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ON ABSOLUTE NÖRLUND SUMMABILITY OF FOURIER SERIES AND THE DERIVED FOURIER SERIES

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Abstract. In this paper two new theorem concerning $| N,p_n |_k$ summability of Fourier series and its derived series have been proved.

1. Introduction.

Let Σa_n be a given infinite series with sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants real or complex, and let us write

$$P_n = \sum_{v=0}^{n} p_v, \qquad P_{-1} = p_{-1} = 0.$$

The sequence to sequence transformation

$$t_n = P_n^{-1} \sum_{v=0}^{n} p_{n-v} s_v \qquad (P_n \neq 0)$$

defines the sequence $\{t_n\}$ of Nörlund mean of $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$. The series Σa_n, or the sequence $\{s_n\}$, is said to be *absolutely summable* (N,p_n), or *summable* $| N,p_n |$, if the sequence $\{t_n\}$ is of bounded variation, that is the series $\Sigma | t_n - t_{n-1} |$ is convergent.

When $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$, $| N,p_n |$ summability reduces to $| C,\alpha |$ summability.

We give the following definition : A series Σa_n is said to be *summable* $| N,p_n |_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} | t_n - t_{n-1} |^k < \infty.$$

Clearly $| N,p_n |_1$ is the same as $| N,p_n |$. We set $\nabla f_n = f_n - f_{n-1}$ for any sequence $\{f_n\}$.

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Let $f(x)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and let its Fouries series be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^{\infty} A_n(x). \tag{1.1}$$

Then the derived series of (1.1) is

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} nB_n(x) \tag{1.2}$$

We write

$$\begin{aligned}
\phi(u) &= f(x+u) + f(x-u) - 2f(x) \\
\psi(u) &= f(x+u) - f(x-u) - 2tf'(x) \\
\phi_1(t) &= \frac{1}{t}\int_0^t \phi(u)du = \frac{1}{t}\Phi(t) \\
\psi_1(t) &= \frac{1}{t}\int_0^t d\psi(u)du = \frac{1}{t}\psi_2(t) \\
\mu_n &= \left(\Pi_{v=1}^{l-1} \log^v n\right)(\log n)^{1+\epsilon}, \qquad \epsilon > 0,
\end{aligned}$$

where

$$\log^l n = \log(\log^{l-1} n), \cdots, \log^2 n = \log \log n.$$

Throughout this paper we are assuming h is a positive function such that for some β, $0 < \beta < 1$, $u^\beta h(u^{-1})$ is nondecreasing. Also we let s_n denote the nth partial sum of the series under consideration.

PANDEY, in 1978, proved the following.

Theorem A. *If*

$$\varphi(t) = \int_t^\delta u^{-1} \mid \phi(u) \mid du = 0\,\{(\log(1/t))^\eta\} \text{ as } t \to +0 \tag{1.3}$$

$0 < \delta \leq \pi$, then the series $\Sigma A_n(x)/\mu_n$ is summable $\mid C, 1 \mid$ for $0 < \eta < \epsilon$.

The object of this paper is to prove the following theorems under conditions weaker than that used in theorem A.

Theorem 1. *Let $\{p_n\}$ be a positive sequence such that $\{\nabla p_n\}$ is bounded, monotonic, nonincreasing, $\{1/p_n^{k-1}P_n\}$ nonincreasing. If*

$$\Phi_1(t) = \int_t^\delta u^{-1}\phi_1(u)du = O\{h(t^{-1})\},\ t \to 0; \tag{1.4}$$

$$\sum \frac{n^k[h(n)]^k}{p_n^{k-1}P_n} < \infty, \tag{1.5}$$

then the series (1.1) is summable $\mid N, p_n \mid_k$.

Theorem 2. *Let $\{p_n\}$ be as defined in theorem 1. If*

$$\Psi(t) = \int_t^\delta u^{-1}\Psi_1(u)du = O\{h(t^{-1})\}, \qquad t \to 0, \tag{1.6}$$

then the series (1.2) is summable $| N, p_n |_k$ provided that (1.4) holds.

The following result is needed

$$\Phi_1(t) = O\{h(t^{-1})\} \quad \Rightarrow \quad \int_0^t \phi_1(u)du = O\{th(t^{-1})\} \tag{1.7}$$

Proof.

$$\begin{aligned}\int_0^t \phi_1(u)du &= \int_0^t -u\Phi_1'(u)du \\ &= \Big[-u\Phi_1(u)\Big]_0^t + \int_0^t \Phi_1(u)du \\ &= O\Big[uh(u^{-1})\Big]_0^t + O\{t^\beta h(t^{-1}) \int_0^t u^{-\beta}du\},\ 0<\beta<1 \\ &= O\{th(t^{-1})\}.\end{aligned}$$

For the fact that (1.4) is weaker than (1.3), when $h(t^{-1}) = (\log(t^{-1}))^\eta$, see [2].

