TAMKANG JOURNAL OF MATHEMATICS
Volume 54, Number 4, 379-404, 2023
DOI:10.5556/j.tkjm.54.2023.4740

Analyzing stability of equilibrium points in impulsive neural network models involving generalized piecewise alternately advanced and retarded argument

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Abstract. In this paper, we investigate the models of the impulsive cellular neural network with piecewise alternately advanced and retarded argument of generalized argument (in short IDEPCAG). To ensure the existence, uniqueness and global exponential stability of the equilibrium state, several new sufficient conditions are obtained. The method is based on utilizing Banach's fixed point theorem and a new IDEPCAG's Grönwall inequality. The criteria given are easy to check and when the impulsive effects do not affect, the results can be extracted from those of the non-impulsive systems. Typical numerical simulation examples are used to show the validity and effectiveness of proposed results.

Keywords. Impulsive neural networks, Piecewise constant argument of generalized type, Equilibrium, Global exponential stability, Grönwall integral inequality.

1 Introduction

Multi-variable feedback systems can exert the retroactive effect on very different time scales. Exemplifying by the extremes, according to the date of the information that is used to feedback, this action can define: (a) a *continuous process* or (b) one *discrete process*. In case (a), the growth rates of the variables are feed backed at each instant, let's say in real time. While, in case (b) there is a set of isolated dates, for example, a succession of instants in which the information is taken, in order to feedback the period between two consecutive sequence elements.

Normally and for mathematical modeling purposes, in case (a) differential equations are used and in case (b), if there is no other dynamics effect between the feedback times, difference equations can be used to express the essence of the dynamics. There are processes (real-world systems, such as some biotechnology-based ones) that can not be categorized into types (a) or (b), as they combine characteristics of both types of scales among other particular effects.

Lately, the new type of feedback systems shows a combination of characteristics from both the continuous-time and the discrete-time systems, which is neither continuous time nor purely discrete-time; among them are dynamical systems with impulses and systems with piecewise constant arguments. This leads to the use of hybrid type equations, for example, *Impulsive differential equations with piecewise constant arguments* (in abbreviation: IDEPCA), that were

Received date: September 21, 2021; Published online: July 29, 2022.
2010 *Mathematics Subject Classification*. 92B20, 34A37, 34K20, 34K34, 26D10.
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first considered by Wiener and Lakshmikantham [40] in 2000, and differential equations with piecewise constant argument without impulsive effect (in short, DEPCA) were studied by Shah and Wiener [34] and Wiener [38] in 1983; and has been investigated by many authors. We highlight the book of J. Wiener [39], pioneer of DEPCA, that recollects much of the research done in DEPCA. In the case, DEPCA of generalized type, were discussed extensively in [1, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 33]. See also T. Li et al. [27], T. Li and G. Viglialoro [28], and Viglialoro and Woolley [36] for models from mathematical biology described by evolutive partial differential equations, also including drift terms.

When scales are mixed these feedback systems can be visualized as control systems, in that, one scale represents the intrinsic of the process and the other is external intervention. However, based on internal parameters. As an example, mentioned in Busenberg and Cooke [3], is the example of the stabilization of hybrid control systems with feedback delay, in which a hybrid system is a dynamical system that presents both continuous and discrete dynamic behavior. The hybrid control systems are very interesting, which depend on the attributes and simplifications of modeling on the process, being the most usual, to represent the intrinsic process with the continuous time scale and to reflect the intervention from the external environment to the system with the discrete scale.

Note that, either as a feedback system or as a system under control, the questions of interest usually refer to the behavior of the variables in the long term, in particular looking for specific patterns according to values in the space of feasible parameters. For reasons of practical necessity for the modeled processes, the most recurrently sought behavior is stability, in some sense, for example, seen as convergence to a steady state or towards dynamic cycles.

In the last decades, the dynamics of artificial neural networks model is one of the most applicable and attractive objects for the mathematical foundations of neuroscience. In 1988, Chua et al. [21] presented a new class of information-processing systems referred to as cellular neural networks (CNNs). It is known that the study of the stability of CNNs, DCNNs (delayed CNNs) and ICNNs (CNNs with impulses) is an important problem in theory and application. Many essential aspects of these networks, such as qualitative features of stability, periodicity, oscillation, and convergence problems have been examined by many other authors (see [2, 5, 7, 14, 22, 23, 25, 26, 29, 30, 31, 32, 37, 41, 42, 43] and the references cited therein). Since stochastic theories of neural networks including statistics of all orders have also been largely studied with various approaches and their efficiency is often based on the convergence of certain moment hierarchies, see [4, 24].

In 2000, J. Cao [5] proposed the problem of neural networks with transmission delays by using the Lyapunov method. Afterwards, considering theory of M-matrices, some stability criteria were established for delayed Hopfield neural networks [7] and the convergence behavior of a unique equilibrium of ICNNs was derived from [22].

In 2003, in view of Halanay-type inequalities and the Lyapunov methods, Mohamad and Gopalsamy [31] discussed the stability of DCNNs with continuous and discrete time; Zhou and Hu [42] (2008) studied periodic and stability conditions for DCNNs with variable and distributed delays. In 2004, by using Mawhin's coincidence degree theory and Grönwall 's inequality, Liu and Liao [29] investigated DCNNs with periodic coefficients.

J. H. Park [32] (2006), B. Wang et al. [37] (2008), Zhang [43] (2009), O.M. Kwona et al. [25] and T. Li [26] (2012) acquired some delay-dependent stability criteria for interval time-varying delays neural networks, by constructing a Lyapunov-Krasovskii functional and linear matrix inequalities. In [30] and [41], some criteria have been derived for high-order neural networks without and with time-varying delays, which were analyzed using the Lyapunov method and analytical technique by linear matrix inequality.

To the best of our knowledge, cellular neural network with piecewise constant argument has been developed by few authors, for example, Huang et al. [23] considered first the following cellular neural network with the DEPCA system

$$x_i'(t) = -a_i([t]) + \sum_{j=1}^{m} b_{ij}([t])\, g_j(x_j([t])) + d_i([t]), \tag{1.1}$$

where $i = 1, 2, ..., m$ and $[\cdot]$ is the greatest integer function. In this case, $x'(t)$ depends during all the interval $[n, n+1)$, n an integer number, only of the value of functions defined at instant n. So, equations type (1.1), with a constant delay of generalized type, are named *differential equation with generalized piecewise constant delay* (DEGPCD). The theory of the IDEGPCD system has been investigated by few authors. See [1, 12, 20].

In the present work, we consider impulsive neural networks models with piecewise alternately advanced and retarded argument of generalized type

$$\begin{cases} \dfrac{dx_i(t)}{dt} = -a_i x_i(t) + \displaystyle\sum_{j=1}^{n} b_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} c_{ij} g_j\big(x_j\left(\gamma(t)\right)\big) + d_i, \quad t \neq t_\kappa, & (1.2a) \\ \Delta x_i(t_\kappa) = \mathfrak{J}_{i\kappa}(x_i(t_\kappa^-)), \quad \kappa \in \mathbb{N}, & (1.2b) \end{cases}$$

with $1 \leq i \leq n$, where

- The constant argument of generalized type is determined by a strictly increasing unbounded sequence of times $\{t_\kappa\}$ and the function $\gamma(\cdot)$ defined by $\gamma(t) = \gamma_\kappa$, $t_\kappa \leq \gamma_\kappa < t_{\kappa+1}$, if $t \in I_\kappa = [t_\kappa, t_{\kappa+1})$.
- The positive constant a_i denotes the relative rate with which the i–th unit resets its potential to the resting state when isolated from other units and inputs. So in (1.2a), it represents an exponential decay.
- The measure of activation of continuous type (resp. piecewise constant type) of the j-th neuron to its incoming potentials is given at any time by the function $f_j(x_j(\cdot))$ (resp.$g_j(x_j(\gamma(\cdot)))$).
- The constant b_{ij} (resp. c_{ij}) represents the weight of continuous type (resp. piecewise type) of the j-th unit on the i-th unit.
- For each neuron, there is an activation flow from outside the system. It is represented by the function d_i for the i-th one.
- $\Delta x_i(t_\kappa) = x_i(t_\kappa) - x_i(t_\kappa^-)$, where $x_i(t_\kappa^-) = \lim_{h \to 0^-} x_i(t_\kappa + h)$ and $\mathfrak{J}_{i\kappa}(x_i(t_\kappa^-))$ at the impulsive moment t_κ.

We say that a deviation argument is of piecewise alternately advanced and retarded argument, and denote $\gamma(t) = \gamma_\kappa$, $t_\kappa \leq \gamma_\kappa < t_{\kappa+1}$, if $t \in I_\kappa$, for all $\kappa \in \mathbb{N}$. One can easily see that, the deviation argument $\ell(t) = t - \gamma(t)$ is assumed to be negative for $t_\kappa < t < \gamma_\kappa$ and positive for $\gamma_\kappa < t < t_{\kappa+1}$, $\kappa \in \mathbb{N}$. Therefore, Eq. (1.2a) is of considerable interest: on each interval $[t_\kappa,\ t_{\kappa+1})$ it is of alternately advanced and retarded type. Eq. (1.2a) is of advanced type on $I_\kappa^+ = [t_\kappa,\ \gamma_\kappa]$ and retarded type on $I_\kappa^- = (\gamma_\kappa, t_{\kappa+1})$. So, equations type (1.2a), with $\gamma(\cdot)$ of alternately advanced and retarded type, are named *differential equation with piecewise alternately advanced and retarded argument of generalized type* (DEPCAG). The equations type can represent anticipatory models. Note that the scientific mathematical community around the DEPCAG with impulsive effect (IDEPCAG) is very limited. See [6, 18].

Taking into account the definition of solutions for the IDEPCAG system [1, 12, 20], we understand that a function $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$, T denotes the transpose of a matrix, is a solution of the IDEPCAG system (1.2a)-(1.2b) in $\mathbb{R}^+ = [0, \infty)$, if $x(t)$ is continuous with possible points of discontinuity of the first kind at t_κ, $\kappa \in \mathbb{N}$, such that the derivative $x'(t)$ exists at each point $t \in \mathbb{R}$, with the possible exception of the points $t_\kappa \in \mathbb{R}$, $\kappa \in \mathbb{N}$, where a one-sided derivative exists, and $x(t_\kappa)$ satisfies the impulsive effects (1.2b), $\kappa \in \mathbb{N}$. Moreover the IDEPCAG system (1.2a)-(1.2b) is satisfied by $x(t)$ on each interval $(t_\kappa, t_{\kappa+1})$, $\kappa \in \mathbb{N}$ as well.

For $x \in \mathbb{R}^n$, its norms are defined as

$$\|x\|_1 = \left(\sum_{i=1}^{n} |x_i| \right) \quad \text{and} \quad \|x\| = \max_{1 \le i \le n} |x_i| .$$

Notice that, to know information about the behavior of solutions of (1.2a)-(1.2b), as a mathematical problem, has an historical evolution, we can point out that:

(1) In 2010, M. U. Akhmet et al. [1] applied linearization method and established stability criterion for the equilibrium and the periodic solution of the IDEGPCD system.

