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Local distance antimagic chromatic number for the union of complete bipartite graphs

V. Priyadharshini and M. Nalliah

Abstract. Let G be a graph on p vertices and q edges with no isolated vertices. A bijection $f : V \to \{1, 2, 3, ..., |V(G)| = p\}$ is called local distance antimagic labeling, if for any two adjacent vertices u and v, we have $w(u) \neq w(v)$, where $w(u) = \sum_{x \epsilon N(u)} f(x)$. The local distance antimagic chromatic number $\chi_{lda}(G)$ is defined to be the minimum number of colors taken over all colorings of G induced by local distance antimagic labelings of G. In this paper, we determine the graph G for the local distance antimagic chromatic number is 2.

Keywords. Distance antimagic graphs, chromatic number, local distance antimagic chromatic number, star graphs.

1 Introduction

Let G be a simple graph with a finite number of vertices. The order and size of G are denoted by p and q, respectively. For more graph-theoretical terms, the reader is referred to Chartrand and Lesniak [6].

Hartsfield and Ringel [8] in their work, introduced antimagic labeling for a graph $G = (V, E)$ as a bijection $f : E \to \{1, 2, ..., |E|\}$ with the property that, for each vertex $u \in V(G)$, the weight $w(u) = \sum_{e \in E(u)} f(e)$, where $E(u)$ is the set of edges incident to u and $w(u) \neq w(v)$ for any two distinct vertices u and $v \in V(G)$. A graph G is called antimagic if G has antimagic labeling.

Hartsfield and Ringel [8] conjectured that (i) every connected graph with at least three vertices admits antimagic labeling and (ii) every tree with at least three vertices admits antimagic labeling. Although these two conjectures are unsolved, the works of many researchers provide evidence that these conjectures are valid for several families of graphs. For a detailed and interesting review of these conjectures, one can see chapter 6 of [5].

Arumugam et al.[3] proposed a new definition as a relaxation of the notion of antimagic labeling. They called a bijection $f : E \to \{1, 2, ..., |E|\}$ is *local antimagic labeling* of G if for any two adjacent vertices u and v in $V(G)$, the condition $w(u) \neq w(v)$ holds. Further, they conjectured that every connected graph with at least three vertices admits local antimagic labeling. This conjecture was solved partially by Bensmail et al.[21] in 2017, and later in 2018, this conjecture was proved to be accurate by Haslegrave [19] using probabilistic tools.

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Corresponding author: M. Nalliah.

Based on the notion of local antimagic labeling, Arumugam et al.[3] introduced a new graph coloring parameter called *local antimagic chromatic number*, $\chi_{la}(G)$, defined as the minimum number of colors taken over all colorings of G induced by local antimagic labelings of G.

Arumugam et al.[3] obtained the local antimagic chromatic number for cycles, paths, friendship graph, complete bipartite graph $K_{m,n}$ and wheel graph. Recently, several authors are studied and investigated the local antimagic chromatic number for several families of graphs [5, 20, 22].

In 2012, Arumugam and Kamatchi[10] introduced distance antimagic labeling and obtained some basic results on cycles, paths and friendship graph. This work extended to other family of graphs by several authors [1, 2, 5, 9, 11, 16, 10, 13, 17, 15, 14, 12, 7, 18]. Motivated by the notion of local antimagic labeling, Divya and Devi Yamini [4] introduced a new distance type labeling on graphs, as follows:

Definition 1. [4] Let G be a graph of order p and size q having no isolated vertices. A bijection $f : V \to \{1, 2, 3, ..., |V(G)|\}$ is called *local distance antimagic labeling*, if for any two adjacent vertices u and v we have $w(u) \neq w(v)$, where $w(u) = \sum_{x \in N(u)} f(x)$. A graph G is called local distance antimagic if G has local distance antimagic labeling.

Definition 2. [4] *The local distance antimagic chromatic number* $\chi_{lda}(G)$ is defined to be the minimum number of colors taken over all colorings of G induced by local distance antimagic labelings of G and denotes $\chi_{lda}(G)$.

From definition 1, one can notice that, if G is distance antimagic, then G is local distance antimagic. The authors obtained local distance antimagic chromatic number for star, subdivision of star, complete bipartite graph, complete r-partite graph, friendship graph, corona product of star, complete and friendship graphs.

