TAMKANG JOURNAL OF MATHEMATICS
Volume 21, Number 1, Spring 1990

ON SUBCLASSES OF STARLIKE FUNCTIONS OF ORDER α AND TYPE β

M.K. AOUF

Abstract. Let $S^*(\alpha,\beta,A,B)(0\leq\alpha<1,\ 0<\beta\leq 1,\ -1\leq A<B\leq 1, 0<B\leq 1)$, denote the class of functions $f(z)=z+\Sigma_{n=2}^{\infty}a_nz^n$ analytic in $U=\{z:|z|<1\}$ which satisfy for $z=re^{i\theta}\in U$,

$$\left|\frac{\frac{zf'(z)}{f(z)}-1}{(B-A)\beta(\frac{zf'(z)}{f(z)}-\alpha)+A(\frac{zf'(z)}{f(z)}-1)}\right|<1.$$

It is the purpose of this paper to show a representation formula, a distortion theorem, a sufficient condition for this class $S^*(\alpha,\beta,A,B)$. Furthermore, we maximize $|a_3-\mu a_2^2|$ over the class $S^*(\alpha,\beta,A,B)$ and we give the radii of convexity for functions in the class $S^*(\alpha,\beta,A,B)$.

1. Introduction.

Let S denote the class of functions $f(z)=z+\Sigma_{n=2}^{\infty}a_nz^n$ be analytic in the unit disc $U=\{z:|z|<1\}$. We use Ω to denote the class of bounded analytic functions $w(z)$ in U, satisfying the conditions $w(0)=0$ and $|w(z)|<1$ for $z\in U$. A function $f(z)\in S$ is said to be *starlike of order* $\alpha(0\leq\alpha<1)$ in the unit disc U if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \tag{1.1}$$

for $z\in U$. And the above condition (1.1) is equivalent to

$$\left|\frac{\frac{zf'(z)}{f(z)}-1}{2(\frac{zf'(z)}{f(z)}-\alpha)-(\frac{zf'(z)}{f(z)}-1)}\right|<1.$$

Juneja and Mogra [3] introduced the class $S^*(\alpha,\beta)$ of starlike functions of order α $(0\leq\alpha<1)$ and type β $(0<\beta\leq 1)$, defined as follows:

Received December 6, 1988.

Definition 1. Let a function $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ be in the class S. Then $f(z)$ is said to be *starlike of order α and type β* if the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{2\beta(\frac{zf'(z)}{f(z)} - \alpha) - (\frac{zf'(z)}{f(z)} - 1)} \right| < 1 \tag{1.2}$$

is satisfied for some α $(0 \leq \alpha < 1)$, β $(0 < \beta \leq 1)$ and for all $z \in U$.

For $0 \leq \alpha < 1$, and $-1 \leq A < B \leq 1$, $0 < B \leq 1$, let $S^*(\alpha, A, B)$ be the class of those functions $f(z)$ of S for which $\frac{zf'(z)}{f(z)}$ is subordinate to $\frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz}$. In other words $f(z) \in S^*(\alpha, A, B)$ if and only if there exists a function $w(z) \in \Omega$ such that

$$\frac{zf'(z)}{f(z)} = \frac{1 + [B + (A - B)(1 - \alpha)]w(z)}{1 + Bw(z)}.$$

And the above condition is equivalent to

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{B\frac{zf'(z)}{f(z)} - [B + (A - B)(1 - \alpha)]} \right| < 1, \qquad z \in U. \tag{1.3}$$

And also the condition (1.3) is equivalent to

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{(B - A)(\frac{zf'(z)}{f(z)} - \alpha) + A(\frac{zf'(z)}{f(z)} - 1)} \right| < 1, \qquad z \in U. \tag{1.4}$$

The aim of the present paper is to introduce a subclass of $S^*(\alpha, \beta)$, which we denote it by $S^*(\alpha, \beta, A, B)$.

Definition 2. Let a function $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ be in the class S. Then $f(z)$ is in the class $S^*(\alpha, \beta, A, B)$ if the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{(B - A)\beta(\frac{zf'(z)}{f(z)} - \alpha) + A(\frac{zf'(z)}{f(z)} - 1)} \right| < 1 \tag{1.5}$$

is satisfied for some α $(0 \leq \alpha < 1)$, β $(0 < \beta \leq 1)$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and for all $z \in U$.

We note that by giving specific values to α, β, A and B, we obtain the following important subclasses studied by various authors in earlier papers:

(i) $S^*(\alpha, \beta, -1, 1) = S^*(\alpha, \beta)$, is the class of starlike functions of order α and type β introduced by Juneja and Mogra [3] and in [7] Owa showed some results for the class $S^*(\alpha, \beta)$.

(ii) $S^*(0,1,-1,1) = S^*$ and $S^*(\alpha,1,-1,1) = S^*(\alpha)$ are, respectively, the well-known class of starlike functions and the class of starlike functions of order α introduced by Robertson [9].

(iii) $S^*(\alpha,\frac{1}{2},-1,1) = \overline{S}_\alpha$, $0 \leq \alpha < 1$, is the subclass of the class of starlike functions of order α, studied by McCarty [5] and Wright [13].

(iv) $S^*(0,\frac{1}{2},-1,1)$, $S^*(0,\frac{2\delta-1}{2\delta},-1,1)$, $\delta > \frac{1}{2}$, are the classes introduced by Ram Singh [10,11].

(v) $S^*(\frac{1-\gamma}{1+\gamma},\frac{1+\gamma}{2},-1,1) = S(\gamma)$, $0 < \gamma \leq 1$, is the class of starlike functions of order γ introduced by Padmanbhan [8].

(vi) $S^*(1-\alpha,\frac{1}{2},-1,1) = S_\alpha$, $0 < \alpha \leq 1$, is the class of starlike functions introduced by Eenigenburg [2].