(2) In 2013, K.-S. Chiu [10] obtained some sufficient conditions for the equilibrium of the IDEPCA system with the particular argument $m\left[\frac{t+l}{m}\right]$, where l and m are positive real numbers such that $l < m$.

(3) In 2022, K.-S. Chiu [20] obtained some sufficient conditions for the equilibrium of the IDEGPCD system with the linear approximation method.

The novelty of our work is to present new and simple sufficient conditions ensuring existence, uniqueness and global exponential stability of the equilibrium state for impulsive neural network models with piecewise alternately advanced and retarded argument of generalized type (ICNN models with the IDEPCAG system). The proposed criteria extend the results of the previous literature. The method is given by the traditional and tailored route of a: IDEPCAG's Grönwall inequality and Banach contraction principle.

The rest of the paper is organized as follows. Firstly, we will introduce some preliminary concepts and propositions. Then by using a new IDEPCAG's Grönwall inequality and the contraction mapping principle, we obtain several criteria for the existence and uniqueness of the equilibrium state of the ICNN models (1.2a)-(1.2b). Moreover under some easily verifiable conditions, our unique equilibrium state of the ICNN models (1.2a)-(1.2b) is globally exponentially stable. Finally, two examples with the numerical simulations are given to show the effectiveness of our results.

2 Preliminary notes

In this section, we present some preliminary concepts and propositions, which are used to prove the stability of solutions of the ICNN models.

For reasons of convenience, certain assumptions are formulated below, which will be convened when necessary.

(H_1) The functions f_i and g_i with $f_i(0) = 0$, $g_i(0) = 0$, $0 \le i \le n$, satisfy the Lipschitz condition:

$$|f_i(u) - f_i(v)| \le \mathfrak{L}_i^f |u - v|, \quad |g_i(u) - g_i(v)| \le \mathfrak{L}_i^g |u - v|$$

for some positive constants $\mathfrak{L}_i^f$, $\mathfrak{L}_i^g$ and for all u, $v \in \mathbb{R}$.

(H_2) The impulsive operator $J_{i\kappa}$, $0 \le i \le n$, $\kappa \in \mathbb{N}$, satisfies

$$|\mathfrak{J}_{i\kappa}(u) - \mathfrak{J}_{i\kappa}(v)| \le \mathfrak{L}_{i\kappa}^J |u - v|,$$

for the positive constant $\mathfrak{L}_{i\kappa}^J$ and for all u, $v \in \mathbb{R}$.

(H_3) For any $\tau > 0$, it is satisfied $\hat{\kappa}(\tau) =: \max\{\kappa_1, \kappa_2\} < 1$, where

$$\kappa_1 = \max_{1 \le i \le n} \left\{ \sup_{1 \le \kappa \le i(\tau)} \left(\frac{e^{a_i \cdot \vartheta_\kappa^-} - 1}{a_i} \right) \left[\sum_{j=1}^n \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^n \mathfrak{L}_j^g |c_{ij}| \right] \right\}$$
$$\kappa_2 = \max_{1 \le i \le n} \left\{ \sup_{i(\tau) \le \kappa} \left(\frac{1 - e^{-a_i \cdot \vartheta_\kappa^+}}{a_i} \right) \left[\sum_{j=1}^n \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^n \mathfrak{L}_j^g |c_{ij}| \right] \right\}$$

here $i(\cdot)$ is an indexer defined by $i(t) = \kappa$ if $t \in I_\kappa = [t_\kappa, t_{\kappa+1})$, and $\vartheta_\kappa^+ = \gamma_\kappa - t_\kappa$, $\vartheta_\kappa^- = t_{\kappa+1} - \gamma_\kappa$, $\kappa \in \mathbb{N}$.

(H_3') For any $\tau > 0$, it is satisfied $\hat{\kappa}(\tau) < 1$, where

$$\hat{\kappa}(\tau) = \max_{1 \le i \le n} \left\{ \sup_{1 \le \kappa \le i(\tau)} \left(\frac{e^{a_i \cdot \vartheta_\kappa} - 1}{a_i} \right) \left[\sum_{j=1}^n \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^n \mathfrak{L}_j^g |c_{ij}| \right] \right\}$$

and $\vartheta_\kappa = t_{\kappa+1} - t_\kappa$, $\kappa \in \mathbb{N}$.

To study the ICNN models with the IDEPCAG system (1.2a)-(1.2b), we need the following proposition.

Proposition 2.1. Integral Representation: *Given a pair* $(\tau, \zeta) \in \mathbb{R}^+ \times \mathbb{R}^n$, *a function* $x = (x_1(\cdot), \cdots, x_n(\cdot))^T : \mathbb{R}^+ \to \mathbb{R}^n$, *such that* $x(\tau) = (x_1(\tau), x_2(\tau) ..., x_n(\tau))^T = \zeta$, *is a solution of the ICNN models with the IDEPCAG system (1.2a)-(1.2b) if and only if their coordinates satisfy on* $\mathbb{R}^+$ *the set of integral equations: for* $i \in \{1, \cdots, n\}$,

$$x_i(t)=\begin{cases} e^{-a_i(t-\tau)}x_i(\tau)+\int_\tau^t e^{-a_i(t-s)}\left[\sum_{j=1}^n b_{ij}f_j(x_j(s))+\sum_{j=1}^n c_{ij}g_j\left(x_j\left(\gamma(s)\right)\right)+d_i\right]ds \\ \qquad +\sum_{k=i(\tau)+1}^{i(t)} e^{-a_i(t-t_\kappa)}\mathfrak{J}_{i\kappa}(x_i(t_\kappa^-)), & i(t)>i(\tau), \\ e^{-a_i(t-\tau)}x_i(\tau)+\int_\tau^t e^{-a_i(t-s)}\left[\sum_{j=1}^n b_{ij}f_j(x_j(s))+\sum_{j=1}^n c_{ij}g_j\left(x_j\left(\gamma_{i(\tau)}\right)\right)\right. \\ \qquad \left.+d_i\right]ds, & i(t)=i(\tau), \\ e^{-a_i(t-\tau)}x_i(\tau)+\int_\tau^t e^{-a_i(t-s)}\left[\sum_{j=1}^n b_{ij}f_j(x_j(s))+\sum_{j=1}^n c_{ij}g_j\left(x_j\left(\gamma(s)\right)\right)+d_i\right]ds \\ \qquad -\sum_{k=i(t)+1}^{i(\tau)} e^{-a_i(t-t_\kappa)}\mathfrak{J}_{i\kappa}(x_i(t_\kappa^-)), & i(t)<i(\tau). \end{cases} \tag{2.1}$$

We do not show the proof of this affirmation, since it can be demonstrated in the same approach as Proposition in [10], Proposition 2.1 in [14] and Theorem 2.3 [35].

The following lemma, which is one of the most important tool will be used in the proofs of our results.

Lemma 2.1. IDEPCAG's Grönwall Inequality: *Let $v:\mathbb{R}^+\to\mathbb{R}^+$ be a non-negative piecewise continuous with possible discontinuity points of the first kind at $t=t_\kappa$, $\kappa\in\mathbb{N}$ for which the inequality satisfying*

$$v(t)\le\begin{cases} v(\tau)+\int_\tau^t\left[\alpha_1 v(s)+\alpha_2 v(\gamma(s))\right]ds+\sum_{\kappa=i(\tau)+1}^{i(t)}\varrho_k v(t_\kappa^-), & i(t)>i(\tau), \\ v(\tau)+\left|\int_\tau^t\left[\alpha_1 v(s)+\alpha_2 v(\gamma(s))\right]ds\right|, & i(t)=i(\tau), \\ v(\tau)+\int_t^\tau\left[\alpha_1 v(s)+\alpha_2 v(\gamma(s))\right]ds+\sum_{\kappa=i(t)+1}^{i(\tau)}\varrho_k v(t_\kappa^-), & i(t)<i(\tau), \end{cases} \tag{2.2}$$

where α_1, α_2, ϱ_k are non-negative constants. Then:

1. *For $i(t)>i(\tau)$,*

$$v(t)\le v(\tau)\left\{\prod_{\kappa=i(\tau)+1}^{i(t)}(1+\varrho_\kappa)\right\}e^{\left(\alpha_1+\frac{\alpha_2}{1-\eta^+}\right)\cdot(t-\tau)}. \tag{2.3}$$

2. *For $i(t)=i(\tau)$,*

$$v(t)\le v(\tau)e^{\left(\alpha_1+\frac{\alpha_2}{1-\eta^+}\right)\cdot(t-\tau)}.$$

3. *For $i(t)<i(\tau)$,*

$$v(t)\le v(\tau)\left\{\prod_{\kappa=i(t)+1}^{i(\tau)}\frac{1}{1-\varrho_k}\right\}e^{\left(\alpha_1+\frac{\alpha_2}{1-\eta^-}\right)\cdot(\tau-t)}, \tag{2.4}$$

where

$$\eta^{+} := \sup_{i(\tau)\leq\kappa} (\gamma_\kappa - t_\kappa)(\alpha_1+\alpha_2) \leq \eta < 1, \tag{2.5}$$

and

$$\eta^{-} := \sup_{1\leq\kappa\leq i(\tau)} (t_{\kappa+1} - \gamma_\kappa)(\alpha_1+\alpha_2) \leq \eta < 1, \qquad \max_{1\leq\kappa\leq i(\tau)} \varrho_k < 1.$$

Proof. First, consider $\tau \leq t$. Suppose that $\psi(t)$ is the right side of the inequality (2.2). Then $\psi(\tau) = v(\tau)$, $v \leq \psi$, ψ is a non-decreasing function and piecewise differentiable, and from (2.2), we have

$$\begin{cases} \psi'(t) \leq \alpha_1\psi(t) + \alpha_2\psi(\gamma(t)), & t \neq t_\kappa, \\ \psi(t_\kappa) \leq (1+\varrho_\kappa)\cdot\psi(t_\kappa^-), & \kappa \in \mathbb{N}. \end{cases} \tag{2.6}$$

If $\tau \leq \ell \leq t$ with $t, \ell \in I_i$, we obtain

$$\psi(t) - \psi(\ell) \leq \int_\ell^t (\alpha_1\psi(s) + \alpha_2\psi(\gamma(s)))\,ds. \tag{2.7}$$

With $t = \gamma_i$, $\ell = t_i$ in (2.7) for $t \in I_i$, as ψ is a non-decreasing function, we get

$$\begin{aligned} \psi(\gamma_i) &\leq \psi(t_i) + \int_{t_i}^{\gamma_i} (\alpha_1\psi(s) + \alpha_2\psi(\gamma_i))ds \\ &\leq \psi(t_i) + \int_{t_i}^{\gamma_i} (\alpha_1+\alpha_2)\psi(\gamma_i)ds = \psi(t_i) + (\gamma_i - t_i)(\alpha_1+\alpha_2)\psi(\gamma_i). \end{aligned} \tag{2.8}$$

