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ON NONNEGATIVE MATRICES WITH A FULLY CYCLIC PERIPHERAL SPECTRUM

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Abstract. Let A be a square complex matrix. We denote by $\rho(A)$ the spectral radius of A. The set of eigenvalues of A with modulus $\rho(A)$ is called the peripheral spectrum of A. The latter set is said to to be fully cyclic if whenever $\rho(A)\alpha x = Ax$, $x \neq 0$, $|\ a\ |= 1$, then $|\ x\ |\ (sgn\ x)^k$ is an eigenvector of A corresponding to $\rho(A)\alpha^k$ for all integers k. In this paper we give some necessary conditions and a set of sufficient conditions for a nonnegative matrix to have a fully cyclic peripheral spectrum. Our conditions are given in terms of the reduced graph of a nonnegative matrix.

1. Introduction and Definitions

The graph-theoretic properties of a (entrywise, square) nonnegative matrix and its algebraic (spectral) properties are intimately connected, as most notably illustrated by the Perron-Frobenius theorem. Indeed, the important class of irreducible nonnegative matrices consists of exactly those nonnegative matrices with a strongly connected associated directed graph. For an excellent survey article on the subject, see Schneider [5]. In this paper we shall investigate the relation between the graph of a nonnegative matrix and the property of having a fully cyclic peripheral spectrum. We shall obtain some necessary conditions and a set of sufficient conditions for a nonnegative matrix to have a fully cyclic peripheral spectrum. This direction of research appears to be new.

We assume elementary knowledge on nonnegative matrices that can be found in many textbooks (see, for instance, Berman and Plemmons [1] or Horn and Johnson [3]). To fix our notation and terminology, we give some definitions.

We denote the *spectrum* and the *spectral radius* of a matrix P respectively by $\sigma(P)$ and $\rho(P)$. According to the famous Perron-Frobenius theorem, if P is a nonnegative matrix then $\rho(P) \in \sigma(P)$. The set of eigenvalues of P with modulus $\rho(P)$ is called the *peripheral spectrum* of P. The latter set is said to be *cyclic* if $\rho(P)\alpha \in \sigma(P)$ and $|\ \alpha\ |= 1$ imply that $\rho(P)\alpha^k \in \sigma(P)$ for all integers k. Note that then α is necessarily a root of

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unity (unless P is nilpotent). If $x = (\xi_1, \cdots, \xi_n)^T \in C^n$, we denote by $| x |$ the real vector $(| \xi_1 |, \cdots, | \xi_n |)^T$. And if also $y = (\eta_1, \cdots, \eta_n)^T \in C^n$, then we denote by xy the vector $(\xi_1\eta_1, \cdots, \xi_n\eta_n)^T$. In particular, we use x^k to denote the vector $(\xi_1^k, \cdots, \xi_n^k)^T$. Also we use $sgn\ x$ to denote the vector $(sgn\ \xi_1, \cdots, sgn\ \xi_n)^T$, where $sgn\ \delta = \delta / | \delta |$ if $\delta \neq 0$ and for $\delta = 0$ define $sgn\ \delta = 1$. The peripheral spectrum of an $n \times n$ matrix P is said to be *fully cyclic* if whenever $\rho(P)\alpha x = Px$, $0 \neq x \in C^n$, $| \alpha | = 1$, then $| x | (sgn\ x)^k$ is an eigenvector of P corresponding to $\rho(P)\alpha^k$ for all integers k. We shall denote a nonnegative (resp. strictly positive) vector x by $x \geq 0$ ($resp. x > 0$).

We now collect the necessary graph-theoretic definitions. For reference, see Schneider [5].

Denote by $< n >$ the set $\{1, \cdots, n\}$. As usual, we define the *associated directed graph* of an $n \times n$ complex matrix $P = (p_{ij})$ to be the graph $G(P)$ with vertex set $< n >$ where (i, j) is an arc if and only if $p_{ij} \neq 0$. The strongly connected components of $G(P)$ are called simply *classes* of P, and are denoted by Greek letters α, β, etc. For any two classes α and β, we say α *has access to* β or β *has access from* α, and we write $\alpha >= \beta$ or $B =< \alpha$, if there is a path in $G(P)$ from a vertex in α to some (and hence every) vertex in β. We write $\alpha \succ \beta$ or $\beta \prec \alpha$ if $\alpha >= \beta$ and $\alpha \neq \beta$. We also say a vertex i has access to a class α if there is a path in $G(P)$ from i to some vertex in α. A class α is said to *initial* (resp. *final*), if there is no class β such that $\beta \succ \alpha$ (resp. $\alpha \succ \beta$). By the *reduced graph* of P, we mean the directed graph with classes of P as vertices, where (α, β) is an arc if and only if $\alpha \neq \beta$ and $P_{\alpha\beta} \neq 0$, where $P_{\alpha\beta}$ denotes the submatrix of P with row indices from α and column indices from β.

