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# CONSTRAINED APPROXIMATION OF A COMPACT SET IN A NORMED SPACE

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## 1. Introduction

In this paper we are going to discuss the constrained approximation, that is, find a best approximation from a convex subset to a given compact subset subject to certain constraints. Essentially, the problem is based on the monotone best approximation given by Roulier [6] and Lorentz and Zeller [4.5], as well as Taylor's papers [7,8,9] on best approximation by algebraic polynomials with restricted ranges. In [6] Roulier has discussed the monotone best approximation to a given function with same property. Later on Lorentz and Zeller generalized this problem to a general form by assuming that $\varepsilon_i D^{k_i} p(x) \geq 0,\ i = 1, 2, \cdots, r.$ where $\varepsilon_i = \pm 1$, $1 \leq k_1 \leq k_2 \leq \ldots \leq k_r < n$ and $p(x)$ is polynomial of degree $\leq n$. Meanwhile, Taylor has considered a problem concerning best approximation by algebraic polynomials with restricted ranges, i.e. to find a polynomial $P_0(x)$ of degree $\leq n$ which is a best approximation to a given function $f(x)$ such that $\ell(x) \leq f(x) - P_0(x) \leq u(x)$ where $\ell, u$ are given functions in $C[a,b]$ such that $\ell(x) \leq u(x)$ for all $x \in [a,b]$. This leads us to question that whether it is possible to develop a more general theory with regard to these problems. To answer the question, let us consider the following problem:

(P): Given a compact subset $F \subset X$, a real Banach Space, to find $y' \in Y$, a subset of a subspace $G$ of $X$, such that

$$d_F(y') = \inf_{y \in Y} d_F(y) = \inf_{y \in Y} \max_{f \in F} \max_{k \in K \subset X^*} < k, f - y >$$

where $K$ is a symmetric $\sigma(X^*, X)$ - compact subset of $X^*$, the real dual space of $X$, and $Y = Y_1 \cap Y_2$ with

$$\begin{aligned} Y_1 &= \{y \in G\ : < h, y - \psi_i > \geq\ 0\ \forall\ h \in B_i,\ i = 1, 2, \cdots, m\} \\ Y_2 &= \{y \in G\ : < h, y - \rho_i > \leq\ 0\ \forall\ h \in C_i,\ i = 1, 2, \cdots, m\} \end{aligned}$$

for some elements $\psi_i$ and $\rho_i$ in $X$.

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We introduce, for $y \in X$,

$$p(y) = \max_{k \in K} < k, y >$$
$$\text{and} \quad d_F(y) = \max_{f \in F} p(f - y)$$

By a proper choice of $B_i, C_i, \psi_i$ and $\rho_i$, we may assume that $Y \neq \emptyset$. A solution for (P) will be called a best approximation to $F$ from $Y$, or briefly, a "best approximation".

We begin with discussing some general questions on the existence and characterization of the solution. Indeed, the theory in which we shall develop will give a definite answer to the previous question.

## 2. General results

**Lemma 2.1.** *Let $G$ be n-dimensional subspace of $X$ and $Y$ be defined as above. Assume that the restriction of $p(\cdot)$ to $G$ is a norm and each $B_i, C_i \subset X^*(i = 1, 2, \cdots, m)$. Then, there exists $y' \in Y$ such that*

$$d_F(y') = \inf_{y \in Y} d_F(y).$$

**Proof.** Let $y_j$ be a sequence in $Y$ such that $\lim_{j \to \infty} d_F(y_j) = \inf_{y \in Y} d_F(y)$. Moreover,

$p(y_j) \leq \max_{f \in F} p(f - y_j) + \max_{f \in F} p(f) \leq M$, for some real $M$, since $\max_{f \in F} p(f)$ is fixed and $d_F(y_j)$ are terms of a convergent sequence. As the restriction is a norm, $\{y_j\}$ is a bounded sequence in $G$. Hence, there exists $y'$ in $G$ such that $\lim_{j \to \infty} y_j = y'$. We will show that, in fact, $y' \in Y$. For each $i$, we have

$< h, y_j - \psi_i > \geq 0 \ \forall h \in B_i$ and $< h, y_j - \rho_i > \leq 0 \ \forall h \in C_i$ and all $j$. Since $B_i, C_i \subset X^*$, $< h, y' - \psi_i > = \lim_{j \to \infty} < h, y_j - \psi_i > \geq 0 \ \forall h \in B_i$ and $< h, y' - \rho_i > = \lim_{j \to \infty} < h, y_j - \rho_i > \leq 0 \ \forall h \in C_i$. Therefore, $y' \in Y$. Further, for each $j$,

$$0 \leq d_F(y') - \inf_{y \in Y} d_F(y) \leq \left[\max_{f \in F} p(f - y_j) + p(y_j - y')\right] - \inf_{y \in Y} d_F(y)$$
$$= \left[d_F(y_j) - \inf_{y \in F} d_F(y)\right] + p(y_j - y').$$

Since the term in breaket tends to zero as $j \to \infty$ and also $\lim_{j \to \infty} p(y_j - y') = 0$, this shows that $d_F(y') = \inf_{y \in Y} d_F(y)$, which proves the lemma.