2. The following lemma is required.

Lemma 2. *Let $s_n^{(1)} = \Sigma_{k=0}^n s_k$. If $\{p_n\}$ is a positive sequence such that $\{\nabla p_n\}$ is bounded, monotonic, nonincreasing, $\{1/(p_n^{k-1}P_n\}$ is nonincreasing, and*

$$\sum_{n=0}^{\infty} \frac{| s_n^{(1)} |^k}{p_n^{k-1}P_n} < \infty,$$

then the series Σa_n is summable $| N, p_n |_k$.

Proof. Since $\Delta_v p_{n-v} = \nabla_n p_{n-k}$, then we have

$$
\begin{aligned}
& t_{n-1} - t_n \\
= & \frac{1}{P_{n-1}} \sum_{v=0}^{n-1} p_{n-1-v} s_v - \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \\
= & \Delta(\frac{1}{P_{n-1}}) \sum_{v=0}^{n-1} p_{n-1-v} s_v + \frac{1}{P_n} \sum_{v=0}^{n-1} (p_{n-1-v} - p_{n-v}) s_v - \frac{p_0 s_n}{P_n} \\
= & \Delta(\frac{1}{P_{n-1}}) \sum_{v=0}^{n-2} \Delta_v (p_{n-1-v}) s_v^{(1)} + \Delta(\frac{1}{P_{n-1}}) p_0 s_{n-1}^{(1)} \\
& + \frac{1}{P_n} \sum_{v=0}^{n-2} \Delta_v (p_{n-1-v} - p_{n-v}) s_v^{(1)} + \frac{1}{P_n} (p_0 - p_1) s_{n-1}^{(1)} \\
& - \frac{p_0 s_n^{(1)}}{P_n} + \frac{p_0 s_{n-1}^{(1)}}{P_n} \\
= & \Delta(\frac{1}{P_{n-1}}) \sum_{v=0}^{n-2} \nabla_n (p_{n-1-v}) s_v^{(1)} + \Delta(\frac{1}{P_{n-1}}) p_0 s_{n-1}^{(1)} \\
& + \frac{1}{P_n} \sum_{v=0}^{n-2} \nabla_n (p_{n-1-v} - p_{n-v}) s_v^{(1)} + \frac{1}{P_n} (p_0 - p_1) s_{n-1}^{(1)} \\
& - \frac{p_0 s_n^{(1)}}{P_n} + \frac{p_0 s_{n-1}^{(1)}}{P_n} \\
= & T_1 + T_2 + T_3 + T_4 + T_5 + T_6, \ say.
\end{aligned}
$$

To prove the lemma, it is sufficient by Minkowski's inequality to show that

$$
\sum_{n=1}^{\infty} (\frac{P_n}{p_n})^{k-1} \mid T_r \mid^k < \infty, \qquad for\ r = 1, 2, 3, 4, 5, 6.
$$

Now applying Hölder's inequality, we have

$$
\begin{aligned}
& \sum_{n=1}^{m} (\frac{P_n}{p_n})^{k-1} \mid T_1 \mid^k \\
= & \sum_{n=1}^{m} (\frac{P_n}{p_n})^{k-1} \mid \Delta(\frac{1}{P_{n-1}}) \sum_{v=0}^{n-2} p_v^{-1} \nabla_n (p_{n-1-v}) s_v^{(1)} p_v \mid^k
\end{aligned}
$$

$$\le \sum_{n=1}^{m} \frac{p_n}{P_n P_{n-1}} \sum_{v=0}^{n-2} p_v^{1-k} \mid \nabla_n(p_{n-1-v}) \mid^k \mid s_v^{(1)} \mid^k \left\{ \sum_{v=0}^{n-2} \frac{p_v}{P_{n-1}} \right\}^{k-1}$$

$$= O(1) \sum_{n=1}^{m} \frac{p_n}{P_n P_{n-1}} \sum_{v=0}^{n-2} p_v^{1-k} \mid s_v^{(1)} \mid^k$$

$$= O(1) \sum_{v=0}^{m-2} p_v^{1-k} \mid s_v^{(1)} \mid^k \sum_{n=v+1}^{m} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=0}^{m} \frac{\mid s_v^{(1)} \mid^k}{p_v^{k-1} P_v}$$

$$\sum_{n=1}^{m} (\frac{P_n}{p_n})^{k-1} \mid T_2 \mid^k = \sum_{n=1}^{m} (\frac{P_n}{p_n})^{k-1} \mid \Delta(\frac{1}{P_{n-1}}) p_0 s_{n-1}^{(1)} \mid^k$$