Let $x = (\xi_1, \cdots, \xi_n)^T \in C^n$. By the *support* of x we mean the set $supp(x) = \{i \in < n >: \xi_i \neq 0\}$. For any class α of P, we denote by $x_\alpha \in C^{|\alpha|}$ the corresponding subvector of x. A class α is said to be *in supp*(x) if $x_\alpha \neq 0$; α is said to be *final in supp*(x) if α is in $supp(x)$ and there is no class β in $supp(x)$ such that $\alpha \succ \beta$.

Let P be an $n \times n$ nonnegative matrix. A class α (of P) is *distinguished* if $\rho(P_{\alpha\alpha}) > \rho(P_{\beta\beta})$ for any class $\beta \succ \alpha$. A class α is said to be *basic* if $\rho(P_{\alpha\alpha}) = \rho(P)$. Clearly basic classes and initial classes are all distinguished classes.

A real matrix A is called a *singular M-matrix* if $A = \rho(P)I - P$ for some nonnegative matrix P. Note that then A and P have the same classes. We call a class α of a singular M-matrix *basic* (resp. *distinguished*) if and only if α is a basic (resp. distinguished) class of an associated nonnegative matrix.

2. Results and Proofs

It is known that the peripheral spectrum of every nonnegative matrix is cyclic (see Schaefer[4, Theorem 2.7, Chap I]). Also it follows from Schaefer[4, Lemma 2.6, Chapter I] (where the proof is elementary and computational in nature) that we have the following useful equivalent condition for a nonnegative matrix to have a fully cyclic peripheral spectrum.

Lemma 2.1. *Let P be a nonnegative matrix. Then the peripheral spectrum of P is fully cyclic if and only if P satisfies the following: if $Px = \lambda x$, $x \neq 0$, $\mid \lambda \mid = \rho(P)$, then $P \mid x \mid = \rho(P) \mid x \mid$.*

The following sufficient condition for a nonnegative matrix to have a fully cyclic peripheral spectrum is also known (see Schaefer [4, Prop.2.8, Chapter I]). From this sufficient condition it follows that the peripheral spectrum of every irreducible nonnegative matrix is fully cyclic.

Theorem 2.2. *Let P be a nonnegative matrix. If there exists $y_0 >> 0$ satisfying $P^T y_0 \leq \rho(P) y_0$, then the peripheral spectrum of P is fully cyclic.*

The proof given by Schaefer [4] for the preceding result is analytic in nature, and depends on the concept of a P-invariant ideal. Here we give an alternative elementary proof that makes use of the reduced graph of a nonnegative matrix.

Proof. First, note that P and P^T have the same basic classes, and that the initial classes of P are exactly the final classes of P^T. By Schneider [5, Theorem 4.1] (where the result is stated in terms of a singular M-matrix), the given condition on P is equivalent to the condition that every basic class of P is initial.

Now let P be $n \times n$, and let $0 \neq x \in C^n$, $\lambda \in C$, with $\mid \lambda \mid = \rho(P)$ such that $Px = \lambda x$. Then $P \mid x \mid \geq \mid Px \mid = \rho(P) \mid x \mid$. Since every basic class of P is initial, by Tam and Wu [6, Theorem 4.7], it follows that $P \mid x \mid = \rho(P) \mid x \mid$. By Lemma 2.1, this proves that the peripheral spectrum of P is fully cyclic.

We shall need the following result.

Lemma 2.3. *Let P be a nonnegative matrix. Let x be an eigenvector of P corresponding to an eigenvalue λ with modulus $\rho(P)$. If α is a class of P final in $supp(x)$, then α is a basic class, and x_α is an eigenvector of $P_{\alpha\alpha}$ corresponding to λ.*

Proof. Our assertion follows from $\lambda x_\alpha = (Px)_\alpha = P_{\alpha\alpha} x_\alpha + \Sigma_{\alpha > -\beta} P_{\alpha\beta} x_\beta = P_{\alpha\alpha} x_\alpha$, where the last equality holds as α is final in $supp(x)$.

We can now give our main results:

Theorem 2.4. *Let P be a nonnegative matrix with a fully cyclic peripheral spectrum. Then each of the following conditions holds:*

(a) *The eigenspace of P corresponding to $\rho(P)$ has a basis consisted of nonnegative vectors.*

(b) *There does not exist a nonbasic class (of P) which has access to two different distinguished basic classes.*

(c) *If λ is in the peripheral spectrum of $P_{\alpha\alpha}$ for some basic class α which is not distinguished then λ is also in the peripheral spectrum of $P_{\beta\beta}$ for some distinguished basic class β which has access to α.*

Proof. For convenience, denote by E(P) the eigenspace of P corresponding to $\rho(P)$ and by $E^*(P)$ the subspace of $E(P)$ which is spanned by its nonnegative vectors.