In this paper, we characterize the class of graphs with local distance antimagic chromatic number 2.

2 Local distance antimagic labeling of graphs

Let $G = (V, E)$ be a graph and $v \in V$. The open neighborhood of v is defined as $N(v) = \{u \in V : uv \in E\}$.

Theorem 2.1. *Let T be a tree on $n \geq 2$ vertices with k leaves and let*

$$\mathcal{L} = \{N(l), \textit{where } l \textit{ is a leaf}\}$$

be the the set of vertices of T. Let $|\mathcal{L}| = t$. Then $\chi_{lda}(T) \geq t + 1$.

Proof. Let f be any local distance antimagic labeling of T. Then, by the coloring induced by f, the color of a leaf l is $w(l) = f(v)$ where $1 \leq f(v) \leq p$ and $v \in N(l)$. Therefore, the leaves receive the colors $w_1, w_2, \ldots, w_t$. If leaf x with $f(x) = p$, then its adjacent vertex y has a weight $w(y) > p + 1$. Therefore, the vertex y receives a new color w_{t+1}. If for any non-leaf x with $f(x) = p$, then there exists a vertex $y \notin \mathcal{L}$ and y is adjacent to x. Clearly, the weight of vertex y, $w(y) > p + 1$ and that, the vertex y receives a new color w_{t+1}. Thus $\chi_{lda}(T) \geq t + 1$. □

Theorem 2.2. *Let G be connected graph of order p. Then $\chi_{lda}(G) = 2$ if and only if $G \cong K_{m,n}$.*

Proof. Let $G = (V, E)$ be a connected graph of order p. Suppose $\chi_{lda}(G) = 2$. Then there exists a local distance antimagic labeling f with 2-colors w_1 and w_2. Let V_1 be the set of all vertices which receives the color w_1 and V_2 be the set of all vertices which receives the color w_2. Then $|V| = |V_1| + |V_2|$ and hence there is no edge in the same partite V_1 and V_2. Therefore, G is bipartite. If G is a tree, then by Theorem 2.1, we get $|\mathcal{L}| = 1$ and hence $|V_1(G)| = 1$, $|V_2(G)| = |V(G)| - 1$. Thus $G \cong K_{1,|V(G)|-1}$. If G is not a tree and bipartite then $|V_1(G)| \geq 2$ and $|V_2(G)| = |V(G)| - |V_1(G)|$. Since every vertex of $V_1(G)$ weight is w_1, it follows that every vertex of $V_1(G)$ must be adjacent to every vertex of $V_2(G)$. Therefore, G is isomorphic to complete bipartite graph with $|V_1(G)| = m$ and $|V_2(G)| = |V(G)| - |V_1(G)| = n$, $m \leq n$.

Conversely, suppose $G \cong K_{m,n}$. Since $\chi(K_{m,n}) = 2$, it follows, we get $\chi_{lda}(K_{m,n}) \geq 2$. So, for proving $\chi_{lda}(K_{m,n}) = 2$ it suffices to provide a local distance antimagic labeling of $K_{m,n}$ that induces a local distance antimagic vertex coloring using 2-colors. Now, we define a vertex labeling $f' : V(K_{m,n}) \to \{1, 2, 3, \ldots, m+n\}$ by $f'(u_i) = i, 1 \leq i \leq m$ and $f'(v_i) = m + i, 1 \leq i \leq n$. Then the vertex weights are $w'(v_i) = \frac{m(m+1)}{2}, 1 \leq i \leq m$ and $w'(u_i) = \frac{n(n+1)}{2} + mn, 1 \leq i \leq n$. Therefore, f' is a local distance antimagic labeling of $K_{m,n}$ that induces a local distance antimagic vertex coloring using 2-colors and hence $\chi_{lda}(K_{m,n}) \leq 2$. Thus $\chi_{lda}(K_{m,n}) = 2$. □

3 χ_{lda} for the union of complete bipartite graphs

In this section, we prove that the local distance antimagic chromatic number for the union of m copies of the complete bipartite graph $mK_{r,s}$ is two when $r + s$ is even for any m and $r + s$ is odd and m is odd. First, we present the following theorem with $r = s = n$.