By (2.5), we have

$$\psi(\gamma_i) \leq \frac{\psi(t_i)}{1-\eta^{+}}. \tag{2.9}$$

Take now in (2.7) with $t \in I_i$ and $\ell = t_i$, we give

$$\begin{aligned} \psi(t) &\leq \psi(t_i) + \int_{t_i}^{t} (\alpha_1\psi(s) + \alpha_2\psi(\gamma_i))ds \\ &\leq \psi(t_i) + \int_{t_i}^{t} \left(\alpha_1\psi(s) + \frac{\alpha_2}{1-\eta^{+}}\psi(t_i)\right) ds \\ &\leq \psi(t_i) + \int_{t_i}^{t} \left\{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^{+}}\right)\psi(s)\right\} ds. \end{aligned} \tag{2.10}$$

Then, applying the Grönwall 's Lemma, we have:

$$\psi(t) \leq \psi(t_i) e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^{+}}\right)\cdot(t-t_i)} \quad \text{for} \quad t \in I_i.$$

By the impulsive condition (2.6), we obtain:

$$\psi(t_{i+1}) \leq (1+\varrho_{i+1})\,\psi(t_i) e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^{+}}\right)\cdot(t_{i+1}-t_i)}. \tag{2.11}$$

From (2.11), recursively we have

$$v(t) \leq \psi(t) \leq \psi(\tau)\left\{\prod_{k=i(\tau)+1}^{i(t)} (1+\varrho_k)\right\} e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^{+}}\right)\cdot(t-\tau)},$$

by $\psi(\tau) = v(\tau)$, we obtain (2.3).

Now, if $0 \le t \le \tau$. Suppose that $w(t)$ is the right side of the inequality (2.2). So $w(\tau) = v(\tau)$, $v \le w$, w is a non-increasing function and piecewise differentiable and from (2.2), we give

$$\begin{cases} w'(t) \le -\left[\alpha_1 w(t) + \alpha_2 w\left(\gamma(t)\right)\right], \\ w(t_\kappa^-) \le (1-\varrho_\kappa)^{-1} \cdot w(t_\kappa). \end{cases} \tag{2.12}$$

If $\tau \ge \ell \ge t \ge 0$ with $t, \ell \in I_j$, we obtain

$$w(t) - w(\ell) \le -\int_\ell^t \left(\alpha_1 w(s) + \alpha_2 w\left(\gamma(s)\right)\right) ds. \tag{2.13}$$

With $t = \gamma_j$, in (2.13) for $t \in I_j$ and $\ell = t_{j+1}^-$, since w is a non-increasing function, we have

$$\begin{aligned} w(\gamma_j) &\le w(t_{j+1}^-) - \int_{t_{j+1}}^{\gamma_j} (\alpha_1 w(s) + \alpha_2 w(\gamma_j)) ds \\ &\le w(t_{j+1}^-) + w(\gamma_j) \cdot (\alpha_1 + \alpha_2)(t_{j+1} - \gamma_j). \end{aligned}$$

By (2.5), we have

$$w(\gamma_j) \le \frac{w(t_{j+1}^-)}{1-\eta^-}. \tag{2.14}$$

Take now (2.14) in (2.13) with $t \in I_j$ and $\ell = t_{j+1}^-$, to get

$$\begin{aligned} w(t) &\le w(t_{j+1}^-) + \int_t^{t_{j+1}} (\alpha_1 w(s) + \alpha_2 w(\gamma_j)) ds \\ &\le w(t_{j+1}^-) + \int_t^{t_{j+1}} \left(\alpha_1 w(s) + \frac{\alpha_2}{1-\eta^-} w(t_{j+1}^-)\right) ds \\ &\le w(t_{j+1}^-) + \int_t^{t_{j+1}} \left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right) w(s) ds \end{aligned}$$

because w is a non-increasing function. Then, applying the Grönwall 's Lemma, we have:

$$w(t) \le w(t_{j+1}^-) e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right) \cdot (t_{j+1} - t)} \quad \text{for} \quad t \in I_j.$$

By (2.12) and $t = t_j$ we have:

$$w(t_j) \le (1-\varrho_{j+1})^{-1} w(t_{j+1}) e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right)(t_{j+1} - t_j)}. \tag{2.15}$$

From (2.15), recursively we obtain

$$\begin{aligned} v(t) \le w(t) &\le (1-\varrho_{j+1})^{-1} w(t_{j+1}) e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right)(t_{j+1} - t)} \\ &\le (1-\varrho_{j+1})^{-1} (1-\varrho_{j+2})^{-1} w(t_{j+2}) e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right)(t_{j+2} - t)} \\ &\le \dots \\ &\le w(\tau) \left\{ \prod_{\kappa=j+1}^{i(\tau)} (1-\varrho_\kappa)^{-1} \right\} e^{\left(\alpha_1 + \frac{\alpha_2}{1-\eta^-}\right) \cdot (\tau - t)}, \end{aligned} \tag{2.16}$$

by $w(\tau) = v(\tau)$ we obtain (2.4). The proof is complete. The IDEPCAG's Grönwall inequality appears to be new. □

Remark 1. If $\gamma(t) = m\left[\frac{t+l}{m}\right]$ with $l < m$, then the inequality (2.2) with constant coefficients is an IDEPCA's Grönwall Inequality which has been studied in [10]. If $\varrho_\kappa \equiv 0$, $\kappa \in \mathbb{N}$, then we get the inequality (2.2) without impulsive effect in [9]. So our results also extend the conclusion in them.

We can see that the ICNN models with the IDEPCAG system (1.2a)-(1.2b) do not have impulsive condition within the intervals $[t_i, t_{i+1})$, $i \in \mathbb{N}$, which is just like the DEPCAG system. Then applying the identical technique of Grönwall inequality with piecewise constant argument (see [8] and [9]). We have the following Proposition.

Proposition 2.2. *Let the conditions (H_1), (H_2) and (H_3) be fulfilled. Then, given an initial condition $(\tau, \zeta) \in \mathbb{R}^+ \times \mathbb{R}^n$, the ICNN models with the IDEPCAG system (1.2a)-(1.2b) on $[t_{i(\tau)}, t_{i(\tau)+1})$ has a unique solution $x(\cdot) = x(\cdot, \tau, \zeta) = (x_1(\cdot), ..., x_n(\cdot))^T$ such that $x(\tau) = (x_1^0, ..., x_n^0)^T = \zeta$.*

The previous proposition assures the existence and uniqueness of solutions in a local sense. The following theorem provides the existence of a unique solution when the initial moment is an arbitrary positive real number τ.

Theorem 2.1. *Let the conditions (H_1), (H_2) and (H_3) be fulfilled. Then, given an initial condition $(\tau, \zeta) \in \mathbb{R}^+ \times \mathbb{R}^n$, the ICNN models with the IDEPCAG system (1.2a)-(1.2b) has a unique solution $x(\cdot) = x(\cdot, \tau, \zeta) = (x_1(\cdot), ..., x_n(\cdot))^T$ such that $x(\tau) = (x_1^0, ..., x_n^0)^T = \zeta$.*

Proof. Letting $\tau \in \mathbb{R}^+$, then we can see that $\tau \in [t_{i(\tau)}, t_{i(\tau)+1})$. Using Proposition 2.2, the ICNN models with the IDEPCAG system (1.2a)-(1.2b) has a unique solution $x(\cdot) = x(\cdot, \tau, \zeta) = (x_1(\cdot), ..., x_n(\cdot))^T$ on $[t_{i(\tau)}, t_{i(\tau)+1})$ such that $x(\tau) = (x_1^0, ..., x_n^0)^T = \zeta$.
Applying the condition (1.2b), we have

$$x(t_{i(\tau)+1}, \tau, \zeta) = x(t^-_{i(\tau)+1}, \tau, \zeta) + \mathfrak{I}_{i(\tau)+1}\big(x(t^-_{i(\tau)+1}, \tau, \zeta)\big).$$

Now, in the following interval $[t_{i(\tau)+1}, t_{i(\tau)+2})$ the solution of the ICNN models with the IDEPCAG system (1.2a)-(1.2b) satisfies

$$\frac{dy_i(t)}{dt} = -a_i y_i(t) + \sum_{j=1}^{n} b_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(y_j(\gamma(t))) + d_i, \quad i = 1, ..., n,$$

and the ICNN models with the IDEPCAG system (1.2a)-(1.2b) admit a unique solution $y(\cdot) = y(\cdot, t_{i(\tau)+1}, y^0) = (y_1(\cdot), ..., y_n(\cdot))^T$ with the initial condition $y^0 = x(t_{i(\tau)+1}, \tau, \zeta)$. By definition of the solution of the ICNN model $x(t, \tau, \zeta) = y(t, t_{i(\tau)+1}, y^0)$ on $[t_{i(\tau)+1}, t_{i(\tau)+2})$. As $\mathbb{R}^+ = \bigcup_{i=1}^{\infty} [t_i, t_{i+1})$, this completes the proof by the mathematical induction. □

Remark 2. If we consider the deviation argument that is of the constant delay of generalized type, i.e., $\gamma(t) = \gamma_i = t_i$, if $t \in [t_i, t_{i+1})$, $i \in \mathbb{N}$. The ICNN models with the IDEPCAG system (1.2a)-(1.2b) can be reduced to the following IDEGPCD system:

$$\begin{cases} \dfrac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} c_{ij} g_j\big(x_j(\beta(t))\big) + d_i, \quad t \neq t_\kappa, & (2.17a) \\ \Delta x_i(t_\kappa) = \mathfrak{I}_{i\kappa}(x_i(t^-_\kappa)), \quad \kappa \in \mathbb{N}, & (2.17b) \end{cases}$$

with $1 \leq i \leq n$, where $\beta(t) = t_\kappa$ if $t \in I_\kappa = [t_\kappa, t_{\kappa+1})$. Then we have the following observations.

i) The ICNN models with the IDEGPCD system is neither more nor less than system (1.1) in [1]. Since those works not have a global IDEGPCD's Grönwall -type inequality, the results for this system have more stronger conditions, see [20, Example 1 and Remark 4.1].

ii) The **IDEPCAG's Grönwall Inequality** of this paper reduces to the result of the **IDEG-PCD's Grönwall Inequality** in [20, Lemma 2.1].

iii) The condition (H_3) with $\kappa_1 < 1$ reduces to the condition (H_3') which is the same condition **(E)** in [20].

From Theorem 2.1 and Remark 2, we can conclude the following results.