2. A representation formula.

Let Q denote the class of functions Ψ which are analytic in the unit disc U and which satisfy $|\Psi(z)| \leq 1$ for all $z \in U$. We require the following lemma:

Lemma 1. *Let $H(z) = 1+c_1z+c_2z^2+\cdots$ be analytic in U and satisfy the condition*

$$\left| \frac{H(z)-1}{(B-A)\beta(H(z)-\alpha)+A(H(z)-1)} \right| < 1,$$

$$(0 \leq \alpha < 1, 0 < \beta \leq 1 \text{ and } -1 \leq A < B \leq 1, 0 < B \leq 1) \tag{2.1}$$

for all $z \in U$. Then we have

$$H(z) = \frac{1+[(B-A)\alpha\beta+A]z\Psi(z)}{1+[(B-A)\beta+A]z\Psi(z)} \tag{2.2}$$

for some $\Psi \in Q$. Conversely, any function H given by the formula (2.2) where $\Psi \in Q$ is analytic in U and satisfies (2.1) for all $z \in U$.

The lemma follows in a straight forward manner. So we omit the proof.

Theorem 1. *Let $f(z) = z+\Sigma_{n=2}^{\infty}a_nz^n$ be analytic in the unit disc U. Then $f(z) \in S^*(\alpha,\beta,A,B)$ if and only if*

$$f(z) = z\exp\left\{-(B-A)\beta(1-\alpha)\int_0^z \frac{\Psi(t)}{1+[(B-A)\beta+A]t\Psi(t)}\,dt\right\}, \tag{2.3}$$

for some $\Psi \in Q$.

Proof. Let $f \in S^*(\alpha,\beta,A,B)$, it is easily seen that $\frac{zf'(z)}{f(z)}$ satisfies the hypothesis of the first part of Lemma 1. Therefore we can write

$$\frac{zf'(z)}{f(z)} = \frac{1+[(B-A)\alpha\beta+A]z\Psi(z)}{1+[(B-A)\beta+A]z\Psi(z)},$$

where $\Psi \in Q$, $z \in U$. Thus we have

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = -\frac{(B-A)\beta(1-\alpha)z\Psi(z)}{1+[(B-A)\beta+A]z\Psi(z)}.$$

Integration gives (2.3) easily. Conversely, if f has the representation (2.3) for some $\Psi \in Q$ then, it follows that

$$\frac{zf'(z)}{f(z)} = \frac{1+[(B-A)\alpha\beta+A]z\Psi(z)}{1+[(B-A)\beta+A]z\Psi(z)}$$

so that by converse part of Lemma 1, we have $f \in S^*(\alpha,\beta,A,B)$. Hence the theorem.

3. Distortion theorems.

Theorem 2. *Let $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc U and suppose that $f \in S^*(\alpha,\beta,A,B)$. Then, for $0 \le \alpha < 1$, $\beta \neq (\frac{A}{A-B})$, $-1 \le A < B \le 1$, $0 < B \le 1$, $z \in U$*

$$|f(z)| \le \frac{|z|}{(1-[(B-A)\beta+A]|z|)^{\frac{(B-A)\beta(1-\alpha)}{(B-A)\beta+A}}}, \tag{3.1}$$

$$|f(z)| \ge \frac{|z|}{(1+[(B-A)\beta+A]|z|)^{\frac{(B-A)\beta(1-\alpha)}{(B-A)\beta+A}}} \tag{3.2}$$

whereas for $0 \le \alpha < 1$, $\beta = (\frac{A}{A-B})$, $-1 \le A < B \le 1$, $0 < B \le 1$, $z \in U$

$$|f(z)| \le |z| e^{-A(1-\alpha)|z|}, \tag{3.3}$$

$$|f(z)| \ge |z| e^{A(1-\alpha)|z|}. \tag{3.4}$$

All the above estimates are sharp.

Proof. Since $f \in S^*(\alpha,\beta,A,B)$, we observe that the condition (1.5), coupled with an application of Schwarz's Lemma [6], implies that, for $z \in U$, $\frac{zf'(z)}{f(z)}$ assumes values lying in the disc K' obtained by taking the line segment joining the points

$$\frac{1+[(B-A)\alpha\beta+A]|z|}{1+[(B-A)\beta+A]|z|} \quad \text{and} \quad \frac{1-[(B-A)\alpha\beta+A]|z|}{1-[(B-A)\beta+A]|z|} \tag{3.5}$$

as diameter. Hence we have

$$\frac{1+[(B-A)\alpha\beta+A]\mid z\mid}{1+[(B-A)\beta+A]\mid z\mid} \leq Re\Big\{\frac{zf'(z)}{f(z)}\Big\} \leq \frac{1-[(B-A)\alpha\beta+A]\mid z\mid}{1-[(B-A)\beta+A]\mid z\mid}. \quad (3.5)$$

Let $\mid z\mid= r$, then (3.5) gives

$$\begin{aligned}\log(\mid \frac{f(z)}{z}\mid) &= Re(\log(\frac{f(z)}{z})) = Re\int_0^z [\frac{f'(s)}{f(s)}-\frac{1}{s}]ds \\ &= \int_0^{|z|}\frac{1}{t}Re\Big\{te^{i\theta}\frac{f'(te^{i\theta})}{f(te^{i\theta})}-1\Big\}dt \\ &\leq \int_0^{|z|}\frac{(B-A)\beta(1-\alpha)}{1-[(B-A)\beta+A]t}\,dt. \end{aligned} \quad (3.6)$$

Now two cases arise. (i) If $0\leq\alpha<1$, $\beta\neq(\frac{A}{A-B})$, $-1\leq A<B\leq 1$, $0<B\leq 1$, then (3.6) gives

$$\log(\mid \frac{f(z)}{z}\mid) \leq -\frac{(B-A)\beta(1-\alpha)}{(B-A)\beta+A}\log(1-[(B-A)\beta+A]\mid z\mid).$$

Thus we have

$$\mid \frac{f(z)}{z}\mid \leq \frac{1}{(1-[(B-A)\beta+A]\mid z\mid)^{\frac{(B-A)\beta(1-\alpha)}{(B-A)\beta+A}}}$$

which gives (3.1).