$$= \sum_{n=1}^{m} \frac{p_n p_0^k}{P_n P_{n-1}^k} \mid s_{n-1}^{(1)} \mid^k$$

$$= O(1) \sum_{n=1}^{m} \frac{1}{p_{n-1}^{k-1} P_{n-1}} \mid s_{n-1}^{(1)} \mid^k$$

$$\sum_{n=1}^{m} (\frac{P_n}{p_n})^{k-1} \mid T_3 \mid^k$$

$$= \sum_{n=1}^{m} (\frac{P_n}{p_n})^{k-1} \mid \frac{1}{P_n} \sum_{v=0}^{n-2} \nabla_n(p_{n-1-v} - p_{n-v}) s_v^{(1)} \mid^k$$

$$\le \sum_{n=1}^{m} \frac{1}{p_n^{k-1} P_n} \sum_{v=0}^{n-2} \nabla_n(p_{n-1-v} - p_{n-v}) \mid s_v^{(1)} \mid^k$$

$$\times \left\{ \sum_{v=0}^{n-2} \nabla_n(p_{n-1-v} - p_{n-v}) \right\}^{k-1}$$

$$= O(1) \sum_{v=0}^{m-2} \mid s_v^{(1)} \mid^k \sum_{n=v+1}^{m} \frac{\nabla_n(p_{n-1-v} - p_{n-v})}{p_n^{k-1} P_n}$$

$$= O(1) \sum_{v=0}^{m} \frac{\mid s_v^{(1)} \mid^k}{p_{v+1}^{k-1} P_{v+1}} \sum_{n=v+1}^{m} \nabla_n(p_{n-1-v} - p_{n-v})$$

$$= O(1) \sum_{v=0}^{m} \frac{\mid s_v^{(1)} \mid^k}{p_v^{k-1} P_v}$$

$$\begin{aligned}\sum_{n=1}^{m}(\frac{P_n}{p_n})^{k-1} \mid T_4 \mid^k &= O(1)\sum_{n=1}^{m}\frac{\mid s_{n-1}^{(1)} \mid^k}{p_n^{k-1}P_n} \\ &= O(1)\sum_{n=1}^{m}\frac{\mid s_{n-1}^{(1)} \mid^k}{p_{n-1}^{k-1}P_{n-1}}.\end{aligned}$$

Other parts could be treated similarly and the lemma follows.

3. Proof of the theorems.

Proof of theorem 1. We have

$$\begin{aligned}\sum_{v=0}^{n}[s_v(x)-f(x)] &= \frac{2}{\pi}\int_0^{\pi} t^{-1}\phi(t)\sum_{v=0}^{n}\sin vt\ dt + o(n) \\ &= \frac{2}{\pi}\int_0^{\pi}\{t^{-1}\phi_1(t)+\phi_1'(t)\}\sum_{v=0}^{n}\sin vt\ dt + o(n) \\ &= \frac{2}{\pi}\{I_1+I_2\},\ say.\end{aligned}$$

$$I_1 = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{1.1}+I_{1.2},\ say.$$

$$I_{1.1} = O(n^2)\int_0^{1/n} t^{-1}\phi_1(t)dt = O\{nh(n)\},\ as\ (1.4)$$

$$\Rightarrow \int_0^{t}\phi_1(t)dt = O\{th(t^{-1})\} \tag{3.1}$$

$$I_{1.2} = O(n)\int_{1/n}^{\pi} t^{-1}\phi_1(t)dt = O\{nh(n)\},\ by\ (1.4) \tag{3.2}$$

$$\begin{aligned}I_2 &= [\sum_{v=0}^{n}\sin\ vt\cdot\phi_1(t)]_0^{\pi} - \int_0^{\pi}\phi_1(t)\sum_{v=0}^{n} v\cos\ vt\ dt \\ &= -\int_0^{\pi}\phi_1(t)\sum_{v=0}^{n} v\cos\ vt\ dt \\ &= -\int_0^{1/n} - \int_{1/n}^{\pi} = -I_{2.1} - I_{2.2},\ say,\ as\end{aligned}$$

$\sin\ v\pi = 0$, and $\Phi(0)=0$ implies

$$[\sum_{v=0}^{n}\sin\ vt\cdot\phi_1(t)]_{t=0} = [\sum_{v=0}^{n}\frac{\sin\ vt}{t}\cdot O(t)]_{t=0} \rightarrow \sum_{v=0}^{n} v\cdot\Phi(0)=0.$$

$$I_{2.1} = O(n^2)\int_0^{1/n} \phi_1(t)dt = O\{nh(n)\}, \tag{3.3}$$

$$\begin{aligned} I_{2.2} &= O(1)\int_{1/n}^{\pi} \phi_1(t) \mid \sum_{v=0}^{n} v \cos vt \mid dt \\ &= O(n)\int_{1/n}^{\pi} \phi_1(t) \cdot \max_{0\leq r\leq n} \mid \sum_{v=0}^{n} \cos vt \mid dt \\ &= O(n)\int_{1/n}^{\pi} t^{-1}\phi_1(t)dt = O\{nh(n)\}. \end{aligned} \tag{3.4}$$