Theorem 3.1. *Let $mK_{n,n}$ be the union of m copies of complete bipartite graph. Then $\chi_{lda}(mK_{n,n}) = 2$.*

Proof. Let $V(mK_{n,n}) = \{u_i^j, v_i^j, 1 \leq i \leq n, 1 \leq j \leq m\}$ and let $E(mK_{n,n}) = \{u_i^j v_i^j, 1 \leq i \leq n, 1 \leq j \leq m\}$. Then $|V(mK_{n,n})| = 2nm$. Define a labeling $f_1 : V(mK_{n,n}) \to \{1, 2, 3, \ldots, 2mn\}$ by

Case 1: n is even

$$f_1(u_i^j) = \begin{cases} 2m(i-1) + 2j - 1, & \text{i is odd}, 1 \leq j \leq m \\ 2mi - (2j-1), & \text{i is even}, 1 \leq j \leq m \end{cases}$$

$$f_1(v_i^j) = \begin{cases} 2j + 2m(i-1), & \text{i is odd}, 1 \leq j \leq m \\ 2mi - (2j-2), & \text{i is even}, 1 \leq j \leq m \end{cases}$$

Then the vertex weights are $w_1(u_i^j) = mn^2 + n$ and $w_1(v_i^j) = mn^2, 1 \leq i \leq n, 1 \leq j \leq m$.

Case 2: n is odd

Define a labeling $f_2 : V(mK_{n,n}) \to \{1, 2, 3, \ldots, 2mn\}$ by

$$f_2(u_i^j) = \begin{cases} j, & i = 1, 1 \leq j \leq m \\ j + m, & i = 2, 1 \leq j \leq m \\ 4m - (2j-2), & i = 3, 1 \leq j \leq m \\ 6m + 2j - 1 + 2m(i-4), & i \geq 4 \text{ and i is even}, 1 \leq j \leq m \\ 2im - (2j-1), & i \geq 5 \text{ and i is odd}, 1 \leq j \leq m \end{cases}$$

$$f_2(v_i^j) = \begin{cases} 4m+1-2j, & i=1, 1 \le j \le m \\ 4m+j, & i=2, 1 \le j \le m \\ 5m+j, & i=3, 1 \le j \le m \\ 6m+2j+2m(i-4), & i \ge 4 \text{ and i is even}, 1 \le j \le m \\ 2im-(2j-2), & i \ge 5 \text{ and i is odd}, 1 \le j \le m \end{cases}$$

Then the vertex weights are $w_2(u_i^j) = mn^2+4m+n-2$ and $w_2(v_i^j) = mn^2-4m+2$. Therefore, f_1 and f_2 admit a local distance antimagic labeling of $mK_{n,n}$ that induces a local distance antimagic vertex coloring using 2-colors. Hence $\chi_{lda}(mK_{n,n}) \le 2$. Since $\chi(K_{n,n}) = 2$, it follows, we get $\chi_{lda}(K_{n,n}) \ge 2$. Hence $\chi_{lda}(mK_{n,n}) = 2$. □

Example 1. The local distance antimagic labeling with 2-colors for $3K_{5,5}$ and $4K_{4,4}$ are presented in Figures 1 and 2.

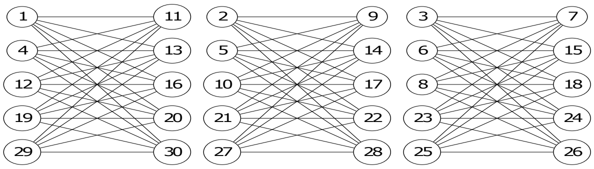

Figure 1: $3K_{5,5}$ with vertex colors 90 and 65.

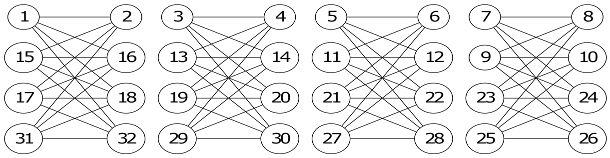

Figure 2: $4K_{4,4}$ with vertex colors 68 and 64.