(ii) If $0\leq\alpha<1$, $\beta=(\frac{A}{A-B})$, $-1\leq A<B\leq 1$, $0<B\leq 1$, then (3.6) gives

$$\log(\mid \frac{f(z)}{z}\mid) \leq -A(1-\alpha)\int_0^{|z|}dt = -A(1-\alpha)\mid z\mid.$$

This proves (3.3). To prove the remaining estimates, (3.5) gives

$$\begin{aligned} rRe\Big\{\frac{\partial}{\partial r}(\log\frac{f(z)}{z})\Big\} &= Re\Big\{\frac{zf'(z)}{f(z)}-1\Big\} \\ &\geq -\frac{(B-A)\beta(1-\alpha)\mid z\mid}{1+[(B-A)\beta+A]\mid z\mid}. \end{aligned}$$

Thus, we have

$$\log(\mid \frac{f(z)}{z}\mid) = Re(\log(\frac{f(z)}{z})) \geq \int_0^z \frac{-(B-A)\beta(1-\alpha)}{1+[(B-A)\beta+A]t}dt. \quad (3.7)$$

Again two cases arise. (i) If $0 \leq \alpha < 1$, $\beta \neq (\frac{A}{A-B})$, $-1 \leq A < B \leq 1,\ 0 < B \leq 1$, then (3.7) gives

$$\log(\mid \frac{f(z)}{z} \mid) \ \geq \ -\frac{(B-A)\beta(1-\alpha)}{(B-A)\beta + A} \log(1 + [(B-A)\beta + A] \mid z \mid).$$

This proves (3.2).

(ii) If $0 \leq \alpha < 1$, $\beta = (\frac{A}{A-B})$, $-1 \leq A < B \leq 1,\ 0 < B \leq 1$, then (3.7) gives

$$\log(\mid \frac{f(z)}{z} \mid) \ \geq \ A(1-\alpha)\int_0^{|z|} dt \ = \ A(1-\alpha) \mid z \mid$$

which gives (3.4). Hence the theorem.

Equality in (3.1) and (3.2) holds for the function

$$f(z) \ = \ \frac{z}{(1 - [(B-A)\beta + A]z)^{\frac{(B-A)\beta(1-\alpha)}{(B-A)\beta+A}}}$$

whereas in (3.3) and (3.4) it holds for the function

$$f(z) \ = \ ze^{-A(1-\alpha)z}.$$

4. A sufficient condition for a function to be in $S^*(\alpha, \beta, A, B)$.

Theorem 3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in the unit disc* U*. If for some* α, β, A, B $(0 \leq \alpha < 1,\ 0 < \beta \leq (\frac{A}{A-B}),\ -1 \leq A < B \leq 1,\ 0 < B \leq 1)$,

$$\sum_{n=2}^{\infty} \{n[(1-A) - (B-A)\beta] - (1+A) + (A-B)\alpha\beta\} \mid a_n \mid \ \leq \ (B-A)\beta(1-\alpha). \quad (4.1)$$

Then $f(z)$ *belongs to* $S^*(\alpha, \beta, A, B)$.

Proof. we employ the same technique as used by Clunie and Keogh [1]. Thus suppose that (4.1) holds, then, for $\mid z \mid = r < 1$,

$$
\begin{aligned}
&| zf'(z)-f(z) | \ - \ | (B-A)\beta(zf'(z)-\alpha f(z))+A(zf'(z)-f(z)) | \\
&= | \sum_{n=2}^{\infty}(n-1)a_n z^n | \ - \ | (B-A)\beta(1-\alpha)z \\
&\quad + \sum_{n=2}^{\infty}[-A+(A-B)\alpha\beta]a_n z^n + \sum_{n=2}^{\infty}[(B-A)\beta+A]na_n z^n | \\
&\leq \sum_{n=2}^{\infty}(n-1) | a_n | r^n - \Big\{ | (B-A)\beta(1-\alpha)z \\
&\quad + \sum_{n=2}^{\infty}[-A+(A-B)\alpha\beta]a_n z^n | - \sum_{n=2}^{\infty}[-A-(B-A)\beta]n | a_n | r^n \} \\
&\leq \sum_{n=2}^{\infty}(n-1) | a_n | r^n - \Big\{(B-A)\beta(1-\alpha)r \\
&\quad - \sum_{n=2}^{\infty}[-A+(A-B)\alpha\beta] | a_n | r^n - \sum_{n=2}^{\infty}[-A-(B-A)\beta]n | a_n | r^n \} \\
&< [\sum_{n=2}^{\infty}(n-1) | a_n | -(B-A)\beta(1-\alpha) \\
&\quad + \sum_{n=2}^{\infty}(-A+(A-B)\alpha\beta-An-(B-A)\beta n) | a_n |] \ r \leq 0, \ by\ (4.1).
\end{aligned}
$$

Hence it follows that

$$
\left| \frac{\frac{zf'(z)}{f(z)}-1}{(A-B)\beta(\frac{zf'(z)}{f(z)}-\alpha)+A(\frac{zf'(z)}{f(z)}-1)} \right| < 1,
$$

so that $f \in S^*(\alpha,\beta,A,B)$.

Remark. Since $f \in S^*(\alpha,(\frac{A}{A-B}),A,B)$ implies $f \in S^*(\alpha,\beta,A,B)$ for $(\frac{A}{A-B}) \leq \beta \leq 1$, the condition (4.1) for $\beta = (\frac{A}{A-B})$ i.e., the condition

$$
\sum_{n=2}^{\infty}\{n-1-A(1-\alpha)\} | a_n | \leq -A(1-\alpha) \tag{4.2}
$$

can also be used as a sufficient condition for a function to be in $S^*(\alpha,\beta,A,B)$ for $0 \leq \alpha < 1$, $(\frac{A}{A-B}) \leq \beta \leq 1$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$,

5. Coefficient estimates.

We need in our discussion the following two lemmas:

Lemma 2 [6]. *If $w(z) \in \Omega$, the $| w(z) | \leq | z |$ and if $w(z) = \Sigma_{n=1}^{\infty} c_n z^n$ then*

$$| c_1 | \leq 1$$

and

$$| c_2 | \leq 1 - | c_1 |^2 . \tag{5.1}$$

Lemma 3 [4]. *Let $w(z) = \Sigma_{n=1}^{\infty} c_n z^n$ be analytic with $| w(z) | < 1$ in U. If ν is any complex number then*

$$| c_2 - \nu c_1^2 | \leq \max\{1, | \nu |\}. \tag{5.2}$$

Equality may be attained with functions $w(z) = z^2$ and $w(z) = z$.