Corollary 2.2. *Let the conditions (H_1), (H_2) and (H_3') be fulfilled. Then, given an initial condition $(\tau,\zeta)\in\mathbb{R}^+\times\mathbb{R}^n$, the ICNN models with the IDEGPCD system (2.17a)-(2.17b) has a unique solution $x(\cdot)=x(\cdot,\tau,\zeta)=(x_1(\cdot),...,x_n(\cdot))^T$ such that $x(\tau)=(x_1^0,...,x_n^0)^T=\zeta$.*

Applying our results to CNN models with the DEPCAG system (1.2a) and CNN models with the DEGPCD system (2.17a) without impulsive effects, we have:

Corollary 2.3. *Let the conditions (H_1) and (H_3) be fulfilled. Then, given an initial condition $(\tau,\zeta)\in\mathbb{R}^+\times\mathbb{R}^n$, there exists a unique solution $x(\cdot)=x(\cdot,\tau,\zeta)=(x_1(\cdot),...,x_n(\cdot))^T$ of the CNN models with the DEPCAG system (1.2a), such that $x(\tau)=(x_1^0,...,x_n^0)^T=\zeta$.*

Corollary 2.4. *Let the conditions (H_1) and (H_3') be fulfilled. Then, given an initial condition $(\tau,\zeta)\in\mathbb{R}^+\times\mathbb{R}^n$, there exists a unique solution $x(\cdot)=x(\cdot,\tau,\zeta)=(x_1(\cdot),...,x_n(\cdot))^T$ of the CNN models with the DEGPCD system (2.17a), such that $x(\tau)=(x_1^0,...,x_n^0)^T=\zeta$.*

Remark 3. Theorem 2.1 reduces to the result of [10, Theorem 5] with $\gamma(t)=m\left[\frac{t+l}{m}\right]$, and Corollary 2.2 reduces to the result of [20, Theorem 2.1] with generalized piecewise constant delay. It is shown that our results are general and they complement the previously known results.

3 Main results

In this section, we shall establish the sufficient criteria for global exponential stability of the equilibrium state of the ICNN models with the IDEPCAG system (1.2a)-(1.2b).

3.1 Existence of a unique equilibrium state

In this subsection, without asking for the conditions of differentiability, monotonicity or boundedness, we present sufficient conditions that are easily verifiable for the existence and uniqueness of the equilibrium of the ICNN models with the IDEPCAG system (1.2a)-(1.2b).

Notice that an equilibrium of the ICNN models with the IDEPCAG system (1.2a)-(1.2b) is the vector $x^*=(x_1^*,x_2^*...,x_n^*)^T\in\mathbb{R}^n$ satisfying

$$a_i x_i^* = \sum_{j=1}^n b_{ij} f_j\left(x_j^*\right) + \sum_{j=1}^n c_{ij} g_j\left(x_j^*\right) + d_i, \quad i\in\{1,2,...,n\}, \tag{3.1}$$

when the impulsive jumps $\mathfrak{J}_{ik}(\cdot)$ as assumed to satisfy the condition $\mathfrak{J}_{i\kappa}(x_i^*)=0$, $i\in\{1,2,...,n\}$, $\kappa\in\mathbb{N}$. One can easily observe that if $\mathfrak{J}_{i\kappa}(x_i^*)\neq 0$, the IDEPCAG system (1.2a)-(1.2b) don't exist the equilibrium, see [22] for more details. If we consider $c_{ij} = 0$, the ICNN models with the IDEPCAG system (1.2a)-(1.2b) reduces to the ICNN models in [22].

Now, we establish the conditions for the existence and uniqueness of the equilibrium state, $x^* = (x_1^*, x_2^*..., x_n^*)^T$, of the ICNN models with the IDEPCAG system (1.2a)-(1.2b).

Theorem 3.1. *Let the conditions* (H_1), (H_2) *and* (H_3) *be fulfilled and the constants* a_i, b_{ij}, c_{ij}, $\mathfrak{L}_i^f$, $\mathfrak{L}_i^g$ *satisfy*

$$a_i > \sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^{n} \mathfrak{L}_j^g |c_{ij}|, \quad i \in \{1, 2, ..., n\}. \tag{3.2}$$

Then the ICNN models with the IDEPCAG system (1.2a)-(1.2b) admit a unique equilibrium state.

Proof. Let a mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$G(v_1, ..., v_n) = \begin{pmatrix} \frac{1}{a_1}\left\{\sum\limits_{j=1}^{n} b_{1j} f_j(v_j) + \sum\limits_{j=1}^{n} c_{1j} g_j(v_j) + d_1\right\} \\ \vdots \\ \frac{1}{a_n}\left\{\sum\limits_{j=1}^{n} b_{nj} f_j(v_j) + \sum\limits_{j=1}^{n} c_{nj} g_j(v_j) + d_n\right\} \end{pmatrix}.$$

We will prove that $G : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction mapping on $\mathbb{R}^n$ with the supremum norm. For $v = (v_1, ..., v_n)^T \in \mathbb{R}^n$, $\mathrm{v} = (\mathrm{v}_1, ..., \mathrm{v}_n)^T \in \mathbb{R}^n$, we have

$$\begin{aligned}
&||G(v_1..., v_n) - G(\mathrm{v}_1..., \mathrm{v}_n)|| \\
&= \max_{1\le i\le n} \left| \frac{1}{a_i}\left[\sum_{j=1}^{n} [b_{ij} f_j(v_j) + c_{ij} g_j(v_j)] + d_i\right] - \frac{1}{a_i}\left[\sum_{j=1}^{n} [b_{ij} f_j(\mathrm{v}_j) + c_{ij} g_j(\mathrm{v}_j)] + d_i\right]\right| \\
&\le \max_{1\le i\le n} \left\{ \frac{1}{a_i}\sum_{j=1}^{n} [|b_{ij}|\,|f_j(v_j) - f_j(\mathrm{v}_j)|] + \frac{1}{a_i}\sum_{j=1}^{n} [|c_{ij}|\,|g_j(v_j) - g_j(\mathrm{v}_j)|] \right\} \\
&\quad \le \max_{1\le i\le n} \left\{ \frac{1}{a_i}\sum_{j=1}^{n} \left[\mathfrak{L}_j^f |b_{ij}|\,|v_j - \mathrm{v}_j|\right] + \frac{1}{a_i}\sum_{j=1}^{n} \left[\mathfrak{L}_j^g |c_{ij}|\,|v_j - \mathrm{v}_j|\right] \right\} \\
&\quad \le \max_{1\le i\le n} \left\{ \frac{1}{a_i}\left[\sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^{n} \mathfrak{L}_j^g |c_{ij}|\right]\right\} \cdot \max_{1\le j\le n} |v_j - \mathrm{v}_j| \\
&\quad \le \rho_1 ||v - \mathrm{v}||,
\end{aligned}$$

where the number

$$\rho_1 = \max_{1\le i\le n} \left[\frac{\sum\limits_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| + \sum\limits_{j=1}^{n} \mathfrak{L}_j^g |c_{ij}|}{a_i} \right]$$

satisfies $0 < \rho_1 < 1$ by virtue of the condition (3.2). Then we have

$$||G(v) - G(\mathrm{v})|| \le \rho_1 ||v - \mathrm{v}||, \quad v, \mathrm{v} \in \mathbb{R}^n,$$

which conclude that G is a contraction mapping on $\mathbb{R}^n$. By the Banach fixed-point theorem, the system (3.1) admits a unique solution x^* such that $G(x^*) = x^*$. Then the ICNN models with the IDEPCAG system (1.2a)-(1.2b) has a unique equilibrium state. $\square$

Theorem 3.2. *Suppose that conditions* (H_1), (H_2) *and* (H_3) *hold, the constants* a_i, b_{ij}, c_{ij}, $\mathfrak{L}_i^f$, $\mathfrak{L}_i^g$ *satisfy*

$$a_j > \mathfrak{L}_j^f \sum_{i=1}^{n} |b_{ij}| + \mathfrak{L}_j^g \sum_{i=1}^{n} |c_{ij}|, \quad j \in \{1, 2, ..., n\}. \tag{3.3}$$

Then the ICNN models with the IDEPCAG system (1.2a)-(1.2b) admit a unique equilibrium state.

Proof. Letting $a_i x_i^* = y_i^*$, $i \in \{1, 2, ..., n\}$ in the system (3.1), we give:

$$y_i^* = \sum_{j=1}^{n} b_{ij} f_j\left(\frac{y_j^*}{a_j}\right) + \sum_{j=1}^{n} c_{1j} g_j\left(\frac{y_j^*}{a_j}\right) + d_i, \qquad i \in \{1, 2, ..., n\}. \tag{3.4}$$

It is enough to demonstrate the existence of a unique solution of the system (3.4).
Let a mapping $\mathcal{G} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\mathcal{G}(v_1, ..., v_n) = \begin{pmatrix} \sum_{j=1}^{n} b_{1j} f_j\left(\frac{v_j}{a_j}\right) + \sum_{j=1}^{n} c_{1j} g_j\left(\frac{v_j}{a_j}\right) + d_1 \\ \vdots \\ \sum_{j=1}^{n} b_{nj} f_j\left(\frac{v_j}{a_j}\right) + \sum_{j=1}^{n} c_{nj} g_j\left(\frac{v_j}{a_j}\right) + d_n \end{pmatrix}.$$

Then, for any $v = (v_1, ..., v_n)^T \in \mathbb{R}^n$, $\mathrm{v} = (\mathrm{v}_1, ..., \mathrm{v}_n)^T \in \mathbb{R}^n$, we have

$$\begin{aligned} \|\mathcal{G}(v) - \mathcal{G}(\mathrm{v})\|_1 &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} b_{ij}\left(f_j\left(\frac{v_j}{a_j}\right) - f_j\left(\frac{\mathrm{v}_j}{a_j}\right)\right) + \sum_{j=1}^{n} c_{ij}\left(g_j\left(\frac{v_j}{a_j}\right) - g_j\left(\frac{\mathrm{v}_j}{a_j}\right)\right) \right| \\ &\le \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \left(\frac{\mathfrak{L}_j^f}{a_j} |b_{ij}| \, |v_j - \mathrm{v}_j| \right) + \sum_{j=1}^{n} \left(\frac{\mathfrak{L}_j^g}{a_j} |c_{ij}| \, |v_j - \mathrm{v}_j| \right) \right\} \\ &\le \left[\max_{1 \le j \le n} \left(\frac{\mathfrak{L}_j^f}{a_j} \sum_{i=1}^{n} |b_{ij}| + \frac{\mathfrak{L}_j^g}{a_j} \sum_{i=1}^{n} |c_{ij}| \right) \right] \sum_{j=1}^{n} |v_j - \mathrm{v}_j| \\ &:= \rho_2 \|v - \mathrm{v}\|_1. \end{aligned}$$

By the assumption $\rho_2 < 1$, this implies that the mapping $\mathcal{G} : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction mapping. By Banach fixed-point theorem $\mathcal{G}$ has exactly one fixed point x^* in $\mathbb{R}^n$ such that $\mathcal{G}(x^*) = x^*$. Thus, the ICNN models with the IDEPCAG system (1.2a)-(1.2b) has exactly one equilibrium state. The proof is now complete. $\square$

3.2 Global exponential stability of equilibrium state

In this subsection, we want to discuss the stability of the ICNN models with the IDEPCAG system (1.2a)-(1.2b).