(a) By the results of Cooper [2, Lemmas 4,5 and 6] (but reformulated in our language) or of Schneider [5, Theorems 3.1 and 7.1], we have,

$$\begin{aligned} E^*(P) &= span\{e^{(\alpha)} : \alpha \text{ is a distinguished basic class}\} \\ &= \{x \in E(P) : \text{for each class } \alpha \text{ final in } supp(x), \\ &\qquad \alpha \text{ is a distinguished basic class}\}, \end{aligned}$$

where for each distinguished basic class α, $e^{(\alpha)}$ denotes the unique (up to multiples) nonnegative eigenvector of P corresponding to $\rho(P)$ with the property that $e^{(\alpha)}_\beta >> 0$ if $\beta >= \alpha$ and is zero vector, otherwise. So if $E(P) \neq E^*(P)$, there would exist a vector $x \in E(P)$ with a class α final in $supp(x)$, and hence a basic class by Lemma 2.3, which is not distinguished. Since the peripheral spectrum of P is fully cyclic, by definition the vector $| x |$ belongs to $E^*(P)$. But $supp(x) = supp(| x |)$, so α is a class final in $supp(| x |)$. This contradicts the above-mentioned second characterization of vectors in $E^*(P)$.

(b) Suppose that there exists a nonbasic class β which has access to two different distinguished basic classes α_1 and α_2. Let $e^{(\alpha_1)}$ and $e^{(\alpha_2)}$ be respectively the Perron vectors corresponding to the classes α_1 and α_2 as mentioned in the proof for part(a). Since the peripheral spectrum of P is fully cyclic, the vector $| e^{(\alpha_1)} - e^{(\alpha_2)} |$ belongs to $E^*(P)$ and hence must be a linear combination of the Perron vectors of P corresponding to distinguished basic classes. It is not difficult to see that then $| e^{(\alpha_1)} - e^{(\alpha_2)} | = a_1 e^{(\alpha_1)} + a_2 e^{(\alpha_2)}$ for some scalars a_1 and a_2. By considering the α_1- and α_2-subvector of $| e^{(\alpha_1)} - e^{(\alpha_2)} |$, we deduce that $a_1 = a_2 = 1$. But this leads to a contradiction, because $| e^{(\alpha_1)} - e^{(\alpha_2)} |_\beta = | e^{(\alpha_1)}_\beta - e^{(\alpha_2)}_\beta | \neq e^{(\alpha_1)}_\beta + e^{(\alpha_2)}_\beta$, as $e^{(\alpha_1)}_\beta, e^{(\alpha_2)}_\beta >> 0$.

(c) Suppose that λ is in the peripheral spectrum of $P_{\alpha\alpha}$ for some basic class α which is not distinguished, and that for each distinguished basic class $\beta \succ \alpha$, λ is not an eigenvalue of $P_{\beta\beta}$. Replacing α by some suitable non-distinguished basic class having access to it, if necessary, we may assume that λ is not an eigenvalue of $P_{\beta\beta}$ for every basic class $\beta \succ \alpha$. We now construct an eigenvector x of P corresponding to the eigenvalue λ. Set $x_\beta = 0$ for all classes β having no access to α. Choose x_α to be an eigenvector of $P_{\alpha\alpha}$ corresponding to λ and for each class $\beta \succ \alpha$, define x_β inductively by "tracing down" the reduced graph of P : if x_γ has been defined for all $\gamma \prec \beta$, define x_β to be $(\lambda I - P_{\beta\beta})^{-1}(\Sigma_{\beta \succ \gamma} P_{\beta\gamma} x_\gamma)$, where the matrix $\lambda I - P_{\beta\beta}$ is nonsingular as λ is not an eigenvalue of $P_{\beta\beta}$. Then $(Px)_\beta = P_{\beta\beta} x_\beta + \Sigma_{\beta \succ \gamma} P_{\beta\gamma} x_\gamma = \lambda x_\beta$. Hence x is an eigenvector of P corresponding to λ. Since the peripheral spectrum of P is fully cyclic, $| x | \in E^*(P)$. But by our construction, the class α is final in $supp(x)$ and hence in $supp(| x |)$. As α is not a distinguished basic class, we have arrived at a contradiction.

The following result is an improvement of Theorem 2.2.

