TAMKANG JOURNAL OF MATHEMATICS
Volume 21, Number 1, Spring 1990

TWO RESULTS ON C-CONGRUENCE NUMERICAL RADII

CHE-MAN CHENG

Abstract. Let M_n denote the set of $n \times n$ complex matrices. For A and C in M_n, define the C-congruence numerical radius of A by

$$\rho_C(A) = \max\{| tr(CUAU^t) |: U \text{ is unitary}\}.$$

First, we show that ρ_C is a norm on M_n if and only if C is neither symmetric nor skew-symmetric. Secondly, we use ρ_C to characterize two matrices A and B in M_n to be unitarily congruent (i.e. $A = UBU^t$ for some unitary U).

1. Introduction.

Let M_n denote the set of all $n \times n$ complex matrices and $\mathcal{U}_n$ the subset consisting of all $n \times n$ unitary matrices. Denote by $tr\ A, A^t$ and A^* the trace, transpose and conjugate transpose of $A(\in M_n)$ respectively. For A and C in M_n, the C-numerical range of A and the C-numerical radius of A defined by

$$W_C(A) = \{tr(CUAU^*) : U \in \mathcal{U}_n\}$$

and

$$r_C(A) = \max\{| z |: z \in W_C(A)\}$$

respectively have been studied extensively. In particular, Goldberg and Straus [2] showed that r_C is a norm on M_n if and only if C is nonscalar and $tr\ C \neq 0$ (for other proofs, see [6] and [8]). Li and Tsing [5] showed that $A = \alpha UBU^*$ with $\alpha \in C$ and $| \alpha |= 1$ if and only if $r_{A^*}(A) = r_{A^*}(B)$ and $r_{B^*}(A) = r_{B^*}(B)$.

Parallel to the C-numerical range, Thompson [10] introduced the C-congruence numerical range of A defined by

$$R_C(A) = \{tr(CUAU^t) : U \in \mathcal{U}_n\}.$$

Various results have also been obtained, see [1] and [9].

Received April 11, 1989.

The author wishes to express his thanks to Dr. Yik-Hoi Au-Yeung for his valuable advice and encouragement.

It is natural to define the C-congruence numerical radius of A by

$$\rho_C(A) = \max\{| z |: z \in R_C(A)\}.$$

By taking C to be any symmetric matrix (i.e. $C = C^t$) and A to be any skew-symmetric matrix (i.e. $A = -A^t$), or vice versa, we can easily check that $\rho_C(A) = 0$. So, ρ_C is not a norm of M_n if C is symmetric or skew symmetric. We shall prove, for other cases of C, ρ_C is a norm on M_n.

Two matrices A and B are said to be *unitarily congruent* (resp. *unitarily similar*) if $A = UBU^t$ (resp. $A = UBU^*$) for some $U \in \mathcal{U}_n$. Hong and Horn [3] gave a characterization for two matrices to be unitarily congruent in terms of unitary similarities. As an application of ρ_C, we shall prove that A and B are unitarily congruent if and only if $\rho_{A^*}(A) = \rho_{A^*}(B)$ and $\rho_{B^*}(A) = \rho_{B^*}(B)$.

In what follows, we shall assume $n > 1$ so as to avoid trivial modifications.

2. Two results on C-congruence numerical radii.

Since the relation $< A, B >= tr(AB^*)$ is an inner product on M_n, the following lemma is obvious.

Lemma 1. *Let $C \in M_n$. Then the following two statements are equivalent:*

(i) *For any $A \in M_n$, if $\rho_C(A) = 0$, then $A = 0$;*
(ii) *$span\{U^tCU : U \in \mathcal{U}_n\} = M_n$, where $span\{U^tCU : U \in \mathcal{U}_n\}$ is the linear subspace of M_n spanned by the set $\{U^tCU : U \in \mathcal{U}_n\}$.*

Let $\mathcal{S}_n$ (resp. $\mathcal{K}_n$) denote the set of all $n \times n$ symmetric (resp. skew-symmetric) matrices. Let $\text{diag}(s_1, \cdots, s_n)$ denote the diagonal matrix with the i^{th} diagonal entry being s_i. The direct sum of two square matrices A and B is written as $A \oplus B$. The following two lemmas are due to Takagi [7] and Youla[11] respectively.