Let the following change of variables $z_i(t) = x_i(t) - x_i^*$,

$$\tilde{f}_j(z_j(t)) = f_j(z_j(t) + x_j^*) - f(x_j^*),$$

$$\tilde{g}_j\left(z_j\left(\gamma(t)\right)\right) = g_j\left(z_j(\gamma(t)) + x_j^*\right) - g_j(x_j^*),$$

$$\tilde{\mathfrak{J}}_{i\kappa}(z_i(t_\kappa^-)) = \mathfrak{J}_{i\kappa}(z_i(t_\kappa^-) + x_i^*) - \mathfrak{J}_{i\kappa}(x_i^*) = \mathfrak{J}_{i\kappa}(z_i(t_\kappa^-) + x_i^*),$$

for $i = 1, 2, ..., n$, $\kappa \in \mathbb{N}$, so that the ICNN models with the IDEPCAG system (1.2a)-(1.2b) can be rewritten as

$$\begin{cases} \dfrac{dz_i(t)}{dt} = -a_i z_i(t) + \displaystyle\sum_{j=1}^{n} b_{ij}\tilde{f}_j(z_j(t)) + \sum_{j=1}^{n} c_{ij}\tilde{g}_j(z_j\left(\gamma(t)\right)), \quad t \neq t_\kappa, & (3.5a) \\ \Delta z_i(t_\kappa) = \tilde{\mathfrak{J}}_{ik}(z_i(t_\kappa^-)), \quad i = 1, 2, ..., n, \quad \kappa \in \mathbb{N}. & (3.5b) \end{cases}$$

We can see that $\tilde{f}_i(\cdot)$ and $\tilde{g}_i(\cdot)$, with $\tilde{f}_i(0) = \tilde{g}_i(0) = 0$, satisfy the Lipschitz condition:

$$|\tilde{f}_i(u) - \tilde{f}_i(v)| \leq \mathfrak{L}_i^f |u - v|, \qquad |\tilde{g}_i(u) - \tilde{g}_i(v)| \leq \mathfrak{L}_i^g |u - v|,$$

and $\tilde{\mathfrak{J}}_{i\kappa}$ satisfies

$$\tilde{\mathfrak{J}}_{i\kappa}(0) = 0, \quad |\tilde{\mathfrak{J}}_{i\kappa}(u) - \tilde{\mathfrak{J}}_{i\kappa}(v)| \leq \mathfrak{L}_{i\kappa}^J |u - v|,$$

for v, $v \in \mathbb{R}$, $i \in \{1, ..., n\}$, $\kappa \in \mathbb{N}$.

The stability of the trivial solution for the IDEPCAG system (3.5a)-(3.5b) is then studied in the same way as that of the equilibrium state $x^* = (x_1^*, x_2^* ..., x_n^*)^T$ of the ICNN models with the IDEPCAG system (1.2a)-(1.2b).

The following notations are required in the section:

$$a_* = \min_{1 \leq i \leq n} a_i, \qquad \mathfrak{L}_\kappa^J = \max_{1 \leq i \leq n} \mathfrak{L}_{i\kappa}^J, \qquad \vartheta = \sup_{\kappa \in \mathbb{N}} \vartheta_\kappa,$$

$$\vartheta^- = \sup_{\kappa \in \mathbb{N}} (t_{\kappa+1} - \gamma_\kappa), \quad \vartheta^+ = \sup_{\kappa \in \mathbb{N}} (\gamma_\kappa - t_\kappa), \quad \vartheta_\kappa = t_{\kappa+1} - t_\kappa,$$

$$\mu_i = \sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^{n} \mathfrak{L}_j^g |c_{ij}| \frac{e^{a_* \cdot \vartheta^-}}{1 - \hat{\nu}}$$

and

$$\max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^{n} \mathfrak{L}_j^g \left|c_{ij}\right| e^{a_* \cdot \vartheta^-} \right) \cdot \vartheta^+ \leq \hat{\nu} < 1.$$

Now, we will introduce the definition and lemma, so as to be used within proof of the stability of the trivial solution for the ICNN models with IDEPCAG system.

Definition 1. The equilibrium state $x^* = (x_1^*, x_2^* ..., x_n^*)^T$ of the ICNN models with the IDEPCAG system (1.2a)-(1.2b) is globally exponentially stable, if there exist constants $\alpha, \lambda > 0$ such that

$$\max_{1 \leq i \leq n} |x_i(t) - x_i^*| \leq \alpha \cdot \max_{1 \leq i \leq n} |x_i(\tau) - x_i^*| e^{-\lambda \cdot (t - \tau)}, \quad \tau \leq t.$$

Lemma 3.1. *Let the conditions (H_1), (H_2) and (H_3) be fulfilled and ψ, φ be the solutions of the ICNN models with the IDEPCAG system (1.2a)-(1.2b). Then the following inequality holds*

$$\max_{1\le i\le n}|\psi_i(t)-\varphi_i(t)|\le \max_{1\le i\le n}|\psi_i(\tau)-\varphi_i(\tau)| \times\exp\Bigg\{-\Bigg(a_*-\max_{1\le i\le n}\left(\sum_{j=1}^n\mathfrak{L}_j^f|b_{ij}|+\sum_{j=1}^n\mathfrak{L}_j^g|c_{ij}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}}\right) - \sup_{\kappa\in\mathbb{N}}\frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa}\Bigg)\cdot(t-\tau)\Bigg\},\quad t\ge\tau. \tag{3.6}$$

Proof. Suppose that $\psi(t)=(\psi_1,..,\psi_n)^T$ and $\varphi(t)=(\varphi_1,..,\varphi_n)^T$ are arbitrary solutions of the ICNN models with the IDEPCAG system (1.2a)-(1.2b). Letting $z(t)=\psi(t)-\varphi(t)$, by the IDEPCAG system (1.2a)-(1.2b), we have

$$\begin{cases} z_i'(t)=-a_iz_i(t)+\displaystyle\sum_{j=1}^n b_{ij}\cdot\Big(f_j(z_j(t)+\varphi_j(t))-f_j(\varphi_j(t))\Big)\\ \qquad\qquad+\displaystyle\sum_{j=1}^n c_{ij}\cdot\Big(g_j(z_j(\gamma(t))+\varphi_j(\gamma(t)))-g_j(\varphi_j(\gamma(t)))\Big),\quad t\ne t_\kappa,\\ \Delta z_i(t_\kappa)=\mathfrak{J}_{i\kappa}(z_i(t_\kappa^-)+\varphi_i(t_\kappa^-))-\mathfrak{J}_{i\kappa}(\varphi_i(t_\kappa^-)),\quad i=1,2,...,n,\quad \kappa\in\mathbb{N}.\end{cases} \tag{3.7}$$

Using Proposition 2.1, we obtain the following integral equations

$$z_i(t)=e^{-a_i(t-\tau)}z_i(\tau)+\int_\tau^t e^{-a_i(t-s)}\mathfrak{R}_i(s,z(s),z(\gamma(s)))ds + \sum_{k=i(\tau)+1}^{i(t)} e^{-a_i(t-t_\kappa)}\mathfrak{J}_{i\kappa}(z_i(t_\kappa^-)),\qquad i(t)>i(\tau), \tag{3.8}$$

where

$$\mathfrak{R}_i(s,z(s),z(\gamma(s))):=\sum_{j=1}^n b_{ij}\cdot\Big(f_j(z_j(s)+\varphi_j(s))-f_j(\varphi_j(s))\Big) + \sum_{j=1}^n c_{ij}\cdot\Big(g_j(z_j(\gamma(s))+\varphi_j(\gamma(s)))-g_j(\varphi_j(\gamma(s)))\Big),$$

and

$$\mathfrak{J}_{i\kappa}(z_i(t_\kappa^-)):=\mathfrak{J}_{i\kappa}(z_i(t_\kappa^-)+\varphi_i(t_\kappa^-))-\mathfrak{J}_{i\kappa}(\varphi_i(t_\kappa^-)).$$

By the conditions (H_1) and (H_2), we have

$$|\mathfrak{R}_i(s,z(s),z(\gamma(s)))|\le\left(\sum_{j=1}^n\mathfrak{L}_j^f|b_{ij}||z_j(s)|+\sum_{j=1}^n\mathfrak{L}_j^g|c_{ij}|\,|z_j(\gamma(s))|\right),$$

and

$$\left|\mathfrak{J}_{i\kappa}(z_i(t_\kappa^-))\right|\le\mathfrak{L}_{i\kappa}^J\left|z_i(t_\kappa^-)\right|.$$

Using (3.8), we can obtain that $u_i(t) = e^{a_*\cdot(t-\tau)}|z_i(t)|$ satisfies

$$|u_i(t)| \le |\psi_i(\tau) - \varphi_i(\tau)| + \int_\tau^t \left(\sum_{j=1}^n \mathfrak{L}_j^f |b_{ij}||u_j(s)| + \sum_{j=1}^n \mathfrak{L}_j^g |c_{ij}| \left|u_j(\gamma(s))\right| e^{a_*\cdot(s-\gamma(s))} \right) ds + \sum_{\kappa=i(\tau)+1}^{i(t)} \mathfrak{L}_{i\kappa}^J \left|u_i(t_\kappa^-)\right|,$$

for $t \in [\tau, \infty)$.
Therefore

$$\begin{aligned}\max_{1\le i\le n} |u_i(t)| \le \max_{1\le i\le n} |\psi_i(\tau) - \varphi_i(\tau)| &+ \int_\tau^t \max_{1\le i\le n} \left\{ \left(\sum_{j=1}^n \mathfrak{L}_j^f |b_{ij}| \right) \cdot \max_{1\le j\le n} |u_j(s)| \right. \\ &\left. + \max_{1\le i\le n} \left(\sum_{j=1}^n \mathfrak{L}_j^g |c_{ij}| e^{a_*\cdot\vartheta^-} \right) \cdot \max_{1\le j\le n} \left|u_j(\gamma(s))\right| \right\} ds \\ &+ \sum_{\kappa=i(\tau)+1}^{i(t)} \max_{1\le i\le n} \mathfrak{L}_{i\kappa}^J \cdot \max_{1\le i\le n} \left|u_i(t_\kappa^-)\right|.\end{aligned}$$

Applying the IDEPCAG's Grönwall Inequality (Lemma 2.1), we have

$$\begin{aligned}\max_{1\le i\le n} |u_i(t)| \le& \max_{1\le i\le n} |\psi_i(\tau) - \varphi_i(\tau)| \\ &\times \left\{ \prod_{\kappa=i(\tau)+1}^{i(t)} \left(1 + \mathfrak{L}_\kappa^J\right) \right\} e^{\max_{1\le i\le n} \left\{ \sum_{j=1}^n \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^n \mathfrak{L}_j^g |c_{ij}| \frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}} \right\}\cdot(t-\tau)}.\end{aligned}$$