For the sake of simplicity, we will assume that $B_i \cap B_j = \emptyset$, $C_i \cap C_j = \emptyset$ for $i \neq j$ and $B_i \cap (-B_j) = \emptyset$, $C_i \cap (-C_j) = \emptyset$ for $i \neq j$, where $-B_j = \{-h : h \in B_j\}$, $-C_j = \{-h : h \in C_j\}$. Define the sets

$$H(y) = \{\varepsilon k \in K : \exists f \in F, < \varepsilon k, f - y >= d_F(y)\} \qquad \text{where } \varepsilon = \pm 1;$$

$$\begin{aligned} N_{1,i}(y) &= \{h \in B_i : < h, y - \psi_i >= 0\}, \\ N_{2,i}(y) &= \{h \in C_i : < h, y - \rho_i >= 0\}, \\ J &= \{1, \cdots, m\}, \\ I_1(y) &= \{i \in J : N_{1,i}(y) \neq \emptyset\}, \\ I_2(y) &= \{i \in J : N_{2,i}(y) \neq \emptyset\}, \end{aligned}$$

and

$$N(y) = \left[\cup_{i \in I_1(y)} N_{1,i}(y)\right] \cup \left[\cup_{i \in I_2(y)} N_{2,i}(y)\right].$$

Let us consider two particular cases which are not of general interest. First, suppose that, for some $y_0 \in Y$ and $k \in K$, there exist $f_1, f_2 \in F$ such that

$$< k, f_1 - y_0 > = d_F(y_0) \quad \textit{and} \quad < k, f_2 - y_0 > = -d_F(y_0).$$

Then $y_0$ is obviously a best approximation, as no approximation can make the error smaller at $k$. We will therefore call $k$ a straddle point of $F$. In case, we have $\{H^+(y') \cap N_{2,i}(y')\} \cup \{H^-(y') \cap N_{1,j}(y')\} \neq \emptyset$ for some $i, j \in J$, where $H^+(y') = \{+k \in H(y')\}$ and $H^-(y') = \{-k \in H(y')\}$, then any attempt to decrease the error $d_F(y')$ would remove $y'$ from $Y$ through failure of the necessary inequalities for $< h, y' - \psi_j >$ and $< h, y' - \rho_i >$ at all $h \in \{H^+(y') \cap N_{2,i}(y')\} \cup \{H^-(y') \cap N_{1,j}(y')\}$. Thus, $y'$ is clearly a best approximation. In the following discussion, unless otherwise stated, we will rule out these two cases.

**Theorem 2.1.** *Let $G$ be a subspace of $X$ and $Y$ a subset of $G$ defined as before. Then $y' \in Y$ is a best approximation to $F$ if and only if there does not exist $y \in G$ such that*

$$< k, y > < 0 \ \forall\, k \in H(y')$$

$$< h, y > \leq < h, y' - \psi_i > \qquad \forall\, h \in B_i, \qquad i = 1, 2, \cdots, m \tag{2.1}$$

and

$$< -h, y > \leq < h, \rho_i - y' > \qquad \forall\, h \in C_i, \qquad i = 1, 2, \cdots, m.$$

**Proof.** Suppose that $y'$ is not a best approximation to $F$, then there is a $y_1$ in $Y$ such that

$$d_F(y_1) < d_F(y').$$

Observe that for each $k \in H(y')$ there is a $f$ in $F$ such that

$$< k, f - y_1 > < < k, f - y' > .$$

Therefore, we have

$$< k, y' - y_1 > < 0 \text{ for all } k \in H(y').$$

Moreover,

$$< h, y' - y_1 > - < h, y' - \psi_i > = < h, -y_1 + \psi_i >= - < h, y_1 - \psi_i > \leq 0$$
$$\text{for all } h \in B_i,\ i = 1, \cdots, m,$$

and

$$< h, y' - y_1 > - < h, y' - \rho_i > = < h, -y + \rho_i > = < h, \rho_i - y_1 > \geq 0$$
$$\forall\ h \in C_i,\ i = 1, 2, \cdots, m.$$

Consequently, if we put $y = y' - y_1$, then we have the required result. This proves the sufficiency.

On the other hand, if there is a $y$ in $G$ such that

$$< k, y > < \ 0 \text{ for all } k \in H(y'),$$

$$< h, y > \leq < h, y' - \psi_i > \text{ for all } h \in B_i,\ i = 1, \cdots, m,$$

and

$$- < h, y > \leq < h, \rho_i - y' > \text{ for all } h \in C_i,\ i = 1, \cdots, m.$$