Theorem 4. *If $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n \in S^*(\alpha, \beta, A, B)$, $\beta \neq (\frac{A}{A-B})$, then*

(a) *for any real number μ, we have*

$$| a_3 - \mu a_2^2 | \leq \begin{cases} \frac{(B-A)\beta(1-\alpha)}{2}\{(B-A)\beta(1-\alpha)(1-2\mu) \\ \qquad +[(B-A)\beta + A]\} & \text{if } \mu \leq \frac{1}{2} \\ \frac{(B-A)\beta(1-\alpha)}{2} & \text{if } \frac{1}{2} \leq \mu \leq \frac{1+A+(B-A)\beta(2-\alpha)}{2(B-A)\beta(1-\alpha)} \\ \frac{(B-A)\beta(1-\alpha)}{2}\{(B-A)\beta(1-\alpha)(2\mu-1) \\ \qquad -[(B-A)\beta + A]\} & \text{if } \mu \geq \frac{1+A+(B-A)\beta(2-\alpha)}{2(B-A)\beta(1-\alpha)} \end{cases} \tag{5.3}$$

and

(b) *for any complex number μ, we obtain*

$$| a_3 - \mu a_2^2 | \leq \frac{(B-A)\beta(1-\alpha)}{2} \max\{1, | (B-A)\beta(1-\alpha)(2\mu-1) - [(B-A)\beta + A] |\}. \tag{5.4}$$

The result is sharp for each μ either real or complex.

Proof. Since $f(z) \in S^*(\alpha, \beta, A, B)$, (2.3) gives

$$\frac{zf'(z)}{f(z)} = \frac{1 + [(B-A)\alpha\beta + A]w(z)}{1 + [(B-A)\beta + A]w(z)}, \tag{5.5}$$

where $w(z) = z\Psi(z) = \Sigma_{k=1}^{\infty} c_k z^k \in \Omega$.

From (5.5), we have

$$\begin{aligned} & w(z) \\ = & \frac{zf'(z) - f(z)}{[-A + (A-B)\beta]zf'(z) + [(B-A)\alpha\beta + A]f(z)} \\ = & \frac{\sum_{k=2}^{\infty}(k-1)a_k z^{k-1}}{(A-B)\beta(1-\alpha) + \sum_{k=2}^{\infty}\{[-A + (A-B)\beta]k + [(B-A)\alpha\beta + A]\}a_k z^{k-1}} \\ = & \frac{-1}{(B-A)\beta(1-\alpha)}[a_2 z + 2a_3 z^2 - \frac{(A-B)\beta(2-\alpha) - A}{(A-B)\beta(1-\alpha)} a_2^2 z^2 + \cdots] \end{aligned}$$

and then comparing coefficients of z and z^2 on both sides, we have

$$\begin{aligned} c_1 &= -\frac{a_2}{(B-A)\beta(1-\alpha)}, \\ c_2 &= -\frac{2a_3}{(B-A)\beta(1-\alpha)} + \frac{(B-A)\beta(2-\alpha)+A}{(B-A)^2\beta^2(1-\alpha)^2}a_2^2. \end{aligned}$$

Thus

$$a_2 = -(B-A)\beta(1-\alpha)$$

and

$$a_3 = -\frac{(B-A)\beta(1-\alpha)}{2}c_2 + \frac{(B-A)\beta(2-\alpha)+A}{2(B-A)\beta(1-\alpha)}a_2^2.$$

Hence

$$\begin{aligned} a_3-\mu a_2^2 &= -\frac{(B-A)\beta(1-\alpha)}{2}c_2 + [\frac{(B-A)\beta(2-\alpha)+A}{2(B-A)\beta(1-\alpha)} - \mu]a_2^2 \\ &= -\frac{(B-A)\beta(1-\alpha)}{2}c_2 + [\frac{(B-A)\beta(2-\alpha)+A}{2(B-A)\beta(1-\alpha)} \\ &\quad -\mu]\cdot(B-A)^2\beta^2(1-\alpha)^2c_1^2. \end{aligned} \tag{5.6}$$

Thus taking modulus of both sides of (5.6), we are led to

$$\mid a_3-\mu a_2^2 \mid = \frac{(B-A)\beta(1-\alpha)}{2} \mid c_2-\{-(B-A)\beta(1-\alpha)(2\mu-1)+[(B-A)\beta+A]\}c_1^2 \mid. \tag{5.7}$$

(a) When μ is real.

For real μ, (5.7) becomes

$$\mid a_3-\mu a_2^2 \mid \leq \frac{(B-A)\beta(1-\alpha)}{2}\{\mid c_2 \mid + \mid (B-A)\beta(1-\alpha)(2\mu-1)-[(B-A)\beta+A] \mid\mid c_1 \mid^2\}. \tag{5.8}$$

Applying Lemma 2 for $\mid c_2 \mid$ in (5.8) we obtain

$$\mid a_3-\mu a_2^2 \mid \leq \frac{(B-A)\beta(1-\alpha)}{2}\{1+[\mid (B-A)\beta(1-\alpha)(2\mu-1)-[(B-A)\beta+A] \mid -1] \mid c_1 \mid^2\}. \tag{5.9}$$

Again using Lemma 2 for $\mid c_1 \mid$ in (5.9) we are led to

$$\mid a_3-\mu a_2^2 \mid \leq \frac{(B-A)\beta(1-\alpha)}{2}\{\mid (B-A)\beta(1-\alpha)(2\mu-1)-[(B-A)\beta+A] \mid\}. \tag{5.10}$$

Thus from (5.10) with simple computations we obtain the results of (5.3) stated in (a) of the theorem for various values of real μ.

(b) When μ is a complex number.