Then, we have

$$\begin{aligned}\max_{1\le i\le n} |\psi_i(t) - \varphi_i(t)| &\le \max_{1\le i\le n} |\psi_i(\tau) - \varphi_i(\tau)| \left\{ \prod_{\kappa=i(\tau)+1}^{i(t)} \left(1 + \mathfrak{L}_\kappa^J\right) \right\} e^{-\left(a_* - \max_{1\le i\le n} \mu_i\right)\cdot(t-\tau)} \\ &\le \max_{1\le i\le n} |\psi_i(\tau) - \varphi_i(\tau)| e^{-\left(a_* - \max_{1\le i\le n} \mu_i\right)\cdot(t-\tau) + \ln\left(\prod_{\kappa=i(\tau)+1}^{i(t)} \left(1+\mathfrak{L}_\kappa^J\right)\right)} \\ &\le \max_{1\le i\le n} |\psi_i(\tau) - \varphi_i(\tau)| e^{-\left(a_* - \max_{1\le i\le n} \mu_i\right)\cdot(t-\tau) + \sum_{\kappa=i(\tau)+1}^{i(t)} \frac{\ln\left(1+\mathfrak{L}_\kappa^J\right)}{\vartheta_\kappa}\cdot\vartheta_\kappa},\end{aligned}$$

or

$$\max_{1\le i\le n} |\psi_i(t) - \varphi_i(t)| \le \max_{1\le i\le n} |\psi_i(\tau) - \varphi_i(\tau)| e^{-\left(a_* - \max_{1\le i\le n} \mu_i - \sup_{\kappa\in\mathbb{N}} \frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa}\right)\cdot(t-\tau)},$$

and the statement (3.6) follows. □

Theorem 3.3. *Suppose that the conditions* (H_1)*,* (H_2)*,* (H_3)*,* (3.2) *and*

$$a_* - \max_{1\le i\le n} \left(\sum_{j=1}^n \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^n \mathfrak{L}_j^g |c_{ij}| \frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}} \right) - \sup_{\kappa\in\mathbb{N}} \frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa} > 0, \tag{3.9}$$

hold, then the unique equilibrium state of the ICNN models with the IDEPCAG system (1.2a)-(1.2b) is globally exponentially stable.

Proof. According to the result of Theorem 3.1, the ICNN models with the IDEPCAG system (1.2a)-(1.2b) has a unique equilibrium state x^*. Now consider that $x(t,\zeta)$ is a solution of (1.2a)-(1.2b) with the initial condition ζ and let $\wp(t) = x(t,\zeta) - x^*$. By Lemma 3.1, we have

$$\max_{1\leq i\leq n} |\wp_i(t)| \leq \max_{1\leq i\leq n} |\wp_i(\tau)| \exp\left\{ -\left(a_* - \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} \mathfrak{L}_j^f |b_{ij}| + \sum_{j=1}^{n} \mathfrak{L}_j^g |c_{ij}| \frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}} \right) - \sup_{\kappa\in\mathbb{N}} \frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa} \right) \cdot (t-\tau) \right\}.$$

By the condition (3.9), we can conclude that $\max_{1\leq i\leq n} |\wp_i(t)| \to 0$ as $t \to \infty$. Then the trivial solution of the ICNN models with the IDEPCAG system (3.5) is globally exponentially stable. So, the equilibrium state of the ICNN models with the IDEPCAG system (1.2a)-(1.2b) is globally exponentially stable. □

By the same way to proof Theorem 3.3, we have:

Theorem 3.4. *Suppose that conditions (H_1), (H_2), (H_3),* (3.3) *and* (3.9) *hold, then the unique equilibrium state of the ICNN models with the IDEPCAG system (1.2a)-(1.2b) is globally exponentially stable.*

Remark 4. Theorem 3.3 reduces to the stability result of [10, Theorem 9] with the classic piecewise alternately advanced and retarded argument, [20, Theorem 3.2] and [1, Theorem 3.1] with generalized piecewise constant delay, we are able to see that the results obtained in this article extend and improve the results given in [10] .

Remark 5. The existence criterion (3.1)-(3.2) and the stability criterion (3.9) can be easily solved by using some existing software, for example, the MATLAB.

Remark 6. Different from the methods used in [1], the relationship that $\|y(\beta(t))\| \leq \bar{B}\, \|y(t)\|$, where $\bar{B} = \left\{1 - \bar{\theta}\left[\alpha_2 + \alpha_3(1+\bar{\theta}\alpha_2)e^{\bar{\theta}\alpha_3}\right]\right\}^{-1} > 0$ in [1, Lemma 3.1] is not required in the present paper. Because this relationship is not necessary for the proposed technique of IDEPCAG's Grönwall inequality here.

Remark 7. The stability criteria in [1] are depended on the upper and lower bounds $\bar{\theta}$ and $\underline{\theta}$. It requires that $\gamma - \alpha_1 - \bar{B}\alpha_2 - \frac{\ln(1+l)}{\underline{\theta}} > 0$ in [1, Theorem 3.1]. Thus, those results cannot be used to obtain the stability of neural networks for any $\bar{\theta}\left[\alpha_2 + \alpha_3(1+\bar{\theta}\alpha_2)e^{\bar{\theta}\alpha_3}\right] > 1$. Then, we can choose proper parameter which the stability criteria in [1] are not satisfied. Hence, our results can be applied more convenient than the results in [1].

Without impulsive effects, we have the following corollaries of Lemma 3.1, Theorem 3.3 and Theorem 3.4.

Corollary 3.5. *Let the conditions (H_1) and (H_3) be fulfilled and ψ, φ be the solutions of the CNN models with the DEPCAG system (1.2a). Then the following inequality holds*

$$\max_{1\le i\le n}|\psi_i(t)-\varphi_i(t)|\le\max_{1\le i\le n}|\psi_i(\tau)-\varphi_i(\tau)| \times\exp\left\{-\left[a_*-\max_{1\le i\le n}\left(\sum_{j=1}^n\mathfrak{L}_j^f|b_{ij}|+\sum_{j=1}^n\mathfrak{L}_j^g|c_{ij}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}}\right)\right]\cdot(t-\tau)\right\}. \tag{3.10}$$

Corollary 3.6. *If the conditions (H_1), (H_3), (3.2) (or (3.3)) and*

$$a_*-\max_{1\le i\le n}\left(\sum_{j=1}^n\mathfrak{L}_j^f|b_{ij}|+\sum_{j=1}^n\mathfrak{L}_j^g|c_{ij}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}}\right)>0 \tag{3.11}$$

hold. Then the unique equilibrium state of the CNN models with the DEPCAG system (1.2a) is globally exponentially stable.

If we consider the deviation argument that is of the constant delay of generalized type, i.e., $\gamma(t)=\gamma_i=t_i$, if $t\in[t_i,t_{i+1})$, $i\in\mathbb{N}$. We have the following corollaries.

Corollary 3.7. *Let the conditions (H_1), (H_2), (H_3'), (3.2) (or (3.3)) and*

$$a_*-\max_{i\in[1,.,n]}\left(\sum_{j=1}^n\mathcal{L}_j^f|b_{ij}|+\sum_{j=1}^n\mathcal{L}_j^g|c_{ij}|e^{a_*\cdot\vartheta}\right)-\sup_{\kappa\in\mathbb{N}}\frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa}>0 \tag{3.12}$$

be fulfilled. Then the unique equilibrium state of the ICNN models with the IDEGPCD system (2.17a)-(2.17b) is globally exponentially stable.

Remark 8. Corollary 3.7 reduces to the stability result of [20, Theorem 3.2]. Moreover this corollary generalizes corresponding result obtained by [1, Theorem 3.1] under complicated and stronger conditions. See [20, Example 1].

Without impulsive effects, we have the following result.

Corollary 3.8. *Let the conditions (H_1), (H_3') and (3.2) (or (3.3)) be fulfilled. Then the unique equilibrium state of the CNN models with the DEGPCD system (2.17a) is globally exponentially stable.*

Remark 9. Recently, the existence and stability of the unique equilibrium state of the ICNN models with piecewise constant argument have been studied by few authors. Moreover, we do not find related works concerning the unique equilibrium state for impulsive cellular neural network models with piecewise alternately advanced and retarded argument of generalized type.

From this point, the model considered in this paper is more general than the existing the ICNN models and the CNN models with piecewise constant argument such as those in Refs. [1, 10, 20].

4 Illustrative examples with simulations

In this section we should present two illustrative examples with simulations for our proposed results.

Example 1. Consider the following ICNN models with the IDEPCAG system:

$$\begin{cases} x_1' = -a_1x_1 + b_{11}f_1(x_1) + b_{12}f_2(x_2) + c_{12}g_2(x_2(\gamma(\cdot))) + c_{13}g_3(x_3(\gamma(\cdot))) + d_1 \\ x_2' = -a_2x_2 + b_{21}f_1(x_1) + b_{23}f_3(x_3) + c_{21}g_1(x_1(\gamma(\cdot))) + c_{22}g_2(x_2(\gamma(\cdot))) + d_2 \\ x_3' = -a_3x_2 + b_{31}f_1(x_1) + b_{32}f_2(x_2) + c_{31}g_1(x_1(\gamma(\cdot))) + c_{33}g_3(x_3(\gamma(\cdot))) + d_3, \end{cases} \tag{4.1a}$$

$$\begin{cases} \Delta x_1(t_\kappa) = \mathfrak{J}_{1\kappa}(x_1(t_\kappa^-)) \\ \Delta x_2(t_\kappa) = \mathfrak{J}_{2\kappa}(x_2(t_\kappa^-)) \\ \Delta x_3(t_\kappa) = \mathfrak{J}_{3\kappa}(x_3(t_\kappa^-)), \end{cases} \tag{4.1b}$$

where

$$\begin{aligned} &a_1 = 1.2, && a_2 = 0.7, && a_3 = 0.9, && b_{11} = 0.25, && b_{12} = 0.45, && b_{21} = 0.15, \\ &b_{23} = 0.35, && b_{31} = 0.35, && b_{32} = 0.25, && c_{12} = 0.15, && c_{13} = 0.35, && c_{21} = 0.25, \\ &c_{22} = 0.35, && c_{31} = 0.45, && c_{33} = 0.25, && d_1 = 0.2, && d_2 = 0.1, && d_3 = 0.2, \end{aligned}$$

and $\gamma(t) = \frac{3\pi}{8}\kappa - \frac{\pi}{4}$, if $\frac{3\pi}{8}(\kappa - 1) \le t < \frac{3\pi}{8}\kappa$, $\kappa \in \mathbb{N}$.
The output functions are

$$\begin{aligned} f_1(x_1(t)) &= \tanh\left(\tfrac{x_1(t)}{6}\right), & f_2(x_2(t)) &= \tanh\left(\tfrac{x_2(t)}{4}\right), \\ f_3(x_3(t)) &= \tanh\left(\tfrac{x_3(t)}{8}\right), & g_1(x_1(\gamma(t))) &= \tanh\left(\tfrac{x_1(\gamma(t))}{4}\right), \\ g_2(x_2(\gamma(t))) &= \tanh\left(\tfrac{x_2(\gamma(t))}{8}\right), & g_3(x_3(\gamma(t))) &= \tanh\left(\tfrac{x_3(\gamma(t))}{3}\right). \end{aligned}$$