Then, by compactness of $H(y')$, there exists a real $b > 0$ and an open subset $V$ of $K$ such that

$$\inf_{k \in H(y')} \mid d_F(y') < k, y > \mid = b,$$

and

$$\max_{f \in F} < k, f - y' >< k, y > < -b/2 \qquad \text{for all } k \in V.$$

Obviously, $H(y') \subset V$. Hence, there is a real $c > 0$ such that

$$d_F(y') - \max_{f \in F} \max_{k \in K|V} < k, f - y' > \geq c.$$

Take $r = \min\left[b/2p(y)^2, c/2p(y)\right]$, and observe that, for $k \in V$,

$$\begin{aligned} \max_{f \in F} < k, f y_t > &= \max_{f \in F} < k, f - y' > +t < k, y > \\ &< \max_{f \in F} < k, f - y' > \leq d_F(y') \text{ for sufficiently small } t \in (0, r], \end{aligned}$$

where $y_t = y' - ty$, while for $k \in K \mid V,\ t \in (0, r]$,

$$\begin{aligned} \max_{f \in F} \mid< k, f - y_t >\mid &= \max_{f \in F} \max_{k \in K|V} \mid< k, f - y' >\mid +t \mid< k, y >\mid \\ &\leq d_F(y') - c + c/2 = d_F(y') - c/2. \end{aligned}$$

It follows that $d_F(y_t) < d_F(y')$ for sufficiently small $t \in (0, r]$. Finally, we will show that there exists $t$ in $(0, r]$ such that $y_t$ is in $Y$. Since $< h, y >\leq< h, y' - \psi_i >$ and

$< h, y' - \psi_i > \geq 0$ for all $h \in B_i$, $i = 1, \cdots, m$, we have $< h, ty > \leq < h, y' - \psi_i >$ for all $h \in B_i$, $i = 1, \cdots, m$ and $t \in (0,1]$. Therefore, $< h, y' - ty - \psi_i > \geq 0$ for all $h \in B_i$, $i = 1, \cdots, m$ and $t \in (0,1]$. Similarly, we have $< h, \rho_i - y' + ty > \geq 0$ for all $h \in C_i$, $i = 1, 2, \cdots, m$ and $t \in (0,1]$. Consequently, this shows that there exists $t \in (0, r]$ such that $y_t$ is in $Y$ and $d_F(y_t) < d_F(y')$. Hence, $y'$ is not a best approximation to $F$, proving the theorem.

The condition (2.1) of Theorem 2.1 is too restrictive. Therefore, for the practical purpose we would like to replace condition (2.1) by a less restrictive condition. First, we assume that $B_i, C_i \subset X^*$ $(i = 1, 2, \cdots, m)$ are $\sigma(X^*, X)$-compact.

With the aid of the additional assumptions, we have

**Theorem 2.2.** *Suppose that $B_i, C_i \subset X^*$ $(i = 1, 2, \cdots, m)$ are $\sigma(X^*, X)$ -compact and there exists $y \in G$ such that $< \tau, y > \geq a > 0$ for $\tau \in N_{1,i}(y') \cup \left[-N_{2,i}(y')\right]$, $i = 1, 2, \cdots, m$, for some given $y' \in Y$. Then $y'$ is a best approximation to $F$ if and only if*

$$\overline{co}(H(y') \cup N(y')) \cap G^{\perp} \neq \emptyset$$

*where $G^{\perp} = \{u \in X^* \ : < u, y > = 0 \ \forall \ y \in G\}$.*

**Proof.** For the sake of convenience, let $M = H(y') \cup N(y')$. As in Theorem 2.1, we know that if $y'$ is not a best approximation, there is an $y_1 \in Y$ such that

$$< k, y' - y_1 > < 0 \qquad k \in H(y')$$

$$< h, y' - y_1 > \leq < h, y' - \psi_i > \qquad \forall \ h \in B_i, \ i = 1, 2, \cdots, m$$

and

$$< -h, y' - y_1 > \leq < h, \rho_i - y' > \qquad \forall \ h \in C_i, \ i = 1, 2, \cdots, m.$$

Hence, $< h, y' - y_1 > \leq 0 \ \forall h \in N_{1,i}(y')$, $i \in I_1(y')$ and $< -h, y' - y_1 > \leq 0 \ \forall h \in N_{2,i}(y')$, $i \in I_2(y')$. If

$$< \tau, y' - y_1 > < 0 \qquad \forall \ \tau \in N(y')$$

then, putting $y = y' - y_1$, we get

$$< \tau; y > < 0 \qquad \forall \ \tau \in M.$$

Hence $\overline{co}(M) \cap G^{\perp} = \emptyset$. Otherwise, if exists $\tau \in N(y')$ such that $< \tau, y' - y_1 > = 0$. By assumption, we have a $y \in G$ such that $< \tau, y > \geq a > 0 \ \forall \ \tau \in N(y')$, and, by the compactness of $H(y')$, there exist real $b_1 > 0$ and $b_2 > 0$ such that $\inf_{k \in H(y')} |< k, y' - y_1 >| = b_1$ and $\max_{k \in H(y')} |< k, y >| = b_2$. Take $c = b_1/2b_2$, then, for $y_t = y' - y_1 - ty$, where $t \in (0, c]$, we have

$$< k, y_t > = < k, y' - y_1 > - t < k, y > < 0 \ \forall \ k \in H(y');$$

and

$$< \tau, y_t > = < \tau, y' - y_1 > - t < \tau, y > < 0 \ \forall \ \tau \in N(y').$$

This again shows that $\overline{co}(M) \cap G^{\perp} = \emptyset$, proving the sufficiency.