Lemma 2. *Let $C \in \mathcal{S}_n$. Then there exists $U \in \mathcal{U}_n$ such that $U^tCU = diag(s_1, \cdots, s_n)$, where $s_i, i = 1, \cdots, n$ are non-negative real numbers and $s_i \geq s_{i+1}, i = 1, \cdots, n-1$.*

Lemma 3. *Let $C \in \mathcal{K}_n$. Then there exists $U \in \mathcal{U}_n$ such that*

$$U^tCU = \begin{pmatrix} 0 & s_1 \\ -s_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & s_m \\ -s_m & 0 \end{pmatrix} \oplus [0] \oplus \cdots \oplus [0]$$

where $s_i, i = 1, \cdots, m$ are positive real numbers and $s_i \geq s_{i+1}, i = 1, \cdot, m-1$.

Let E_{ij} denote the matrix in M_n with 1 at its (i, j) entry and zero elsewhere. If σ is a permutation of $\{1, \cdots, n\}$, then $P(\sigma)$ denotes the corresponding permutation matrix (i.e. $P(\sigma) = (\delta_{i,\sigma(i)})$). We also use the standard notation $(i_1, \cdots, i_r)$ to denote the permutation (in fact, a cycle) σ for which $\sigma(i_l) = i_{l+1}$ for $l = 1, \cdots, r-1$, $\sigma(i_r) = i_1$ and $\sigma(i) = i$ otherwise. As usual, $\sigma_1\sigma_2$ denotes the composition of σ_1 and σ_2.

Tam[8] showed that if A is nonscalar and $trA \neq 0$, then span$\{UAU^* : U \in \mathcal{U}_n\} = M_n$. The following Theorem gives a parallel result.

Theorem 1. *Let* $C(\neq 0) \in M_n$. *Then*

$$span\{U^tCU : U \in \mathcal{U}_n\} = \begin{cases} \mathcal{S}_n & if\ C \in \mathcal{S}_n \\ \mathcal{K}_n & if\ C \in \mathcal{K}_n \\ M_n & \text{otherwise.} \end{cases}$$

Proof. Case 1. $C \in \mathcal{S}_n$, then the inclusion span$\{U^tCU : U \in \mathcal{U}_n\} \subset \mathcal{S}_n$is trival. Since span$\{U^tCU : U \in \mathcal{U}_n\}$ is invariant under unitary congruence of C, by Lemma 2, we may assume $C = diag(s_1, \cdots, s_n)$.

Firstly, suppose that $C = E_{11}$. Let

$$V = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) \oplus I_{n-2} \in \mathcal{U}_n,$$

where I_{n-2} is the $(n-2) \times (n-2)$ identity matrix. Then

$$V^tCV = \left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) \oplus O_{n-2}$$

where O_{n-2} is the $(n-2) \times (n-2)$ zero matrix. Direct computations yield

$$P^t((1j))CP((1j)) = E_{jj},\ j = 1, \cdots, n;$$

$$P^t((1r)(2l))(V^tCV)P((1r)(2l)) = \frac{1}{2}(E_{rr} + E_{rl} + E_{lr} + E_{ll}) : 1 \leq r < l \leq n.$$

As the multiplication of permutation matrices in this way preserves unitary congruence, we have shown that $\{E_{jj} : j = 1, \cdots, n\} \cup \frac{1}{2}(E_{rr} + E_{rl} + E_{lr} + E_{ll}) : 1 \leq r < l \leq n\} \subset \{U^tCU : U \in \mathcal{U}_n\}$. Since they also form a basis of $\mathcal{S}_n$, we are done if $C = E_{11}$.