The impulsive functions are

$$\begin{aligned} \mathfrak{J}_{1\kappa}(x_1(t_\kappa^-)) = \mathfrak{J}_{1\kappa}\left(x_1\left(\tfrac{3\pi}{8}(\kappa-1)^-\right)\right) &= \tfrac{x_1\left(\frac{3\pi}{8}(\kappa-1)^-\right) - x_1^*}{5}, \\ \mathfrak{J}_{2\kappa}(x_2(t_\kappa^-)) = \mathfrak{J}_{2\kappa}\left(x_2\left(\tfrac{3\pi}{8}(\kappa-1)^-\right)\right) &= \tfrac{x_2\left(\frac{3\pi}{8}(\kappa-1)^-\right) - x_2^*}{8}, \\ \mathfrak{J}_{3\kappa}(x_3(t_\kappa^-)) = \mathfrak{J}_{3\kappa}\left(x_3\left(\tfrac{3\pi}{8}(\kappa-1)^-\right)\right) &= \tfrac{x_3\left(\frac{3\pi}{8}(\kappa-1)^-\right) - x_3^*}{6}, \end{aligned}$$

where $x_1^* = 0.22081$, $x_2^* = 0.20723$, $x_3^* = 0.30335$.

We can easily verify that the point $x^* = (x_1^*, x_2^*, x_3^*)^T$ satisfies

$$\begin{cases} a_1x_1^* = \sum_{j=1}^{2} b_{1j}f_j(x_j^*) + \sum_{j=1}^{2} c_{1j}g_j\left(x_j^*\right) + d_1, \\ a_2x_2^* = \sum_{j=1}^{2} b_{2j}f_j(x_j^*) + \sum_{j=1}^{2} c_{2j}g_j\left(x_j^*\right) + d_2, \\ a_3x_3^* = \sum_{j=1}^{2} b_{3j}f_j(x_j^*) + \sum_{j=1}^{2} c_{3j}g_j\left(x_j^*\right) + d_3, \end{cases}$$

approximately. And it is clear that $\mathfrak{J}_{i\kappa}\left(x_i^*\right) = 0$ for $i = 1, 2, 3$. By simple calculation, we can see that $a_* = 0.7$, $\vartheta^+ = \vartheta_\kappa^+ = \frac{\pi}{8}$, $\vartheta^- = \vartheta_\kappa^- = \frac{\pi}{4}$, $\vartheta = \vartheta_\kappa = \frac{3\pi}{8}$, $\mathfrak{L}_1^f = \mathfrak{L}_{3\kappa}^J = \frac{1}{6}$, $\mathfrak{L}_2^f = \mathfrak{L}_1^g = \frac{1}{4}$,

$\mathfrak{L}_3^f = \mathfrak{L}_2^g = \mathfrak{L}_{2\kappa}^J = \frac{1}{8}$, $\mathfrak{L}_3^g = \frac{1}{3}$, $\mathfrak{L}_{1\kappa}^J = \frac{1}{5}$, $\mathfrak{L}_\kappa^J = \frac{1}{5}$ and $\sup_{\kappa\in\mathbb{N}} \ln(1+\mathfrak{L}_\kappa^J)/\vartheta_\kappa \approx 0.15476$.

Then

$$\max_{1\le i\le 3}\left\{\left(\frac{e^{a_i\cdot\vartheta^-}-1}{a_i}\right)\left[\sum_{j=1}^{3}\mathfrak{L}_j^f|b_{ij}|+\sum_{j=1}^{3}\mathfrak{L}_j^g|c_{ij}|\right]\right\}\approx 0.377986<1,$$

$$\max_{1\le i\le 3}\left\{\left(\frac{1-e^{-a_i\cdot\vartheta^+}}{a_i}\right)\left[\sum_{j=1}^{3}\mathfrak{L}_j^f|b_{ij}|+\sum_{j=1}^{3}\mathfrak{L}_j^g|c_{ij}|\right]\right\}\approx 0.104754<1,$$

and

$$a_1 = 1.2 > 0.289583 \approx \sum_{j=1}^{3}\mathfrak{L}_j^f\,|b_{1j}|+\sum_{j=1}^{3}\mathfrak{L}_j^g\,|c_{1j}|,$$

$$a_2 = 0.7 > 0.175 = \sum_{j=1}^{3}\mathfrak{L}_j^f\,|b_{2j}|+\sum_{j=1}^{3}\mathfrak{L}_j^g\,|c_{2j}|,$$

$$a_3 = 0.9 > 0.316667 \approx \sum_{j=1}^{3}\mathfrak{L}_j^f\,|b_{3j}|+\sum_{j=1}^{3}\mathfrak{L}_j^g\,|c_{3j}|.$$

By Theorem 3.1, we can conclude that the ICNN models with the IDEPCAG system (4.1a)-(4.1b) has a unique equilibrium state x^*.

On the other hand, we have

$$\hat{\nu} = \max_{1\le i\le 3}\left(\sum_{j=1}^{3}\mathfrak{L}_j^f|b_{ij}|+\sum_{j=1}^{3}\mathfrak{L}_j^g\,|c_{ij}|\,e^{a_*\cdot\vartheta^-}\right)\cdot\vartheta^+\approx 0.1807149<1,$$

$$\mu_1 = \sum_{j=1}^{3}\mathfrak{L}_j^f|b_{1j}|+\sum_{j=1}^{3}\mathfrak{L}_j^g|c_{1j}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}}\approx 0.431113<0.5452406\approx a_*-\sup_{\kappa\in\mathbb{N}}\frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa},$$

$$\mu_2 = \sum_{j=1}^{3}\mathfrak{L}_j^f|b_{2j}|+\sum_{j=1}^{3}\mathfrak{L}_j^g|c_{2j}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}}\approx 0.273166<0.5452406\approx a_*-\sup_{\kappa\in\mathbb{N}}\frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa}$$

and

$$\mu_3 = \sum_{j=1}^{3}\mathfrak{L}_j^f|b_{3j}|+\sum_{j=1}^{3}\mathfrak{L}_j^g|c_{3j}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}}\approx 0.53504<0.5452406\approx a_*-\sup_{\kappa\in\mathbb{N}}\frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa}.$$

Then $a_* - \max_{1\le i\le 3}\mu_i - \sup_{\kappa\in\mathbb{N}}\frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa}\approx 0.0102002>0$. One can see that all conditions (H_1), (H_2), (H_3), (3.2) and (3.9) in Theorem 3.3 are satisfied. So, by Theorem 3.3, the unique equilibrium state of the ICNN models with the IDEPCAG system (4.1a)-(4.1b) is globally exponentially stable. The simulation of the unique equilibrium state x^* of the ICNN models (4.1a)-(4.1b) with and without impulses, are shown in Figures 4.1–4.4.

For the simulation, the initial states $(x_1(0), x_2(0), x_3(0))^T$ are given by the random function. Figures 4.1–4.4 show that the conditions obtained in this article are valid for the ICNN models (4.1a)-(4.1b) with and without impulses.

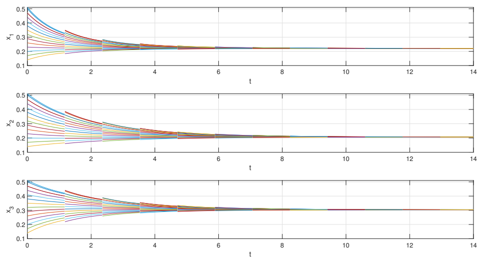

Figure 4.1: Convergence of the unique globally exponentially stable equilibrium state for the ICNN models with the IDEPCAG system (4.1a)-(4.1b).

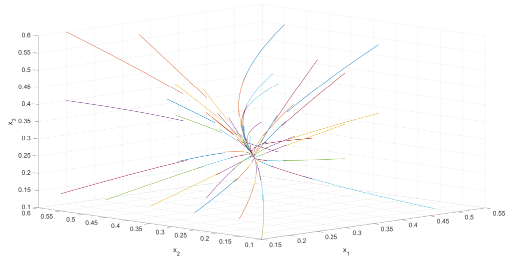

Figure 4.2: Phase portrait of state variables for the ICNN models with the IDEPCAG system (4.1a)-(4.1b).

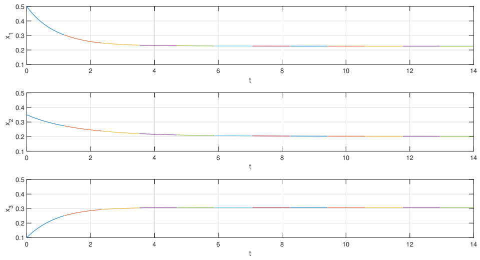

Figure 4.3: Convergence of the unique globally exponentially stable equilibrium state for the CNN models with the DEPCAG system (4.1a) without impulsive effects.

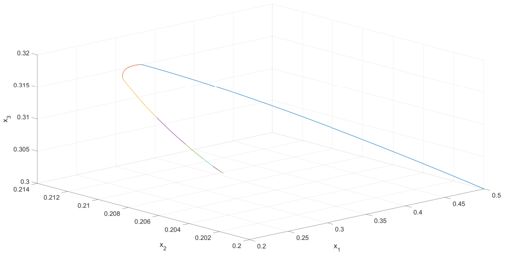

Figure 4.4: Convergence of the unique globally exponentially stable equilibrium state for the CNN models with the DEPCAG system (4.1a) with the initial conditions $(0.5; 0.2; 0.3)^T$.

Note that the simulation illustrates that all trajectories uniformly converge to the unique exponentially stable equilibrium point where $x^* = (0.22081; 0.20723; 0.30335)^T$.