Conversely, suppose $\overline{co}(M) \cap G^{\perp} = \emptyset$, by a separation theorem [2] and the fact that the dual space of $X^*$ under $\sigma(X^*, X)$ - topology is $X$, there is an $y \in G$ such that

$$< k, y > < 0 \qquad \forall\, k \in H(y')$$

and

$$< \tau, y > < 0 \qquad \forall \tau \in N(y').$$

By a similar argument to that in the proof of Theorem 2.1, there exist real $b, c, r > 0$ and an open subset $V$ of $K$ such that

$$\left[\max_{f\in F} < k, f - y_t >\right]^2 \leq \left[d_F(y')\right]^2 - tb/2 \;\forall\, t \in (0, r],\; k \in V,$$

$$\max_{f\in F} < k, f - y_t > \leq d_F(y') - c/2 \qquad \forall k \in K \mid V, \qquad t \in (0, r]$$

where $y_t = y' - ty$.

Since $B_i, C_i$ are compact and $N_{1,i}(y')$, $N_{2,i}(y')$ are closed subsets of $B_i, C_i$, there exist $e_{j,i} > 0$ such that

$$\min_{h\in N_{1,i}(y')} |< h, y >| = e_{1,i} \qquad \forall\, i \in I_1(y')$$

and

$$\min_{h\in N_{2,i}(y')} |< -h, y >| = e_{2,i} \qquad \forall\, i \in I_2(y').$$

Define the open set $U_{j,i}$ as

$$\begin{aligned} U_{1,i} &= \{h \in B_i :< h, y > < -\frac{e_{1,i}}{2}, \qquad \forall\, i \in I_1(y')\} \\ U_{2,i} &= \{h \in C_i :< -h, y > < -\frac{e_{2,i}}{2}, \qquad \forall\, i \in I_2(y')\} \\ U_{1,i} &= \emptyset \qquad \forall\, i \in J \mid I_1(y') \\ &\textit{and} \\ U_{2,i} &= \emptyset \qquad \forall\, i \in J \mid I_2(y'). \end{aligned}$$

Clearly, $N_{j,i}(y') \subset U_{j,i}\; \forall\, i \in I_j(y')$. Hence there exists $c_{j,i} > 0$ such that

$$\begin{aligned} < h, y' - \psi_i > &\geq c_{1,i} \qquad \forall\, h \in B_i \mid U_{1,i},\; i \in J \\ < -h, y' - \rho_i > &\geq c_{2,i} \qquad \forall\, h \in C_i \mid U_{2,i},\; i \in J \end{aligned}$$

Put $\mu_i = \max_{h\in B_i} |< h, y >|$, $\nu_i = \max_{h\in C_i} |< h, y >|$, $a_{j,i} = \min\left\{\frac{c_{j,i}}{2\mu_i}, \frac{c_{j,i}}{2\nu_i}\right\}$ and $r_0 = \left\{\min_{i\in J}\{a_{1,i}\},\ \min_{i\in J}\{a_{2,i}\}, r\right\}$. Observe that, for $h \in U_{1,i}$ and $t \in (0, r_0]$,

$$< h, y' - ty - \psi_i > = < h, y' - \psi_i > - t < h, y > > 0$$

and for $h \in B_i \mid U_{1,i},\ t \in (0, r_0]$

$$< h, y' - ty - \psi_i > = < h, y' - \psi_i > \ - t < h, y > \geq \ c_{1,i} - t < h, y > > \ 0.$$

Similary, for $h \in U_{2,i}$ and $t \in (0, r]$,

$$< -h, y' - ty - \rho_i > = < -h, y' - \rho_i > \ - t < -h, y > > \ 0$$

and for $h \in C_i \mid U_{2,i}, t \in (0, r_0]$

$$< -h, y' - ty - \rho_i > = < -h, y' - \rho_i > -t < -h, y > \geq \ c_{2,i} - t < -h, y > > \ 0.$$

This shows $y' - ty$ is in $Y$ for $t \in (0, r_0]$. Moreover, we have $d_F(y' - ty) < d_F(y')$. Hence $y'$ is not a best approximation to $F$ which proves the theorem.

In the case where $G$ is of dimension $n$, we have the following useful result without the additional assumption of continuity of linear functional in $B_i, C_i,\ i = 1, 2, \cdots, m$.