For the general case $C = diag(s_1, \cdots, s_n)$ with $s_1 \neq 0$ (else $C = 0$) let $W = diag(1, i, \cdots, i) \in \mathcal{U}_n$, where $i^2 = -1$. Then

$$\frac{1}{2s_1}(W^tCW + C) = E_{11}.$$

Hence, we have

$$span\{U^tCU : U \in \mathcal{U}_n\} \supseteq span\{U^tE_{11}U : U \in \mathcal{U}_n\} = \mathcal{S}_n,$$

and so Case 1 is settled.

Case 2. $C \in \mathcal{K}_n$. The proof is similar to case 1 and we give only a sketch. We apply Lemma 3, if $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus O_{n-2}$, then

$$\{P^t((1r)(2l))CP((1r)(2l)) : 1 \leq r < l \leq n\} = \{(E_{rl} - E_{lr}) : 1 \leq r < l \leq n\}$$

and is a basis of $\mathcal{K}_n$. For the general case, we let

$$W = \begin{cases} I_2 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if n is even} \\ I_2 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus [1] & \text{if n is odd.} \end{cases}$$

Then $\frac{1}{2s_1}(W^tCW + C) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus O_{n-2}$. Hence our result follows.

Case 3. C is neither symmetric nor skew-symmetric.

Claim: there exist $\lambda_i \in \mathcal{C}$ and $U_i \in \mathcal{U}_n$, $i = 1, \cdots, l$ such that $\Sigma_{i=1}^{l} \lambda_i U_i^t C U_i = C^t$.

Proof. Since $C + C^t$ is symmetric and non-zero, by case 1, we can find $V_1, \cdots, V_m \in \mathcal{U}_n$, where $m = 1 + \cdots + n = dim\ \mathcal{S}_n$, such that

$$\{V_i^t[\frac{1}{2}(C + C^t)]V_i : i = 1, \cdots, m\}$$

is a basis of $\mathcal{S}_n$. Since $dim\ \mathcal{S}_n > dim\ \mathcal{K}_n$,

$$\{V_i^t[\frac{1}{2}(C - C^t)]V_i : i = 1, \cdots, m\}$$

must be linearly dependent. Let $\beta_i \in \mathcal{C}$, $i = 1, \cdots, m$ be such that not all of them are zero and

$$\sum_{i=1}^{m} \beta_i V_i^t[\frac{1}{2}(C - C^t)]V_i = 0.$$

Then, as $C = \frac{1}{2}(C + C^t) + \frac{1}{2}(C - C^t)$,

$$\sum_{i=1}^{m} \beta_i V_i^t C V_i = \sum_{i=1}^{m} \beta_i V_i^t[\frac{1}{2}(C + C^t)]V_i$$

is symmetric and nonzero. By case 1 again, there exists $\alpha_j \in \mathcal{C}$ and $U_j \in \mathcal{U}_n, j = 1, \cdots, r$ such that

$$\sum_{j=1}^{r} \alpha_j U_j^t (\sum_{i=1}^{m} \beta_i V_i^t C V_i) U_j = C + C^t$$

Hence the claim is valid.

Now, for any $A \in M_n$, $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$.
With $\Sigma_{i=1}^l \lambda_i U_i^t C U_i = C^t$, by Case 1 and Case 2 respectively, we conclude that

$$\frac{1}{2}(A + A^t),\ \frac{1}{2}(A - A^t) \in span\{U^t C U \,:\, U \in \mathcal{U}_n\}.$$

Hence $A \in span\{U^t C U \,:\, U \in \mathcal{U}_n\}$ and the proof is completed.

By Lemma 1 and Theorem 1, we readily have:

Theorem 2. *For $n > 1$, ρ_C is a norm on M_n if and only if C is neither symmetric nor skew-symmetric.*

From Theorem 1 and results similar to Lemma 1, we see that if $C(\neq)$ is symmetric (resp. skew-symmetric), ρ_C is a norm on $\mathcal{S}_n$ (resp. $\mathcal{K}_n$).