For any complex number μ (5.7) may be written as

$$|a_3-\mu a_2^2| \leq \frac{(B-A)\beta(1-\alpha)}{2} \,|\, c_2 - \{-(B-A)\beta(1-\alpha)(2\mu-1)+[(B-A)\beta+A]\}c_1^2 \,| \,. \tag{5.11}$$

Applying Lemma 3 in (5.11) we get

$$|a_3-\mu a_2^2| \leq \frac{(B-A)\beta(1-\alpha)}{2}\max\{1, |(B-A)\beta(1-\alpha)(2\mu-1)-[(B-A)\beta+A]|\} \tag{5.12}$$

which is (5.4) in (b) of the theorem.

The sharpness of (5.3) and (5.4) follows from that of (5.2).

Theorem 5. *Let $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ be in $S^*(\alpha,\beta,A,B)$.*

(a) *If*

$$\beta(1-\alpha)(k-\alpha) > \frac{k-1}{(B-A)^2\beta}\{(k-1)(1-A^2)-(k-\alpha)(B-A)\beta[(B-A)\beta+2A]\},$$

Let

$$M = \frac{\beta(1-\alpha)(k-\alpha)}{\frac{(k-1)}{(B-A)^2\beta}\{(k-1)(1-A^2)-(k-\alpha)(B-A)\beta[(B-A)\beta+2A]\}},$$

$k = 2, 3, \cdots, n-1$. Then

$$|a_n| \leq \frac{\Pi_{k=2}^{n}\{[(B-A)\beta+A]k - 2(A+\frac{B-A}{2}\beta+\frac{B-A}{2}\alpha\beta)\}}{(n-1)!} \tag{5.13}$$

for $n = 2, 3, \cdots, M+2$ and

$$|a_n| \leq \frac{1}{n-1}\frac{\Pi_{k=2}^{M+3}\{[(B-A)\beta+A]k - 2(A+\frac{A-B}{2}\beta+\frac{A-B}{2}\alpha\beta)\}}{(M+1)!}, \qquad n > M+2. \tag{5.14}$$

(b) *If*

$$\beta(1-\alpha)(k-\alpha) \leq \frac{(k-1)}{(B-A)^2\beta}\{(k-1)(1-A^2)-(k-\alpha)(B-A)\beta[(B-A)\beta+2A]\}, \quad \textit{then}$$

$$|a_n| \leq \frac{(B-A)\beta(1-\alpha)}{n-1} \qquad \textit{for } n \geq 2. \tag{5.15}$$

The estimates in (5.13) and (5.15) are sharp.

Proof. Since $f \in S^*(\alpha,\beta,A,B)$, we have

$$\frac{zf'(z)}{f(z)} = \frac{1+[(B-A)\alpha\beta+A]w(z)}{1+[(B-A)\beta+A]w(z)}, \tag{5.16}$$

where $w(z) = \Sigma_{k=1}^{\infty} t_k z^k = z\Psi(z) \in \Omega$. From (5.16), we have

$$[(B-A)\beta(1-\alpha)z + \sum_{k=2}^{\infty}\{[(B-A)\beta+A]k - [(B-A)\alpha\beta+A]\}a_k z^k][\sum_{k=1}^{\infty} t_k z^k]$$
$$= \sum_{k=2}^{\infty}(1-k)a_k z^k. \tag{5.17}$$

Equating corresponding coefficients on both sides of (5.17) we observe that the coefficients a_n depends only on $a_2, a_3, \cdots, a_{n-1}$ for $n \geq 2$. Hence for $n \geq 2$, it follows from (5.17) that

$$[(B-A)\beta(1-\alpha)z + \sum_{k=2}^{n-1}\{[(B-A)\beta+A]k - [(B-A)\alpha\beta+A]\}a_k z^k]w(z)$$
$$= \sum_{k=2}^{n}(1-k)a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k$$

which, since $| w(z) |< 1$, yields

$$| (B-A)\beta(1-\alpha)z + \sum_{k=2}^{n-1}\{[(B-A)\beta+A]k - [(B-A)\alpha\beta+A]\}a_k z^k |$$
$$\geq | \sum_{k=1}^{n}(1-k)a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k | . \tag{5.18}$$

Squaring both sides of (5.18) and integrating round $| z |= r$, $0 < r < 1$, we obtain

$$\{(B-A)^2\beta^2(1-\alpha)^2 r^2 + \sum_{k=2}^{n-1}\{[(B-A)\beta+A]k - [(B-A)\alpha\beta+A]\}^2 | a_k |^2 r^{2k}\}$$
$$\geq \sum_{k=2}^{n}(k-1)^2 | a_k |^2 r^{2k} + \sum_{k=n+1}^{\infty} | b_k |^2 r^{2k}.$$

If we take the limit as r approaches 1, then

$$\{(B-A)^2\beta^2(1-\alpha)^2 + \sum_{k=2}^{n-1}\{[(B-A)\beta+A]k - [(B-A)\alpha\beta+A]\}^2 | a_k |^2\}$$
$$\geq \sum_{k=2}^{n}(k-1)^2 | a_k |^2$$

or

$$(n-1)^2 \mid a_n \mid^2 \leq (B-A)^2\beta^2(1-\alpha)^2 + \sum_{k=2}^{n-1}\{([(B-A)\beta + A]k - [(B-A)\alpha\beta + A])^2 - (k-1)^2\} \mid a_k \mid^2, \qquad n \geq 2. \tag{5.19}$$

Now two cases arise. (a) Let

$$\beta(1-\alpha)(k-\alpha) > \frac{k-1}{(B-A)^2\beta}\{(k-1)(1-A^2) - (k-\alpha)(B-A)\beta[(B-A)\beta + 2A]\}.$$

If $n \leq M+2$, (5.19) gives in particular

$$\begin{aligned} \mid a_2 \mid &\leq (B-A)\beta(1-\alpha), \\ 4 \mid a_3 \mid^2 &\leq (B-A)^2\beta^2(1-\alpha)^2 + \{(2[(B-A)\beta + A] \\ &\quad - [(B-A)\alpha\beta + A])^2 - 1\} \mid a_2 \mid^2 \\ &\leq (B-A)^2\beta^2(1-\alpha)^2 \cdot 2[(B-A)\beta + A] - [(B-A)\alpha\beta + A])^2 \\ i.e., \qquad \mid a_3 \mid &\leq \frac{\Pi_{k=2}^{3}\{[(B-A)\beta + A]k - 2(A + \frac{B-A}{2}\beta + \frac{B-A}{2}\alpha\beta)\}}{2!}. \end{aligned}$$

Mathematical induction shows that

$$\mid a_n \mid \leq \frac{\Pi_{k=2}^{n}\{[(B-A)\beta + A]k - 2(A + \frac{B-A}{2}\beta + \frac{B-A}{2}\alpha\beta)\}}{(n-1)!}$$

for $n = 2, 3, \cdots, M+2$, which completes the proof of (5.13).