**Theorem 2.3.** *Suppose $G$ is an n-dimensional subspace of $X$ and $B_i, C_i \subset X^*(i = 1, 2, \cdots, m)$. Furthermore, assume that the restriction $B_i \mid_G$, $C_i \mid_G$ are closed and bounded. If there exists $y \in G$ such that*

$$< \tau, y > \geq \ a \ > \ 0, \ \forall \ \tau \in N_{1,i}(y') \ \cup \ \left[-N_{2,i}(y')\right], \ i = 1, 2, \cdots, m,$$

*for some given $y' \in Y$, then $y'$ is a best approximation to $F$ if and only if there exist $s$ functionals, $k_1, \cdots, k_s \in H(y')$, $\ell_i$ functionals $h_{i,1}, \ldots, h_{i,\ell_i} \in N_{1,i}(y')$ for $i \in I_1(y')$ and $t_i$ functionals $q_{i,1}, \ldots, q_{i,t_i} \in N_{2,i}(y')$ for $i \in I_2(y')$ and $s + \Sigma_{i \in I_1(y')} \ell_i + \Sigma_{i \in I_2(y')} t_i$ scalars $a_1, \ldots, a_s, b_{i,1}, \cdots, b_{i,\ell_i}, c_{i,1}, \ldots, c_{i,t_i} > 0$, such that*

$$s \ + \sum_{i \in I_1(y')} \ell_i \ + \sum_{i \in I_2(y')} t_i \ \leq \ n \ + \ 1,$$

$$\sum_{i=1}^{s} a_i \ + \sum_{i \in I_1(y')} \sum_{j=1}^{\ell_i} b_{i,j} \ + \sum_{i \in I_2(y')} \sum_{j=1}^{t_i} c_{ij} \ = \ 1$$

and

$$\sum_{i=1}^{s} a_i < k_i, y > + \sum_{i \in I_1(y')} \sum_{j=1}^{\ell_i} b_{ij} < h_{i,j}, y > - \sum_{i \in I_2(y')} \sum_{i=1}^{t_i} c_{i,j} < q_{i,j}, y >= 0 \ \forall \ y \in G.$$

**Proof.** Define the set $W$ of n-tuples as follows:

$$W \ = \ \left\{(< k, y_1 >, \ldots, < k, y_n >) : k \in H(y')\right\}$$
$$\cup \left\{(< h_{i,j}, y_1 >, \ldots, < h_{i,j}, y_n >) : h_{i,j} \in N_{1,i}(y'), \ i \in I_1(y')\right\}$$
$$\cup \left\{(< -q_{i,j}, y_1 >, \ldots, < -q_{i,j}, y_n >) : q_{i,j} \in N_{2,i}(y'), \ i \in I_2(y')\right\}$$

where $y_1, \ldots, y_n$ is a basis for $G$. Obviously, $W$ is a compact subset of $R^n$, since $B_i \mid_G$, $C_i \mid_G$ are closed and bounded.

First of all, we shall show that $y'$ is a best approximation if and only if $O \in co(W)$.

Suppose that $y'$ is not a best approximation, then, as in the proof of sufficiency of Theorem 2.2, there exists a $y \in G$ such that

$$< k, y > < 0 \qquad \forall\, k \in H(y')$$

and

$$< \tau, y > < 0 \qquad \forall\, \tau \in N(y').$$

Therefore, by a known result in [1, p.19], $O \notin co(W)$.

On the other hand, suppose $O \notin co(W)$, then, by a known result in [1, p.19], there exists $y \in G$ such that

$$< k, y > < 0 \qquad \forall\, k \in H(y')$$

and

$< r, y > < 0 \qquad \forall\, \tau \in N(y')$.

By a similar argument to that in the proof of necessity of Theorem 2.2, we can find $y \in Y$ such that $d_F(y) < d_F(y')$. Therefore, $y'$ is not a best approximation.

Thus, we have shown that $y'$ is a best approximation if and only if $O \in co(W)$. By Caratheodory's Theorem, $0 \in co(W)$ if and only if there exist $k_1, \ldots, k_s \in H(y')$, $h_{i,1}, \ldots, h_{i,\ell_i} \in N_{1,i}(y')$ for $i \in I_1(y')$ and $q_{i,1}, \cdots, q_{i,t_i} \in N_{2,i}(y')$ for $i \in I_2(y')$ and $s + \Sigma_{i \in I_1(y')} \ell_i + \Sigma_{i \in I_2(y')} t_i$ scalars $a_1, \cdots, a_s$, $b_{i,1}, \cdots, b_{i,\ell_i}$ and $c_{i,1}, \ldots, c_{i,t_i} > 0$ such that

$$s + \sum_{i \in I_1(y')} \ell_i + \sum_{i \in I_2(y')} t_i \leq n + 1,$$

$$\sum_{i=1}^{s} a_i + \sum_{i \in I_1(y')} \sum_{j=1}^{\ell_i} b_{i,j} + \sum_{i \in I_2(y')} \sum_{j=1}^{t_i} c_{i,j} = 1$$

and

$$\sum_{i=1}^{s} a_i(< k_i, y_1 >, \ldots, < k_i, y_n >) + \sum_{i \in I_1(y')} \sum_{j=1}^{\ell_i} b_{i,j}(< h_{i,j}, y_1 >, \ldots, < h_{i,j}, y_n >)$$
$$- \sum_{i \in I_2(y')} \sum_{j=1}^{t_i} c_{i,j}(< q_{i,j}, y_1 >, \ldots, < q_{i,j}, y_n >) = 0.$$

On multiplying by any vector $\overline{d} = (d_1, d_2, \ldots, d_n)$, it follows that

$$\sum_{i=1}^{s} a_i < k_i, y > + \sum_{i \in I_1(y')} \sum_{j=1}^{\ell_i} b_{i,j} < h_{i,j}, y >$$
$$- \sum_{i \in I_2(y')} \sum_{j=1}^{t_i} c_{i,j} < q_{i,j}, y > = 0 \qquad \forall\, y \in G.$$

which proves the theorem.