The idea of the proof of the following theorem was given by Li and Tsing [5]. However, since the proof is short and for the sake of completeness, we reproduce the proof.

Theorem 3. *Let $A, B \in M_n$. Then A and B are unitarily congruent if and only if $\rho_{A^*}(A) = \rho_{A^*}(B)$ and $\rho_{B^*}(A) = \rho_{B^*}(B)$.*

Proof. ($\Rightarrow$) Since $\rho_C(A)$ is invariant under unitary congruence of A, the result follows. ($\Leftarrow$) Let $\|\cdot\|$ denote the norm on M_n induced by the inner product $< A, B >= tr(AB^*)$ on M_n.

For any $A, C \in M_n$, by Cauchy-Schwarz inequality,

$$\begin{aligned} \rho_C(A) &= \max\{|< U^t C U, A^* >| : U \in \mathcal{U}_n\} \\ &= \max\{|< A, U^* C^* U^{*^t} >| : U \in \mathcal{U}_n\} \\ &\leq \|A\| \cdot \|C^*\|, \end{aligned}$$

with the equality holds if and only if there exists $\alpha \in \mathcal{C}$ and $U \in \mathcal{U}_n$ such that $A = \alpha U C^* U^t$. By the above inequality and given assumption, we have

$$\|A\|^2 = \rho_{A^*}(A) = \rho_{A^*}(B) \leq \|A\| \cdot \|B\|$$

and

$$\|B\|^2 = \rho_{B^*}(B) = \rho_{B^*}(A) \leq \|A\| \, \|B\|.$$

So $\|A\| = \|B\|$ and hence $\rho_{B^*}(A) = \|A\| \cdot \|B\|$. Since the equality holds, $A = \alpha U B U^t$ for some unitary U and complex number α. As $\|A\| = \|B\|$, we can assume $|\alpha| = 1$. Then $A = (\alpha^{1/2} U) B (\alpha^{1/2} U)^t$ and $\alpha^{1/2} U \in \mathcal{U}_n$.

Acknowledgements.

Through private communication, the author learnt that Li and Tsing [4] had also obtained the result of Theorem 1 for the cases $C \in \mathcal{S}_n$ or $C \in \mathcal{K}_n$ when considering another problem. The author also wishes to thank the referee for his valuable comments on the paper.

References

[1] M.D. Choi, C. Laurie, H. Radjavi and P. Rosenthal, On the congruence numerical range and related functions of matrices, *Linear and Multilinear Algebra* 22 (1987), 1-5.

[2] M. Goldberg and E.G. Straus, Norm properties of C-numerical radii, *Linear Algebra and Appl.* 24(1979), 113-131.

[3] Y. Hong and R.A. Horn, A characterization of unitary congruence, (to appear in *Linear and Multilinear Algebra*).

[4] C.K. Li and N. K. Tsing, G-invariant hermitian forms and G-invariant elliptical norms (to appear).

[5] C.K. Li and N.K. Tsing, Norms that are invariant under unitary similarities and the C-numerical radii (to appear in *Linear and Multilinear Algebra*).

[6] M. Marcus and M. Sandy, Three elementary proofs of the Goldberg-Straus theorem on numerical radii, *Linear and Multilinear Algebra* 11 (1982), 243-252.

[7] T. Takagi, On an algebraic problem related to an analytic theorem of Caretheodory and Fejer and on an allied theorem of Landau, *Japan J. Math.* 1(1925),83-93.

[8] B.S. Tam, A simple proof of the Goldberg-Straus theorem on numerical radii, *Glasgow Math. J.* 28(1986), 139-141.

[9] T.Y. Tam, Note on a paper of Thompson: The congruence numerical range, *Linear and Multilinear Algebra* 17 (1985), 107-115.

[10] R.C. Thompson, The congruence numerical range, *Linear and Multilinear Algebra* 8(1980), 197-206.

[11] D.C. Youla, A normal form for a matrix under the unitary group, *Canad. J. Math.* 13(1961), 694-704.

Department of Mathematics, University of Hong Kong.