Theorem 3.2. *Let $mK_{r,s}$ be the union of m copies of complete bipartite graph with $r < s$ and r, s are even. Then $\chi_{lda}(mK_{r,s}) = 2$.*

Proof. Let $V(mK_{r,s}) = \{u_i^j, v_i^j, 1 \le i \le r, s, 1 \le j \le m\}$ and let $E(mK_{r,s}) = \{u_i^j v_i^j, 1 \le i \le r, 1 \le j \le m, 1 \le i \le s\}$. Then $|V(mK_{r,s})| = m(r+s)$. Define a labeling $f_3 : V(mK_{r,s}) \to \{1, 2, 3, \ldots, m(r+s)\}$ by

$$f_3(u_i^j) = \begin{cases} m(i-1)+j, & \text{i is odd}, 1 \le i \le r, 1 \le j \le m \\ mi+1-j, & \text{i is even}, 1 \le i \le r, 1 \le j \le m \end{cases}$$

$$f_3(v_i^j) = \begin{cases} rm+m(i-1)+j, & \text{i is odd}, 1 \le i \le s, 1 \le j \le m \\ rm+mi+1-j, & \text{i is even}, 1 \le i \le s, 1 \le j \le m \end{cases}$$

Then the vertex weights are $w_3(u_i^j) = mrs + \frac{s(ms+1)}{2}$ and $w_3(v_i^j) = \frac{r}{2}(mr+1)$. Therefore f_3 admits a local distance antimagic labeling of $mK_{r,s}$ that induces a local distance antimagic vertex coloring using 2-colors and hence $\chi_{lda}(mK_{r,s}) \le 2$. Since $\chi(K_{r,s}) = 2$, it follows, we get $\chi_{lda}(K_{r,s}) \ge 2$. Hence $\chi_{lda}(mK_{r,s}) = 2$. □

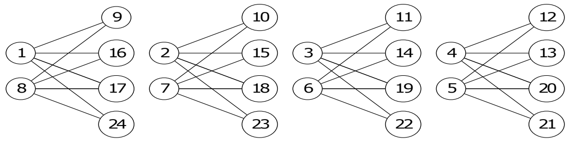

Figure 3: $4K_{2,4}$ with vertex colors 66 and 9

Example 2. The local distance antimagic labeling with 2-colors for $4K_{2,4}$ is given in Figure 3.

Theorem 3.3. *Let $mK_{r,s}$ be the union of m copies of complete bipartite graph with $r < s$ and $r, s \geq 3$ are odd. Then $\chi_{lda}(mK_{r,s}) = 2$.*

Proof. Let $V(mK_{r,s}) = \{u_i^j, v_i^j, 1 \leq i \leq r, s, 1 \leq j \leq m\}$ and let $E(mK_{r,s}) = \{u_i^j v_i^j, 1 \leq i \leq r, 1 \leq j \leq m, 1 \leq i \leq s\}$. Then $|V(mK_{r,s})| = m(r+s)$. Define a labeling $f_4 : V(mK_{r,s}) \to \{1, 2, 3, \ldots, m(r+s)\}$ by

$$f_4(u_i^j) = \begin{cases} m(i-1)+j, & i = 1, 2, 1 \leq j \leq m \\ 4m - 2(j-1), & i = 3, 1 \leq j \leq m \\ m(i+1)-(j-1), & i \geq 5 \text{ is odd}, 1 \leq i \leq r, 1 \leq j \leq m \\ mi + j, & i \geq 4 \text{ is even}, 1 \leq i \leq r, 1 \leq j \leq m \end{cases}$$

$$f_4(v_i^j) = \begin{cases} 4m - (2j-1), & i = 1, 1 \leq j \leq m \\ m(r+2)+j, & i = 3, 1 \leq j \leq m \\ m(r+i)-(j-1), & i \geq 5 \text{ is odd}, 1 \leq i \leq s, 1 \leq j \leq m \\ m(r-1+i)+j, & i \geq 2 \text{ is even}, 1 \leq i \leq s, 1 \leq j \leq m \end{cases}$$

Then the vertex weights are $w_4(u_i^j) = \frac{1}{2}(2mr(s-1) + s(ms+1) + 5m - 1)$ and $w_4(v_i^j) = \frac{1}{2}(mr(r+2) - 5m + r + 1)$. Therefore, f_4 admits a local distance antimagic labeling of $mK_{r,s}$ that induces a local distance antimagic vertex coloring using 2-colors and hence $\chi_{lda}(mK_{r,s}) \leq 2$. Since $\chi(K_{r,s}) = 2$, it follows, we get $\chi_{lda}(K_{r,s}) \geq 2$. Thus $\chi_{lda}(mK_{r,s}) = 2$. □

Example 3. The local distance antimagic labeling with 2-colors for $3K_{3,5}$ is given in Figure 4.