## 3. Application to space $C[a,b]$

We now turn to a concrete application of the results of Section 2. Let $G$ be an n-dimensional subspace of $C^r[a,b]$, the set of all $r$ times continuous differentiable functions in $C[a,b]$. We denote the point evaluation functional of k-th derivative at $x$ by $\hat{x}^k(f) = D^k f(x)$ for all $f \in C^r[a,b]$ $(k \le r)$. The semi-norm $p(\cdot)$ is defined as

$$p(f) \;=\; \max_{x \in T} \mid f(x) \mid$$

where $T$ is a closed subset of $[a,b]$. Obviously, the corresponding set $K = \{\varepsilon \hat{x} : x \in T\}$ where $\varepsilon = \pm 1$. First, we consider $B_i = C_i = \left\{\hat{x}^{k_i} : x \in [a,b]\right\}$ and $\psi_i, \rho_i \in C^r[a,b]$, $i = 1,2,\cdots,m$, such that $\hat{x}^{k_i}(\psi_i) < \hat{x}^{k_i}(\rho_i)$ $\forall \hat{x}^{k_i} \in B_i$, $i = 1,2,\cdots,m$, $1 \le k_1 \le k_2 \le \cdots \le k_m < r$. Hence the set

$$Y \;=\; \left\{g \in G : \; \hat{x}^{k_i}(\psi_i) \le \hat{x}^{k_i}(g) \le \hat{x}^{k_i}(\rho_i), \; \forall \, \hat{x}^{k_i} \in B_i, \; i = 1,2,\cdots,m\right\}.$$

Obviously the restriction $B_i \mid_G$ is closed and bounded. By virtue of Theorem 2.3, we have,

**Theorem 3.1.** *Suppose there exists a $g \in G$ such that $\tau(g) \ge \beta > 0$ $\forall \, \tau \in N_{1,i}(g_0) \cup \left[-N_{2,i}(g_0)\right]$ $i = 1,2,\cdots,m$, for some given $g_0 \in Y$. Then $g_0$ is best approximation to a compact subset $F \subset C[a,b]$ if and only if there exist $x_1,\ldots,x_s \in T$, $y_{i,1},\ldots,y_{i,\ell_i} \in [a,b]$ for $i \in I_1(g_0)$, $z_{i,1},\ldots,z_{i,t_i} \in [a,b]$, for $i \in I_2(g_0)$, $s$ functions $f_1,\ldots,f_s \in F$ (not necessarily distinct) and $s + \Sigma_{i \in I_1(g_0)} \ell_i + \Sigma_{i \in I_2(g_0)} t_i$ scalars $a_1,\ldots,a_s$, $b_{i,1},\cdots,b_{i,\ell_i}$ and $c_{i,1},\ldots,c_{i,t_i} > 0$ such that*

$$s \;+ \sum_{i \in I_1(g_0)} \ell_i \;+ \sum_{i \in I_2(g_0)} t_i \;\le\; n \;+\; 1,$$

$$\sum_{i=1}^{s} a_i \;+\; \sum_{i \in I_1(g_0)} \sum_{j=1}^{\ell_i} b_{i,j} \;+\; \sum_{i \in I_2(g_0)} \sum_{j=1}^{t_i} c_{i,j} \;=\; 1,$$

$$\sum_{i=1}^{s} a_i \sigma(x_i) g(x_i) \;+\; \sum_{i \in I_1(g_0)} \sum_{j=1}^{\ell_i} b_{i,j} D^{k_i} g(y_{i,j})$$
$$-\sum_{i \in I_2(g_0)} \sum_{j=1}^{t_i} c_{ij} D^{k_i} g(z_{i,j}) \;=\; 0 \qquad \forall \, g \in G. \tag{3.1}$$

$$\mid f_i(x_i) - g_0(x_i) \mid \;=\; d_F(g_0), \qquad i = 1,2,\cdots,s,$$

$$D^{k_i}\Big[g_0(y_{i,j})-\psi_i(y_{i,j})\Big] \;=\; 0, \qquad j=1,\cdots,\ell_i,\ i\in I_1(g_0)$$

and

$$D^{k_i}\Big[g_0(z_{i,j})-\rho_i(z_{i,j})\Big] \;=\; 0, \qquad j=1,\cdots,t_i,\ i\in I_2(g_0),$$

where $\sigma(x_i)=\text{sign}(f_i(x_i)-g_0(x_i))$ $i=1,2,\cdots,s$.