Next, we suppose $n > M+2$. Then (5.19) gives

$$\begin{aligned} (n-1)^2 \mid a_n \mid^2 &\leq (B-A)^2\beta^2(1-\alpha)^2 + \sum_{k=2}^{M+2}\{([(B-A)\beta + A]k \\ &\quad - [(B-A)\alpha\beta + A])^2 - (k-1)^2\} \mid a_k \mid^2 \\ &\quad + \sum_{k=M+3}^{n-1}\{([(B-A)\beta + A]k - [(B-A)\alpha\beta + A)^2 - (k-1)^2\} \mid a_k \mid^2 \\ &\leq (B-A)^2\beta^2(1-\alpha)^2 + \sum_{k=2}^{M+2}\{([(B-A)\beta + A]k \\ &\quad - [(B-A)\alpha\beta + A])^2 - (k-1)^2\} \mid a_k \mid^2 . \end{aligned} \tag{5.20}$$

Substituting upper estimates for $a_2, a_3, \cdots, a_{M+2}$ obtained above in (5.20), we get

$$(n-1)^2 \mid a_n \mid^2 \leq ([(B-A)\beta + A](M+2) - [(B-A)\alpha\beta + A])^2 \cdot$$

$$\cdot\left(\frac{\Pi_{k=2}^{M+2}\{[(B-A)\beta+A]k-2(A+\frac{B-A}{2}\beta+\frac{B-A}{2}\alpha\beta)\}}{(M+1)!}\right)^2$$

$$i.e., \mid a_n \mid \leq \frac{1}{n-1}\frac{\Pi_{k=2}^{M+3}\{[(B-A)\beta+A]k-2(A+\frac{B-A}{2}\beta+\frac{B-A}{2}\alpha\beta)\}}{(M+1)!}$$

this proves (5.14).

(b) If

$$\beta(1-\alpha)(k-\alpha) \leq \frac{k-1}{(B-A)^2\beta}\{(k-1)(1-A^2)-(k-\alpha)(B-A)\beta[(B-A)\beta+2A]\},$$

then (5.19) gives

$$(n-1)^2 \mid a_n \mid^2 \leq (B-A)^2\beta^2(1-\alpha)^2 \ for\ n \geq 2$$

$$i.e., \qquad \mid a_n \mid \leq \frac{(B-A)\beta(1-\alpha)}{n-1} \ for\ n \geq 2,$$

which gives (5.15).

The function f given by

$$\frac{zf'(z)}{f(z)} = \frac{1-[(B-A)\alpha\beta+A]z}{1-[(B-A)\beta+A]z}, \ where$$

$$\beta(1-\alpha)(k-\alpha) > \frac{k-1}{(B-A)^2\beta}\{(k-1)(1-A^2)-(k-\alpha)(B-A)\beta\cdot[(B-A)\beta+2A]\}$$

shows that the estimates in (5.13) are sharp while the estimates in (5.15) are sharp for the function

$f(z) = z\exp[(B-A)\beta(1-\alpha)/(n-1)]z^{n-1}$, where

$$\beta(1-\alpha)(k-\alpha) \leq \frac{k-1}{(B-A)^2\beta}\{(k-1)(1-A^2)-(k-\alpha)(B-A)\beta\cdot[(B-A)\beta+2A]\} \ and\ n \geq 2.$$

6. The radius of convexity for functions in the class $S^*(\alpha,\beta,A,B)$.

Singh and Goel [12] showed the following lemma.

Lemma 4. *If $w(z) \in \Omega$, then for $z \in U$*

$$\mid zw'(z)-w(z) \mid \leq \frac{\mid z \mid^2 - \mid w(z) \mid^2}{1-\mid z \mid^2}. \tag{6.1}$$

Lemma 5. *Let $w(z) \in \Omega$. Then we have*

$$Re\left\{\frac{zw'(z)}{(1+[(B-A)\beta+A]w(z))(1+[(B-A)\alpha\beta+A]w(z))}\right\}$$
$$\leq -\frac{1}{(B-A)^2\beta^2(1-\alpha)^2}Re\Big\{[(B-A)\beta+A]p(z) + \frac{[(B-A)\alpha\beta+A]}{p(z)} - 2(\frac{B-A}{2}\alpha\beta+\frac{B-A}{2}\beta+A)\Big\}$$
$$+\frac{r^2\mid[(B-A)\beta+A]p(z)-[(B-A)\alpha\beta+A]\mid^2-\mid 1-p(z)\mid^2}{(B-A)^2\beta^2(1-\alpha)^2(1-r^2)\mid p(z)\mid}, \qquad (6.2)$$

where $p(z)=\frac{1+[(B-A)\alpha\beta+A]w(z)}{1+[(B-A)\beta+A]w(z)}$, $r=\mid z\mid$ *and* $0\leq\alpha<1$, $0<\beta\leq 1$, $-1\leq A<B\leq 1$, $0<B\leq 1$.

The proof of Lemma 5 follows easily from Lemma 4.