Theorem 2.5. *Let P be a nonnegative matrix. Then the peripheral spectrum of P is fully cyclic if the following conditions are all satisfied:*

(a) The eigenspace of P corresponding to $\rho(P)$ contains a basis consisted of nonnegative vectors.

(b) Each distinguished basic class of P is initial.

and (c) Each principal submatrix of P associated with a basic class which is not distinguished is primitive.

Proof. Let P be $n \times n$. Let $0 \neq x \in C^n$, $\lambda \in C$ with $|\lambda| = \rho(P)$ such that $Px = \lambda x$. By Lemma 2.1 it sufficies to show that then $P|x| = \rho(P)|x|$. First, consider the case when $\lambda = \rho(P)$. Then by (a) $x \in E(P) = E^*(P)$, where $E(P)$ and $E^*(P)$ have the same meanings as in the proof of Theorem 2.4, and hence by the previous description of the set $E^*(P)$, x can be expressed in the form $\Sigma_{k=1}^m a_k e^{(\alpha_k)}$, where $e^{(\alpha_1)}, \cdots, e^{(\alpha_m)}$ are the (unique) Perron vectors corresponding to the distinguished basic classes $\alpha_1, \cdots, \alpha_m$ of P. But by (b) the classes $\alpha_1, \cdots, \alpha_m$ are all initial, hence the supports of the vectors $e^{(\alpha_1)}, \cdots, e^{(\alpha_m)}$ are pairwise disjoint. Thus $|x| = \Sigma_{k=1}^m |a_k||e^{(\alpha_k)}| = \Sigma_{k=1}^m |a_k| e^{(\alpha_k)}$, and hence $P|x| = \rho(P)|x|$.

Next, suppose that $\lambda \neq \rho(P)$. If α is a class final in *supp*(x) then by Lemma 2.3, α is a basic class and x_α is an eigenvector of $P_{\alpha\alpha}$ corresponding to λ. Since $|\lambda| = \rho(P)$ and $\lambda \neq \rho(P)$, in view of (c), this implies that α is a distinguished basic class. And by (b) it follows that for any class α in *supp*(x), α is a basic, initial class. Thus, if α is in *supp*(x), then $P_{\alpha\alpha}x_\alpha = (Px)_\alpha = \lambda x_\alpha$. Since $P_{\alpha\alpha}$ is irreducible, $P_{\alpha\alpha}|x_\alpha| = \rho(P)|x|_\alpha$, and hence $(P|x|)_\alpha = P_{\alpha\alpha}|x_\alpha| = \rho(P)|x|_\alpha$. On the other hand, if α is not in *supp*(x), we also obtain, $(P|x|)_\alpha = 0 = \rho(P)|x|_\alpha$. We have shown that $P|x| = \rho(P)|x|$. The proof is complete.

Note that Theorems 2.4 and 2.5 have the same condition (a). See Cooper [2, Theorem 3] for a condition which infers this condition. Also, as can be readily seen, conditions (b) and (c) of Theorem 2.5 imply the respective conditions of Theorem 2.4. Below we give some examples which show that conditions (b) and (c) of Theorem 2.5 are not necessary for a nonnegative matrix to have a fully cyclic peripheral spectrum.

Example 2.6. Let P be the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ a & b & 0 \end{pmatrix}$, where a, b are nonnegative numbers with at least one positive. Then P has two classes, namely, $\alpha_1 = \{1,2\}$ and $\alpha_2 = \{3\}$. The reduced graph of P is $\begin{array}{cl} \bigcirc & \alpha_1 \\ \uparrow & \\ \square & \alpha_2 \end{array}$, where we represent a basic class by an O and a non-basic class by an $\square$. The peripheral spectrum of P is $\{-1, 1\}$. Up to scalar multiples, there is only one eigenvector of P corresponding to 1, namely, $(1, 1, a+b)^T$, and one corresponding to -1, namely $(-1, 1, a-b)^T$. Thus the peripheral spectrum of P is fully cyclic if and only if $|(-1, 1, a-b)^T| = (1, 1, a+b)^T$ if and only if either $a > 0$ and $b = 0$ or $a = 0$ and $b > 0$. This example shows that condition (b) of Theorem 2.5 is not a necessary condition for a nonnegative matrix to have a fully cyclic peripheral spectrum.

Example 2.7. Let $P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then the reduced graph of P_1 and P_2 are both equal to $\begin{matrix}\circ\\ \downarrow\\ \circ\end{matrix}$. However, the peripheral spectrum of P_1 is fully cyclic, whereas that of P_2 is not. Note also that the irreducible blocks of P_1 and P_2 are all imprimitive matrices. So condition (c) of Theorem 2.5 is also not a necessary condition for a nonnegative matrix to have a fully cyclic peripheral spectrum.

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