If we put $B_i=\Big\{\varepsilon_i\hat{x}^{k_i}: x\in[a,b]\Big\}$, $C_i=\emptyset$, and $\psi_i=0$, $i=1,2,\ldots,m$, where $\varepsilon_i=\pm1$, $1\le k_1\le k_2\le\cdots\le k_m<r$, then $Y=\{g\in G:\ \varepsilon_i D^{k_i}g(x)\ge 0,\ \forall\ x\in[a,b],\ i=1,2,\ldots,m\}$. Suppose $G$ is the set of all polynomials of degree $\le n$. Then, since $G$ is the set of all polynomials of degree $\le n$, there always exist $g\in G$ such that $\varepsilon_i\hat{x}^{k_i}(g)>0$, $\forall\varepsilon_i\hat{x}^{k_i}\in N_{1,i}(g_0)$, $i\in I_1(g_0)$, for some $g_0\in Y$, and by virtue of Theorem 2.3, we have

**Theorem 3.2.** *An element $g_0\in Y$ is a best approximation to a compact subset $F\subset C[a,b]$, if and only if there exist $x_1,\ldots,x_s\in T$, $y_{i,1},\cdots,y_{i,\ell_i}\in[a,b]$ for $i\in I_1(g_0)$, $s$ functions $f_1,\cdots,f_s\in F$ (not necessarily distinct) and $s+\Sigma_{i\in I_1(g_0)}\ell_i$ scalars $a_1,\ldots,a_s$ and $b_{i,1},\ldots,b_{i,\ell_i}>0$ such that $s+\Sigma_{i\in I_1(g_0)}\ell_i\le n+2$,*

$$\sum_{i=1}^{s}a_i \;+\; \sum_{i\in I_1(g_0)}\sum_{j=1}^{\ell_i}b_{i,j} \;=\; 1.$$

$$\sum_{i=1}^{s}a_i\sigma(x_i)g(x_i) \;+\; \Sigma_{i\in I_1(g_0)}\sum_{j=1}^{\ell_i}\varepsilon_i b_{i,j}D^{k_i}g(y_{i,j}) \;=\; 0 \qquad \forall\, g\in G \tag{3.2}$$

$$|\ f_i(x_i)-g_0(x_i)\ | = \ d_F(g_0), \qquad i=1,2,\cdots,s$$

*and*

$$D^{k_i}g_0(y_{i,j}) \;=\; 0, \qquad j=1,2,\cdots,\ell_i,\ i\in I_1(g_0),$$

*where $\sigma(x_i)=\text{sign}(f_i(x_i)-g_0(x_i))$, $i=1,2,\cdots,s$.*

In the case $F$ consists of a single function and $p(\cdot)$ is the usual supremum-norm, Theorem 3.2 is, in fact, a known result given in [5].

Now, consider the case. $m=1$, $B_1=C_1=\{\hat{x}: x\in[a,b]\}$ and $G$ is an n-dimensional subspace of $C[a,b]$. Let $\psi_1,\rho_1$ be two given functions in $C[a,b]$ such that $\psi_1(x)<\rho_1(x)$ $\forall x\in[a,b]$. Then, by Theorem 2.3, we have

**Theorem 3.3.** *Suppose there exists a $g\in G$ such that $\tau(g)\ge\beta>0$ $\forall\tau\in N_{1,2}(g_0)\cup\Big[-N_{2,1}(g_0)\Big]$, for some given $g_0\in Y$. Then $g_0$ is a best approximation to a compact subset $F\subset C[a,b]$ if and only if there exist $s$ points, $x_1,\ldots,x_s\in T$, $\ell$ points $y_1,\ldots,y_\ell\in[a,b]$, $t$ points $z_1,\ldots,z_t\in[a,b]$, $s$ functions $f_1,\ldots,f_s\in F$ and $s+\ell+t$ scalars $a_1,\cdots,a_s$, $b_1,\ldots,b_\ell$ and $c_1,\cdots,c_t>0$ such that $s+\ell+t\le n+1$,*

$$\sum_{i=1}^{s}a_i \;+\; \sum_{i=1}^{\ell}b_i \;+\; \sum_{i=1}^{t}c_i \;=\; 1 \qquad \textit{and}$$

$$\sum_{i=1}^{s} a_i\sigma(x_i)g(x_i) + \sum_{i=1}^{\ell} b_i g(y_i) - \sum_{i=1}^{t} c_i g(z_i) = 0 \quad \forall\, g \in G \tag{3.3}$$

$$| f_i(x_i) - g_0(x_i) | = d_F(g_0), \qquad i = 1,2,\ldots,s;$$
$$g_0(y_i) - \psi_1(y_i) = 0, \qquad i = 1,2,\cdots,\ell;$$

*and*

$$g_0(z_i) - \rho_1(z_i) = 0, \qquad i = 1,2,\cdots,t,$$

*where*

$$\sigma(x_i) = \operatorname{sgn}(f_i(x_i) - g_0(x_i)), \qquad i = 1,2,\cdots,s.$$

*In this case, if $G$ is the set of all polynomials of degree $\leq n$, $F$ consists of a single function $f$ say, $\psi_1 = f + \gamma_1$, $\rho_1 = f + \gamma_2$, for some $\gamma_1, \gamma_2 \in C[a,b]$ such that $\gamma_1(x) < \gamma_2(x)$, and $p(\cdot)$ is the usual supremum norm, then Theorem 3.3 is, in fact, a known result in [7].*

