1| + |\alpha_2||, 2||\alpha_1| - |\alpha_2||\}.\end{aligned}$$

By direct computation, we get

$$\|(3,1)\| = \|(3,-1)\| = 4; \qquad \|(3,1) + (3,-1)\| = \|(6,0)\| = 12.$$

Hence $\|\cdot\|$ does not satisfy (N3) and cannot be a norm.

In the following theorem and Corollary 3, we characterize the functions $\varphi : \mathbf{R}_+^n \to \mathbf{R}$ and $\varphi : \mathbf{R}^n \to \mathbf{R}$, respectively, that can induce norms on X. We shall assume without loss of generality that all component spaces X_i $(1 \le i \le n)$ are nontrivial (that is, $X_i \ne 0$), for if $X_i = 0$ then we can delete X_i from the Cartesian product $X = X_1 \times \cdots \times X_n$ without affecting the structure of X. Our main result is

Theorem. *Let $X_1, \cdots, X_n$ be linear spaces over the same field $\mathbf{F}$ and $\|\cdot\|_1, \cdots, \|\cdot\|_n$ be norms on $X_1, \cdots, X_n$ respectively. Then a function $\varphi : \mathbf{R}_+^n \to \mathbf{R}$ will induce a norm on $X = X_1 \times \cdots \times X_n$ if and only if φ satisfies*

(I) $\varphi(a_1, \cdots, a_n) > 0$ *if* $(a_1, \cdots, a_n) \neq (0, \cdots, 0)$;
(II) $\varphi(\alpha(a_1, \cdots, a_n)) = \alpha\varphi(a_1, \cdots, a_n)$ *if* $\alpha \geq 0$;
(III) $\varphi(c_1, \cdots, c_n) \leq \varphi(a_1, \cdots, a_n) + \varphi(b_1, \cdots, b_n)$ *if* $(c_1, \cdots, c_n) = (a_1, \cdots, a_n) + (b_1, \cdots, b_n)$;
(IV) $\varphi(a_1, \cdots, a_n) \leq \varphi(b_1, \cdots, b_n)$ *if* $a_i \leq b_i$ *for all* i.

Proof. Let $\varphi : \mathbf{R}_+^n \to \mathbf{R}$ be given and $\|\cdot\| : X \to \mathbf{R}$ be induced by φ, that is,

$$\|(x_1, \cdots, x_n)\| = \varphi(\|x_1\|_1, \cdots, \|x_n\|_n)$$

for all $(x_1, \cdots, x_n) \in X$. Suppose φ satisfies (I) through (IV). By (I) and (II). one easily shows that $\|\cdot\|$ satisfies (N1) and (N2).

Now if $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in X$, then

$$\begin{aligned} \|x + y\| &= \|(x_1 + y_1, \cdots, x_n + y_n)\| \\ &= \varphi(\|x_1 + y_1\|_1, \cdots, \|x_n + y_n\|_n). \end{aligned} \tag{3}$$

Since $\|x_i + y_i\|_i \leq \|x_i\|_i + \|y_i\|_i$ for all i and φ satisfies (IV), the expression in (3) cannot be greater than

$$\begin{aligned} &\varphi(\|x_1\|_1 + \|y_1\|_1, \cdots, \|x_n\|_n + \|y_n\|_n) \\ =\ &\varphi((\|x_1\|_1, \cdots, \|x_n\|_n) + (\|y_1\|_1, \cdots, \|y_n\|_n)) \\ \leq\ &\varphi(\|x_1\|_1, \cdots, \|x_n\|_n) + \varphi(\|y_1\|_1, \cdots, \|y_n\|_n) \qquad (by (III)) \\ \leq\ &\|(x_1, \cdots, x_n)\| + \|(y_1, \cdots, y_n)\| \\ =\ &\|x\| + \|y\|. \end{aligned}$$

Hence $\|\cdot\|$ satisfies (N3) also and is a norm then.

Conversely, suppose $\|\cdot\|$ satisfies (N1) through (N3). Since each x_i is nontrivial, we can find nonzero $x_i \in X_i$ such that $\|x_i\|_i = 1$. Then for any $(a_1, \cdots, a_n) \in \mathbf{R}_+^n$,

$$\|a_i x_i\|_i = a_i \qquad \text{for all } i$$

and hence

$$\varphi(a_1, \cdots, a_n) = \varphi(\|a_1 x_1\|_1, \cdots, \|a_n x_n\|_n) = \|(a_1 x_1, \cdots, a_n x_n)\|.$$

Using this relation and the fact that $\|\cdot\|$ is a norm, one can prove that φ satisfies (I) and (II) readily. The condition (III) is also satisfied because for any $(a_1,\cdots,a_n)$, $(b_1,\cdots,b_n)\in\mathbf{R}_+^n$,

$$\begin{aligned}
&\varphi((a_1,\cdots,a_n)+(b_1,\cdots,b_n))\\
&= \varphi(a_1+b_1,\cdots,a_n+b_n)\\
&= \|(a_1x_1+b_1x_1,\cdots,a_nx_n+b_nx_n)\|\\
&= \|(a_1x_1,\cdots,a_nx_n)+(b_1x_1,\cdots,b_nx_n)\|\\
&\le \|(a_1x_1,\cdots,a_nx_n)\|+\|(b_1x_1,\cdots,b_nx_n)\|\\
&= \varphi(a_1,\cdots,a_n)\ +\ \varphi(b_1,\cdots,b_n).
\end{aligned}$$