Compining (3.1),(3.2),(3.3) and (3.4) we obtain

$$\sum_{v=0}^{n}[s_v(x) - f(x)] = O\{nh(n)\},$$

which implies

$$\mid \sum_{v=0}^{n} s_v(x) \mid \leq \mid \sum_{v=0}^{n}[s_v(x) - f(x)] \mid + \mid \sum_{v=0}^{n} f(x) \mid = O\{nh(n)\}.$$

The theorem follows by the lemma.

Proof of theorem 2. We have

$$\begin{aligned} \sum_{v=0}^{n}[s_v(x) - f'(x)] &= \frac{1}{2\pi}\int_0^{\pi} \frac{\sum_{v=0}^{n}\sin(v+\frac{1}{2})t}{\sin\frac{1}{2}t} d\psi(t) \\ &= \frac{1}{2\pi}\int_0^{\pi} \{\psi_1(t) + t\psi_1'(t)\}\frac{\sum_{v=0}^{n}\sin(v+\frac{1}{2})t}{\sin\frac{1}{2}t} dt \\ &= \frac{1}{2\pi}\{I_1 + I_2\}, \ say. \end{aligned}$$

$$I_1 = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{1.1} + I_{1.2}, \ say.$$

$$I_{1.1} = O(n^2)\int_0^{1/n} \psi_1(t)dt = O\{nh(n)\}, \ as \ (1.6)$$

$$\Rightarrow \int_0^{t} \psi_1(t)dt = O\{th(t^{-1})\} \tag{3.5}$$

$$I_{1.2} = O(n)\int_{1/n}^{\pi} t^{-1}\psi_1(t)dt = O\{nh(n)\},\ by\ (1.6) \tag{3.6}$$

$$I_2 = [\sum_{v=0}^{n}\frac{\sin(v+\frac{1}{2})t}{\sin\frac{1}{2}t}\cdot\psi_2(t)]_{t=0}^{\pi} - \int_0^{\pi} t\psi_1(t)\frac{\sum_{v=0}^{n}(v+\frac{1}{2})t\ \cos(v+\frac{1}{2})t}{\sin\frac{1}{2}t}$$

$$+\frac{1}{4}\int_0^{\pi} t\psi_1(t)\frac{\cos\frac{1}{2}t\sum_{v=0}^{n}\sin(v+\frac{1}{2})t}{\sin^2\frac{1}{2}t}dt - \int_0^{\pi}\psi_1(t)\frac{\sum_{v=0}^{n}\sin(v+\frac{1}{2})t}{\sin\frac{1}{2}t}dt$$

$$= -I_{2.1} + \frac{1}{4}I_{2.2} - I_{2.3},$$

as the quantity in the brackets equals zero for the same reason given in the proof of theorem 1.

$$I_{2.3} = O\{nh(n)\},\ as\ before. \tag{3.7}$$

$$I_{2.1} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{2.1.1} + I_{2.1.2},\ say.$$

$$I_{2.1.1} = O(n^2)\int_0^{1/n}\psi_1(t)dt = O\{nh(n)\}. \tag{3.8}$$

$$I_{2.1.2} = O(n)\int_{1/n}^{\pi}\psi_1(t)\max_{0\le r\le n}|\sum_{v=0}^{r}\cos(v+\frac{1}{2})t|\,dt$$

$$= O(n)\int_{1/n}^{\pi} t^{-1}\psi_1(t)dt = O\{nh(n)\} \tag{3.9}$$

$$I_{2.2} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{2.2.1} + I_{2.2.2},\ say.$$

$$I_{2.2.1} = O(n)\int_0^{1/n}\psi_1(t)dt = O\{nh(n)\} \tag{3.10}$$

$$I_{2.2.2} = O(n)\int_{1/n}^{\pi} t^{-1}\psi_1(t)dt = \{nh(n)\} \tag{3.11}$$

Combining (3.5), $\cdots$, (3.11), we obtain

$$\sum_{v=0}^{n}[s_v(x) - f'(x)] = O\{nh(n)\},$$

which implies

$$|\sum_{v=0}^{n} s_v(x)| = O\{nh(n)\}.$$

The theorem also follows by the lemma.

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