## 4. Appication to space $C[a,b]$ endowed with $L_\mu$-norm $(\mu \geq 1)$

Let $G$ be an n-dimensional subspace of $C^r[a,b]$ and $B_i, C_i, \psi_i, \rho_i, Y$ be defined as in the beginning of Section 3. Suppose $p(\cdot)$ is defined to be a $L_\mu$-norm. Then, by virtue of Theorem 2.3, we have

**Theorem 4.1.** *Suppose there exists a $g \in G$ such that $\tau(g) \geq \beta > 0\ \forall\, \tau \in N_{1,i}(g_0) \cup \left[-N_{2,i}(g_0)\right]$, $i \in J$ for some given $g_0 \in Y$. Then $g_0$ is a best approximation to a compact subset $F \subset C[a,b]$ if and only if there exist $s$ functions $u_1(x), \ldots, u_s(x) \in L_\nu[a,b]$, $s$ functions $f_1, \ldots, f_s \in F$ (not necessarily distinct), $\Sigma_{i\in I_1(g_0)}\ell_i$ points $y_{i,1}, \ldots, y_{i,\ell_i} \in [a,b]$, $\Sigma_{i\in I_2(g_0)}t_i$ points $z_{i,1}, \ldots, z_{i,t_i} \in [a,b]$ and $s + \Sigma_{i\in I_1(g_0)}\ell_i + \Sigma_{i\in I_2(g_2)}t_i$ scalars $a_1, \cdots, a_s$, $b_{i,1}, \ldots, b_{i,\ell_i}$ and $c_{i,1}, \ldots, c_{i,t_i} > 0$ such that $s + \Sigma_{i\in I_1(g_0)}\ell_i + \Sigma_{i\in I_2(g_0)}t_i \leq n+1$ and*

$$\sum_{i=1}^{s} a_i \int_a^b g(x)u_i(x)dx + \sum_{i\in I_1(g_0)} \sum_{j=1}^{\ell_i} b_{i,j} D^{k_i} g(y_{i,j})$$
$$+ \sum_{i\in I_2(g_0)} \sum_{j=1}^{t_i} c_{i,j} D^{k_i} g(z_{i,j}) = 0 \quad \forall\, g \in G \tag{4.1}$$

*where $\frac{1}{\mu} + \frac{1}{\nu} = 1$, $\int_a^b (f_i - g_0)u_i = d_F(g_0)$, $i = 1,2,\cdots,s$,*

$$D^{k_i}\left[g_0(y_{i,j}) - \psi_i(y_{i,j})\right] = 0, \qquad j = 1,2,\cdots,\ell_i,\ i \in I_1(g_0)$$

*and*

$$D^{k_i}\left[g_0(z_{i,j}) - \rho_i(z_{i,j})\right] = 0, \qquad j = 1,2,\cdots,t_i,\ i \in I_2(g_0).$$

*Finally, we consider the case $m = 1$, $\varepsilon_1 = -1$, $B_1 = \{\varepsilon_1 \hat{x} : x \in [a,b]\}$, $C_1 = \emptyset$ and $F = \{f\}$. Suppose $G$ is an $n$-dimensional subspace of $C[a,b]$ containing constant functions. Define the set $Y$ as before by taking $\psi_1 = f$. This again leads to one-sided approximation in $C[a,b]$ with $L_\mu$-norm. However, it is clear that $f(x) - g(x)$ does not change sign for each $g \in Y$, hence, if $u(x) \in L_\nu[a,b]$ such that $\int (f-g)u = \|f-g\|_\mu$, then $u(x) = (f-g)^{\mu-1}(x)/\|f-g\|_\mu^{\mu-1}$. Consequently, we have*

**Theorem 4.3.** *A function $g_0 \in Y$ is a best approximation to $f \in C[a,b]$ if and only if there exist $y_1, \ldots, y_\ell \in N_{1,1}(g_0)$ and $\ell$ scalars $b_1, \ldots, b_\ell > 0$ such that $\ell \leq n$ and*

$$\int_a^b g(x)(f(x) - g_0(x))^{\mu-1}dx \;-\; \sum_{i=1}^{\ell} b_i g(y_i) \;=\; 0 \qquad \text{for all } g \in G. \tag{4.2}$$

*If $\mu = 1$, then the equality (4.2) can be written as*

$$\int_a^b g(x)dx \;=\; \sum_{i=1}^{\ell} b_i g(y_i) \qquad \text{for all } g \in G_n.$$

**Remark** In the case $G$ is the set of all polynomials of degree $\leq n-1$, then $2\ell - e_1 \geq n$ where $\ell$ and $e_1$ are the number of points in $N_{1,1}(g_0)$ and $N_{1,1}(g_0) \cap \{a,b\}$, respectively, for otherwise, we would find a $g \in G$ such that $g$ has double zeros on $N_{1,1}(g_0) \mid \{a,b\}$ and simple zeros on $N_{1,1}(g_0) \cap \{a,b\}$, which would contradict the relation (4.2). Moreover, if $Dg_0 \neq 0$, then $2\ell - e_1$ is, in fact, actually equal to $n$.

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