Remark. The transformation $p(z)=\frac{1+[(B-A)\alpha\beta+A]w(z)}{1+[(B-A)\beta+A]w(z)}$, maps the circle $\mid w(z)\mid\leq r$ onto the circle

$$\mid p(z)-\frac{1-[(B-A)\beta+A][(B-A)\alpha\beta+A]r^2}{1-[(B-A)\beta+A]^2r^2}\mid\leq\frac{(B-A)\beta(1-\alpha)r}{1-[(B-A)\beta+A]^2r^2}.$$

Theorem 5. *If $f(z)\in S^*(\alpha,\beta,A,B)$, then for $\mid z\mid=r$, $0<r<1$,*

$$Re\{1+\frac{zf''(z)}{f'(z)}\}\geq$$
$$\begin{cases}\dfrac{1-(2A+(B-A)\beta-3(B-A)\alpha\beta)r+(-A-(B-A)\alpha\beta)^2r^2}{(1+[(B-A)\beta+A]r)(1+[(B-A)\alpha\beta+A]r)} \quad for\ R_0\leq R_1,\\ \dfrac{2}{(B-A)\beta(1-\alpha)(1-r^2)}\{\sqrt{[(B-A)\alpha\beta+A+1](1-[(B-A)\alpha\beta+A]r^2)}\cdot \\ \sqrt{[(B-A)\beta(2-\alpha)+A+1]-(B-A)\beta[(B-A)\beta-\alpha+2(1+A)]r^2} \\ -(1+[-A-(B-A)\alpha\beta][(B-A)\beta+A]r^2)+ \\ +(-A-\frac{B-A}{2}\beta-\frac{B-A}{2}\alpha\beta)(1-r^2)\}\quad for\ R_0\geq R_1,\end{cases} \qquad (6.3)$$

where

$$a = \frac{1-[(B-A)\beta+A][(B-A)\alpha\beta+A]r^2}{1-[(B-A)\beta+A]^2r^2},$$
$$d = \frac{(B-A)\beta(1-\alpha)r}{1-[(B-A)\beta+A]^2r^2},$$
$$R_0 = [\frac{[(B-A)\alpha\beta+A+1](1-[(B-A)\alpha\beta+A]r^2)}{[(B-A)\beta(2-\alpha)+A+1]-(B-A)\beta[(B-A)\beta-\alpha+2(1+A)]r^2}]^{\frac{1}{2}}$$

and

$$R_1 = a-d = \frac{1+[(B-A)\alpha\beta+A]r}{1+[(B-A)\beta+A]r}.$$

All these bounds are sharp.

Proof. Since the function $f(z)$ belongs to the class $S^*(\alpha,\beta,A,B)$, by using Theorem 1, we get

$$\frac{zf'(z)}{f(z)} = \frac{1+[(B-A)\alpha\beta+A]z\Psi(z)}{1+[(B-A)\beta+A]z\Psi(z)} \tag{6.4}$$

where $\Psi\in Q$ for all $z\in U$. Writing $z\Psi(z)=w(z)$, where $w\in\Omega$, we get

$$\frac{zf'(z)}{f(z)} = \frac{1+[(B-A)\alpha\beta+A]w(z)}{1+[(B-A)\beta+A]w(z)}. \tag{6.5}$$

Differentiating (6.5) logarithmically, we have

$$\begin{aligned}1+\frac{zf''(z)}{f'(z)} &= \frac{1+[(B-A)\alpha\beta+A]w(z)}{1+[(B-A)\beta+A]w(z)} - \\ (B-A)\beta(1-\alpha)&\{\frac{zw'(z)}{(1+[B-A)\beta+A]w(z))(1+[(B-A)\alpha\beta+A]w(z))}\}.\end{aligned} \tag{6.6}$$

An application of Lemma 5 gives

$$\begin{aligned}&Re\{1+\frac{zf''(z)}{f'(z)}\}\\ \geq\ &\frac{1}{(B-A)\beta(1-\alpha)}[Re\{((B-A)\beta(2-\alpha)+A)p(z)+\frac{(B-A)\alpha\beta+A}{p(z)}\}\\ &-\frac{r^2\mid[(B-A)\beta+A]p(z)-[(B-A)\alpha\beta+A]\mid^2-\mid 1-p(z)\mid^2}{(1-r^2)\mid p(z)\mid}]\\ &-\frac{(B-A)\beta(1+\alpha)+2A}{(B-A)\beta(1-\alpha)},\end{aligned} \tag{6.7}$$

where

$$p(z) = \frac{1+[(B-A)\alpha\beta+A]w(z)}{1+[(B-A)\beta+A]w(z)}.$$

By setting $p(z)=a+\xi+i\eta$; $R^2=(a+\xi)^2+\eta^2$, where $a=\frac{1-[(B-A)\beta+A][(B-A)\alpha\beta+A]r^2}{1-[(B-A)\beta+A]^2r^2}$ and denoting the expression on the right hand side of (6.7) by $G(\xi,\eta)$, we get

$$\begin{aligned}G(\xi,\eta) &= \frac{1}{(B-A)\beta(1-\alpha)}\Big[[(B-A)\beta(2-\alpha)+A](a+\xi)\\ &+[(B-A)\alpha\beta+A](a+\xi)R^{-2}-\frac{1-[(B-A)\beta+A]^2r^2}{1-r^2}(d^2-\xi^2-\eta^2)R^{-1}\Big]\\ &-\frac{(B-A)\beta(1+\alpha)+2A}{(B-A)\beta(1-\alpha)},\end{aligned} \tag{6.8}$$

where

$$d = \frac{(B-A)\beta(1-\alpha)r}{1-[(B-A)\beta+A]^2r^2}.$$

Differentiating (6.8) partially w.r.t. η, we get

$$\frac{\partial G}{\partial \eta} = \frac{1}{(B-A)\beta(1-\alpha)}\eta R^{-4}H(\xi,\eta), \tag{6.9}$$

where

$$\begin{aligned} H(\xi,\eta) &= 2(-A-(B-A)\alpha\beta)(a+\xi) \\ &+ \frac{(d^2-\xi^2-\eta^2)(1-[(B-A)\beta+A]^2r^2)}{1-r^2}R + 2\frac{1-[(B-A)\beta+A]^2r^2}{1-r^2}R^3. \end{aligned}$$