Finally, for any i with $0\le a_i\le b_i$, we can find t such that $1/2\le t\le 1$ and $a_i=(2t-1)b_i=tb_i+(1-t)(-b_i)$. As a result,

$$\begin{aligned}
&\varphi(a_1,\cdots,a_n)\\
&= \|(a_1x_1,\cdots,a_nx_n)\|\\
&= \|t(a_1x_1,\cdots,a_{i-1}x_{i-1},b_ix_i,a_{i+1}x_{i+1},\cdots,a_nx_n)\\
&\quad +(1-t)(a_1x_1,\cdots,a_{i-1}x_{i-1},-b_ix_i,a_{i+1}x_{i+1},\cdots,a_nx_n)\|\\
&\le t\|(a_1x_1,\cdots,a_{i-1}x_{i-1},b_ix_i,a_{i+1}x_{i+1},\cdots,a_nx_n)\|\\
&\quad +(1-t)\|(a_1x_1,\cdots,a_{i-1}x_{i-1},-b_ix_i,a_{i+1}x_{i+1},\cdots,a_nx_n)\|\\
&= t\varphi(a_1,\cdots,a_{i-1},b_i,a_{i+1},\cdots,a_n)\\
&\quad +(1-t)\varphi(a_1,\cdots,a_{i-1},b_i,a_{i+1},\cdots,a_n)\\
&= \varphi(a_1,\cdots,a_{i-1},b_i,a_{i+1},\cdots,a_n).
\end{aligned}$$

Since this is true for all i, we have

$$\begin{aligned}
\varphi(a_1,\cdots,a_n)\ &\le\ \varphi(b_1,a_2,\cdots,a_n)\\
\le\ \varphi(b_1,b_2,a_3,\cdots,a_n)\ &\le\ \cdots\ \le\ (b_1,\cdots,b_n)
\end{aligned}$$

if $0\le a_i\le b_i$ for all i. Hence (IV) is satisfied.

A particular application of the above theorem gives

Corollary 1. *Suppose $\Phi:\mathbf{F}^n\to\mathbf{R}$ is a function that satisfies*

$$\Phi(\alpha_1,\cdots,\alpha_n)\ =\ \Phi(|\,\alpha_1\,|,\cdots,|\,\alpha_n\,|)\ \textit{for all}\ (\alpha_1,\cdots,\alpha_n)\in\mathbf{F}^n. \qquad (V)$$

Let φ denote the restriction on Φ on $\mathbf{R}_+^n$. Then Φ is a norm if and only if φ satisfies (I) through (IV).

Proof. We look at each component space F in the Cartesian product $\mathbf{F}^n$ as equipped with the norm $|\cdot|$. Then Φ is induced by its restriction on $\mathbf{R}_+^n$. Applying the theorem, we get the result.

Another interpretation of Corollary 1 is the following. Let $\varphi : \mathbb{R}_+^n \to \mathbb{R}$ be a given function. We extend the domain of φ to $\mathbb{F}^n$ by defining

$$\varphi(\alpha_1, \cdots, \alpha_n) = \varphi(|\alpha_1|, \cdots, |\alpha_n|) \qquad for \quad (\alpha_1, \cdots, \alpha_n) \in \mathbb{F}^n.$$

Then Corollary 1 says that φ can be extended to a norm on $\mathbb{F}^n$ in the above way if and only if φ satisfies (I) through (IV).

A norm on $\mathbb{F}^n$ satisfies (V) is known as an *absolute* norm. A norm $\|\cdot\|$ on $\mathbb{F}^n$ is said to be *monotone* if for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $\mathbb{F}^n$, $|x_i| \leq |y_i|$ for all $i = 1, \cdots, n$, will imply $\|x\| \leq \|y\|$. By Corollary 1, we easily deduce the following well-known result (see [1, p.285]).

Corollary 2. *A norm on $\mathbb{F}^n$ is monotone if and only if it is absolute.*

Now we let Φ be a norm on F^n. Recall that Φ is said to induce a norm on $X = X_1 \times \cdots \times X_n$ if the function $\|\cdot\| : X \to \mathbb{R}$ defined by

$$\|(x_1, \cdots, x_n)\| = \Phi(\|x_1\|_1, \cdots, \|x_n\|_n)$$

is a norm on X. As the above definition relies solely on the restriction of Φ on $\mathbb{R}_+^n$, we see that Φ will induce a norm on X if and only if its restriction on $\mathbb{R}_+^n$ does so. By Corollary 1 and our Theorem, we have

Corollary 3. *Let Φ be a norm on $\mathbb{F}^n$ that satisfies (V). Then Φ induces a norm on $X = X_1 \times \cdots \times X_n$.*

By Corollary 3, one sees why the functions $\varphi_p(\cdot)$ defined in (1) can induce norms in (2) on $X = X_1 \times \cdots \times X_n$.

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Chi-Kwong Li

Department of Mathematics, The College of William and Mary, Williamsburg, VA 23185.

Nam-Kiu Tsing

Systems Research Center and Electrical Engineering Department, University of Maryland, College Park, MD 20742.