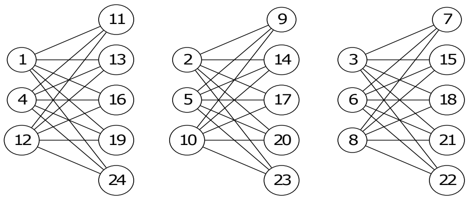

Figure 4: $3K_{3,5}$ with vertex colors 83 and 17

Theorem 3.4. *Let $mK_{r,s}$ be the union of m copies of complete bipartite graph with $r+s$ is odd and m is odd. Then $\chi_{lda}(mK_{r,s}) = 2$.*

Proof. Let $V(mK_{r,s}) = \{u_i^j, v_i^j, 1 \le i \le r, s, 1 \le j \le m\}$ and let $E(mK_{r,s}) = \{u_i^j v_i^j, 1 \le i \le r, 1 \le j \le m, 1 \le i \le s\}$. Then $|V(mK_{r,s})| = m(r+s)$. For r is even and s is odd with $r < s$, define a labeling $f_5 : V(mK_{r,s}) \to \{1, 2, 3, \ldots, m(r+s)\}$ by

$$f_5(u_i^j) = \begin{cases} m(s+i-1)+j, & \text{i is odd}, 1 \le i \le r, 1 \le j \le m \\ m(s+i)+1-j, & \text{i is even}, 1 \le i \le r, 1 \le j \le m \end{cases}$$

$$f_5(v_i^j) = \begin{cases} \frac{j+1}{2}, & i = 1, \text{ j is odd}, 1 \le j \le m \\ \frac{m+j+1}{2}, & i = 1, \text{ j is even}, 1 \le j \le m \\ \frac{3m+j}{2}, & i = 2, \text{ j is odd}, 1 \le j \le m \\ \frac{2m+j}{2}, & i = 2, \text{ j is even}, 1 \le j \le m \\ mi-j+1, & i \ge 3 \text{ is odd}, 1 \le i \le s, 1 \le j \le m \\ m(i-1)+j, & i \ge 4 \text{ is even}, 1 \le i \le s, 1 \le j \le m \end{cases}$$

Then the vertex weights are $w_5(u_i^j) = \frac{s(ms+1)}{2}$ and $w_5(v_i^j) = msr + \frac{r(mr+1)}{2}$.

If r is odd and s is even, define a labeling f_5' by $f_5'(u_i^j) = f_5(v_i^j), 1 \le i \le r$ and $f_5'(v_i^j) = f_5(u_i^j), 1 \le i \le s$. This labeling schemes f_5 and f_5' admit a local distance antimagic labeling of $mK_{r,s}$ that induces a local distance antimagic vertex coloring using 2-colors and hence $\chi_{lda}(mK_{r,s}) \le 2$. Since $\chi(K_{r,s}) = 2$, it follows, we get $\chi_{lda}(K_{r,s}) \ge 2$. Thus $\chi_{lda}(mK_{r,s}) = 2$. □

Example 4. The local distance antimagic labeling with 2-colors for $5K_{2,3}$ and $3K_{3,4}$.

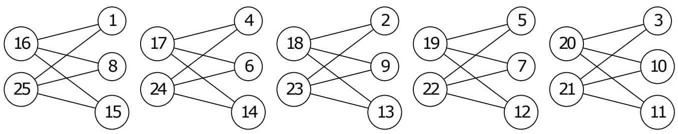

Figure 5: $5K_{2,3}$ with vertex colors 24 and 41

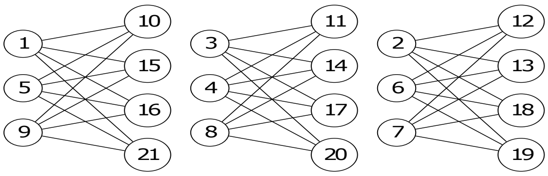

Figure 6: $3K_{3,4}$ with vertex colors 62 and 15

Now, we present an upper bound for $\chi_{lda}(mK_{r,s})$, where $r+s$ is odd and m is even.