It is easily seen that $H(\xi,\eta)>0$ and so (6.9) gives that the minimum of $G(\xi,\eta)$ on every chord ξ=constant is reached when $\eta=0$ and thus the minimum of $G(\xi,\eta)$ in the circle $\xi^2+\eta^2\le d^2$ is attained on the diameter $\eta=0$. Hence putting $\eta=0$ in (6.8), we get

$$\begin{aligned} L(R) = G(\xi,0) &= \frac{1}{(B-A)\beta(1-\alpha)}\Big[([(B-A)\beta(2-\alpha)+A] \\ &+ \frac{1-[(B-A)\beta+A]^2r^2}{1-r^2})R \\ &+ \frac{[(B-A)\alpha\beta+A+1](1-[(B-A)\alpha\beta+A]r^2)}{1-r^2}R^{-1} \\ &- 2a\frac{1-[(B-A)\beta+A]^2r^2}{1-r^2}\Big] - \frac{(B-A)\beta(1+\alpha)+2A}{(B-A)\beta(1-\alpha)}, \end{aligned}$$

where $R=a+\xi$ and $a-d\le R\le a+d$. Thus the absolute minimum of $L(R)$ in $(0,\infty)$ is attained at

$$R_0 = \sqrt{\frac{[(B-A)\alpha\beta+A+1](1-[(B-A)\alpha\beta+A]r^2)}{[(B-A)\beta(2-\alpha)+A+1]-(B-A)\beta[(B-A)\beta-\alpha+2(1+A)]r^2}} \tag{6.10}$$

and equals

$$\begin{aligned} L(R_0) &= \frac{2}{(B-A)\beta(1-\alpha)(1-r^2)}\Big\{ \\ &\sqrt{[(B-A)\alpha\beta+A+1](1-[(B-A)\alpha\beta+A]r^2)}\cdot \\ &\sqrt{[(B-A)\beta(2-\alpha)+A+1]-(B-A)\beta[(B-A)\beta-\alpha+2(1+A)]r^2} \\ &-(1+[-A-(B-A)\alpha\beta][(B-A)\beta+A]r^2) \\ &+(-A-\frac{B-A}{2}\beta-\frac{B-A}{2}\alpha\beta)(1-r^2)\Big\}. \end{aligned} \tag{6.11}$$

It is easily seen that $R_0<a+d$, but R_0 is not always greater than $a-d$. In such a case when $R_0\notin[a-d,a+d]$, the minimum of $L(R)$ on the segment $[a-d,a+d]$ is attained

at $R_1 = a - d$ since $L(R)$ increases with R on this segment. The value of this minimum equals

$$\begin{aligned} L(R_1) &= L(a-d) \\ &= \frac{1-(-2A+(B-A)\beta-3(B-A)\alpha\beta)r+(-A-(B-A)\alpha\beta)^2r^2}{(1+[(B-A)\beta+A]r)(1+[(B-A)\alpha\beta+A]r)}. \end{aligned} \tag{6.12}$$

The two minima given by (6.11) and (6.12) coincide for such values of α, β, A, B $(0 \le \alpha < 1,\ 0 < \beta \le 1,\ -1 \le A < B \le 1, 0 < B \le 1)$ for which $R_0 = R_1$. The ineqality (6.3) follows from (6.11) and (6.12). This completes the proof of the theorem.

The functions given by

$$\frac{zf'(z)}{f(z)} = \frac{1-[(B-A)\alpha\beta+A]z}{1-[(B-A)\beta+A]z} \qquad for\ R_0 \le R_1;$$

and

$$\frac{zf'(z)}{f(z)} = \frac{1-(B-A)\alpha\beta bz+[(B-A)\alpha\beta+A]z^2}{1-(B-A)\beta bz+[(B-A)\beta+A]z^2} \qquad for\ R_0 \ge R_1$$

where b is determined by the relation

$$\frac{1-(B-A)\alpha\beta br+[(B-A)\alpha\beta+A]r^2}{1-(B-A)\beta br+[(B-A)\beta+A]r^2} = R_0 =$$
$$\left[\frac{[(B-A)\alpha\beta+A+1](1-[(B-A)\alpha\beta+A]r^2)}{[(B-A)\beta(2-\alpha)+A+1]-(B-A)\beta[(B-A)\beta-\alpha+2(1+A)]r^2}\right]^{\frac{1}{2}}$$

show that the results obtained in the theorem are sharp.

References

[1] J. Clunie, F.R. Keogh, On starlike and convex schlicht funtions, *J. London Math. Soc.*, 35 (1960), 229-233.

[2] P. J. Eenigenburg, A class of starlike mapping in the unit disc, *Compositio Math.*,(2) 24 (1972), 235-238.

[3] O.P. Juneja and M.L. Morgra, On starlike functions of order α and type β, *Rev. Roumaine Math. Pures Appl.*, 23 (1978), 751-765.

[4] F.R. Keogh and E.P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.*, 20 (1969), 8-12.

[5] C.P. McCarty, Starlike functions, *Proc. Amer. Math. Soc* ., 43 (1974), 361-366.

[6] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York, 1952.

[7] S. Owa, On the starlike functions of order α and type β, *Math. Japonica*, 27 (1982), 723-725.

[8] K.S. Padmanabhan, On certain classes of strlike functions in the unit disc, *J. Indian Math. Soc.* 32 (1968), 89-103.

[9] M.S. Robertson, On the theory of univalent functions, *Ann. Math.*, 37 (1936), 374-408.

[10] R. Singh, On a class of starlike functions, *Compositio Math.*, 19 (1967), 78-82.

[11] R. Singh, On a class of starlike functions II, *Ganita* 19 (1968), 103-110.
[12] V. Singh, R.M. Goel, On radii of convexity and starlikeness of some classes of functions, *J. Math. Soc. Japan*, 29 (1971), 323-339.
[13] D.J. Wright, On a class of starlike functions, *Compositio Math.*, 21 (1969), 122-124.

Dept. of Maths., Faculty of Science, Univ. of Mansoura, Mansoura, Egyst.