Example 2. Consider the following ICNN models with the IDEPCAG system:

$$\begin{cases} x'_1 = -a_1x_1 + b_{11}f_1(x_1) + b_{12}f_2(x_2) + c_{11}g_1(x_1(\gamma(\cdot))) + c_{12}g_2(x_2(\gamma(\cdot))) + d_1 \\ x'_2 = -a_2x_2 + b_{21}f_1(x_1) + b_{22}f_2(x_2) + c_{21}g_1(x_1(\gamma(\cdot))) + c_{22}g_2(x_2(\gamma(\cdot))) + d_2, \end{cases} \tag{4.2a}$$

$$\begin{cases} \Delta x_1(t_\kappa) = \mathfrak{J}_{1\kappa}(x_1(t_\kappa^-)) \\ \Delta x_2(t_\kappa) = \mathfrak{J}_{2\kappa}(x_2(t_\kappa^-)), \end{cases} \tag{4.2b}$$

where $a_1 = 0.9$, $a_2 = 0.6$, $b_{11} = 0.16$, $b_{12} = 0.25$, $b_{21} = 0.25$, $b_{22} = 0.18$, $c_{11} = 0.23$, $c_{12} = 0.25$, $c_{21} = 0.15$, $c_{22} = 0.27$, $d_1 = 3$, $d_2 = 2$ and $\gamma(t) = 3(\kappa - 1) + 1$, if $3(\kappa - 1) \le t < 3\kappa$, $\kappa \in \mathbb{N}$.
The output functions are

$$f_1(x_1(t)) = \tanh\left(\frac{x_1(t)}{6}\right), \qquad f_2(x_2(t)) = \tanh\left(\frac{x_2(t)}{5}\right),$$

$$g_1(x_1(\gamma(t))) = \frac{|x_1(\gamma(t))+1| - |x_1(\gamma(t))-1|}{8}, \qquad g_2(x_2(\gamma(t))) = \frac{|x_2(\gamma(t))+1| - |x_2(\gamma(t))-1|}{16}.$$

The impulsive functions are

$$\mathfrak{J}_{1\kappa}(x_1(t_\kappa^-)) = \mathfrak{J}_{1\kappa}\left(x_1(3(\kappa-1)^-)\right) = \tfrac{x_1(3(\kappa-1)^-)-x_1^*}{8},$$
$$\mathfrak{J}_{2\kappa}(x_2(t_\kappa^-)) = \mathfrak{J}_{2\kappa}\left(x_2(3(\kappa-1)^-)\right) = \tfrac{x_2(3(\kappa-1)^-)-x_2^*}{6},$$

where $x_1^* = 3.7103$, $x_2^* = 3.8762$.
We can easily verify that the point $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ satisfies

$$\begin{cases} a_1 x_1^* = \sum_{j=1}^{2} b_{1j} f_j(x_j^*) + \sum_{j=1}^{2} c_{1j} g_j\left(x_j^*\right) + d_1, \\ a_2 x_1^* = \sum_{j=1}^{2} b_{2j} f_j(x_j^*) + \sum_{j=1}^{2} c_{2j} g_j\left(x_j^*\right) + d_2, \end{cases}$$

approximately. And it is clear that $\mathfrak{J}_{i\kappa}\left(x_i^*\right) = 0$ for $i = 1, 2$. By simple calculation, we can see that $a_* = 0.6$, $\vartheta^+ = \vartheta_\kappa^+ = 1$, $\vartheta^- = \vartheta_\kappa^- = 2$, $\vartheta = \vartheta_\kappa = 3$, $\mathfrak{L}_1^f = \mathfrak{L}_{1\kappa}^J = \frac{1}{6}$, $\mathfrak{L}_2^f = 0.2$, $\mathfrak{L}_1^g = 0.25$, $\mathfrak{L}_2^g = \mathfrak{L}_{2\kappa}^J = 0.125$, $\mathfrak{L}_\kappa^J = \frac{1}{6}$ and $\sup_{\kappa\in\mathbb{N}} \ln(1+\mathfrak{L}_\kappa^J)/\vartheta_\kappa \approx 0.05138$.
Then

$$\max_{1\le i\le 2}\left\{\left(\frac{e^{a_i\cdot\vartheta^-}-1}{a_i}\right)\left[\sum_{j=1}^{2}\mathfrak{L}_j^f|b_{ij}| + \sum_{j=1}^{2}\mathfrak{L}_j^g|c_{ij}|\right]\right\} \approx 0.928106 < 1,$$

$$\max_{1\le i\le 2}\left\{\left(\frac{1-e^{-a_i\cdot\vartheta^+}}{a_i}\right)\left[\sum_{j=1}^{2}\mathfrak{L}_j^f|b_{ij}| + \sum_{j=1}^{2}\mathfrak{L}_j^g|c_{ij}|\right]\right\} \approx 0.111982 < 1,$$

and

$$a_1 = 0.9 > 0.1883333 \approx \mathfrak{L}_1^f \sum_{j=1}^{2}|b_{1j}| + \mathfrak{L}_1^g \sum_{j=1}^{2}|c_{1j}|,$$

$$a_2 = 0.6 > 0.1385 = \mathfrak{L}_2^f \sum_{j=1}^{2}|b_{2j}| + \mathfrak{L}_2^g \sum_{j=1}^{2}|c_{2j}|.$$

By Theorem 3.2, we can conclude that the ICNN models with the IDEPCAG system (4.2a)-(4.2b) has a unique equilibrium state x^*.
On the other hand, we have

$$\hat{\nu} = \max_{1\le i\le 2}\left(\sum_{j=1}^{2}\mathfrak{L}_j^f|b_{ij}| + \sum_{j=1}^{2}\mathfrak{L}_j^g\,|c_{ij}|\,e^{a_*\cdot\vartheta^-}\right)\cdot\vartheta^+ \approx 0.371327 < 1,$$

$$\mu_1 = \sum_{j=1}^{2}\mathfrak{L}_j^f|b_{1j}| + \sum_{j=1}^{2}\mathfrak{L}_j^g|c_{1j}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}} \approx 0.545368 < 0.548616 \approx a_* - \sup_{\kappa\in\mathbb{N}}\frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa},$$

and

$$\mu_2 = \sum_{j=1}^{2}\mathfrak{L}_j^f|b_{2j}| + \sum_{j=1}^{2}\mathfrak{L}_j^g|c_{2j}|\frac{e^{a_*\cdot\vartheta^-}}{1-\hat{\nu}} \approx 0.422617 < 0.548616 \approx a_* - \sup_{\kappa\in\mathbb{N}}\frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa}.$$

Then

$$a_* - \max_{1\le i\le 2} \mu_i - \sup_{\kappa\in\mathbb{N}} \frac{\ln(1+\mathfrak{L}_\kappa^J)}{\vartheta_\kappa} \approx 0.0032476 > 0.$$

One can see that all conditions (H_1), (H_2), (H_3), (3.3) and (3.9) in Theorem 3.4 are satisfied. So, by Theorem 3.4, the unique equilibrium state of the ICNN models with the IDEPCAG system (4.2a)-(4.2b) is globally exponentially stable.

The simulation of the unique equilibrium state x^* of the ICNN models (4.2a)-(4.2b) with and without impulses, are shown in Figures 4.5–4.8.

For the simulation, the initial states $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$ are given by the random function. Figure 4.5 shows that the conditions obtained in this article are valid for the ICNN models with the IDEPCAG system (4.2a)-(4.2b).

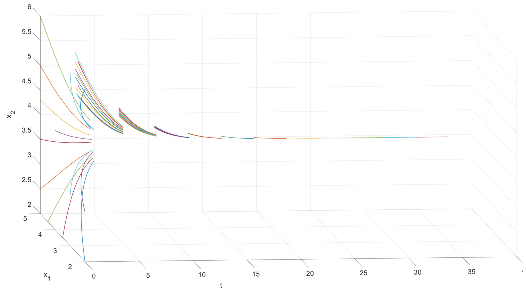

Figure 4.5: Some trajectories uniformly convergent to the unique equilibrium state for the ICNN models with the IDEPCAG system (4.2a)-(4.2b).

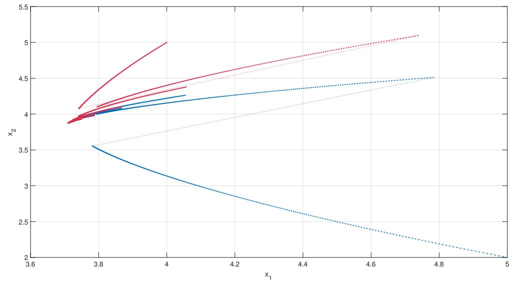

Figure 4.6: Exponential convergence of two trajectories towards the unique equilibrium state for the ICNN models with the IDEPCAG system (4.2a)-(4.2b). Initial conditions: (i) (4; 5) in red and (ii) (5; 2) in blue.

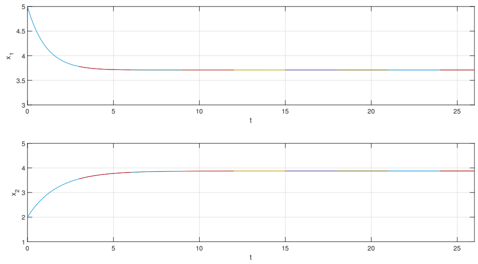

Figure 4.7: Convergence of the unique globally exponentially stable equilibrium state for the CNN models with the DEPCAG system (4.2a) without impulsive effects.

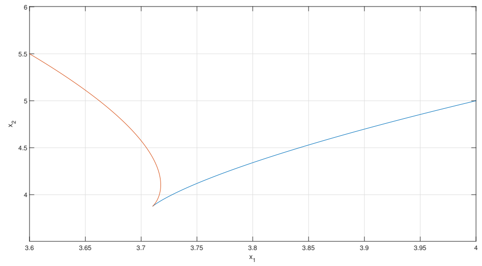

Figure 4.8: Convergence of the unique globally exponentially stable equilibrium state for the CNN models with the DEPCAG system (4.2a) without impulsive effects. Initial conditions: (i) (3.6; 5.5) in red and (ii) (4; 5) in blue.

Remark 10. 1. Note that the simulation shows that some trajectories converge to the unique equilibrium state $\begin{pmatrix} 3.7103 \\ 3.8762 \end{pmatrix}$ of the CNN models with the DEPCAG system (4.2a).
2. When considering system (4.2a)-(4.2b) with generalized piecewise constant delay, the param-

eters of the system (4.2a)-(4.2b) do not satisfy the Theorem 3.1 in [1]. It implies that the results in the present paper are less conservative than the results in [1].

5 Conclusions

In this paper, the unique globally exponentially stable equilibrium state for the impulsive cellular neural network models with piecewise alternately advanced and retarded argument of generalized type have been investigated. By using the equivalent integral equation, a new IDEPCAG's Grönwall inequality and Banach fixed-point theorem, some new sufficient conditions have been developed to ensure the existence, uniqueness and global exponential stability of the equilibrium state for general non-autonomous ICNN models with the IDEPCAG system. The proposed criteria for the existence and stability theorems are easily tested by analyzing multiple relationships between neural network parameters and Lipschitz constants without asking for the conditions of differentiability, monotonicity or boundedness. Based on the proposed approach, it is unnecessary to utilize Razumikhin-type technique or construct a Lyapunov function that is applied from the previous literature. Moreover, illustrative simulation examples show that the approach used is more efficient and extend the results of the previous literature [1], [10] and [20].

Acknowledgment

The author wishes to express your sincere gratitude to the referee for the valuable suggestions, which helped to improve the paper.

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