Theorem 3.5. *Let $mK_{r,s}$ be the union of m copies of complete bipartite graph with $r+s$ is odd and m is even. Then $\chi_{lda}(mK_{r,s}) \leq 3$.*

Proof. Let $V(mK_{r,s}) = \{u_i^j, v_i^j, 1 \leq i \leq r, s, 1 \leq j \leq m\}$ and let $E(mK_{r,s}) = \{u_i^j, v_i^j, 1 \leq i \leq r, s, 1 \leq j \leq m\}$. Then $|V(mK_{r,s})| = m(r+s)$. If r is even and s is odd, we define a labeling $f_6 : V(mK_{r,s}) \to \{1, 2, 3, ..., m(r+s)\}$ by

$$f_6(u_i^j) = \begin{cases} 2m+1+(2j-2), & i = 1, 1 \leq j \leq m \\ m(s+2)-(2j-2), & i = 2, 1 \leq j \leq \frac{m}{2} \\ m(s+i-1)+j, & i \geq 3 \text{ is odd}, 1 \leq j \leq m \\ m(s+i)-(j-1), & i \geq 4 \text{ is even}, 1 \leq j \leq m \end{cases}$$

$$f_6\left(u_i^{\frac{m}{2}+j}\right) = m(s+2)+1-2j, i = 2 \text{ and } 1 \leq j \leq \frac{m}{2}$$

$$f_6(v_i^j) = \begin{cases} m(i-1)+j, & i = 1, 2, 1 \leq j \leq m \\ 4m-2(j-1), & i = 3, 1 \leq j \leq m \\ m(i+1)-(j-1), & i \geq 5 \text{ is odd}, 1 \leq j \leq m \\ mi+j, & i \geq 4 \text{ is even}, 1 \leq j \leq m \end{cases}$$

Then the vertex weights are $w_6(u_i^j) = \frac{1}{2}(ms(s+2)-5m+s+1)$, $w_6(v_i^j) = \frac{r}{2}(mr+1)+ms(r-1)+2m, 1 \leq j \leq \frac{m}{2}$ and $w_6(v_i^j) = \frac{r}{2}(mr+1)+ms(r-1)+3m-1, \frac{m}{2} \leq j \leq m$. If r is odd and s is even, define a labeling f_6' by $f_6'(u_i^j) = f_6(v_i^j), 1 \leq i \leq r$ and $f_6(v_i^j) = f_6(u_i^j), 1 \leq i \leq s$. Now, f_6 and f_6' admit a local distance antimagic labeling of $mK_{r,s}$ that induces a local distance antimagic vertex coloring using 3-colors. Hence $\chi_{lda}(mK_{r,s}) \leq 3$. □

Finally, we obtain an exact χ_{lda} for $mK_{r,s}$, where $r = 2, s = 3$ and $m = 2$.

Theorem 3.6. *Let $G \cong 2K_{2,3}$ be a disconnected graph. Then $\chi_{lda}(2K_{2,3}) = 3$.*

Proof. Suppose $\chi_{lda}(2K_{2,3}) = 2$. Then there exists a local distance antimagic labeling f with 2-colors. Let $W = \{w_1, w_2\}$ be the set of colors. The minimum possible vertex weight is 3 and the maximum possible vertex weight is at most 27. Therefore, $3 \leq w \leq 27$, where $w \in W$. Let $w_1 = w(u_i^j)$ and $w_2 = w(v_i^j)$. Then the vertex labels are partitioned into 2 three-element sets with their sum is w_1 and the rest of the four vertices labels are partitioned into 2 two-element sets with their sum is w_2. The list of all possible combinations of weights w_1 and w_2 are presented in Tables 1-4. Clearly, there is no 2 two-element sets with same vertex weight w_2, a contradiction. Thus $\chi_{lda}(2K_{2,3}) \geq 3$.

Let $V(2K_{2,3}) = \{u_1^j, u_2^j, v_1^j, v_2^j, v_3^j, j = 1, 2\}$. Define a bijection $f_7 : V(2K_{2,3}) \to \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ by $f_7(u_1^1) = 2, f_7(u_2^1) = 6, f_7(u_1^2) = 4, f_7(u_2^2) = 5,$ $f_7(v_1^1) = 1, f_7(v_2^1) = 8, f_7(v_3^1) = 10, f_7(v_1^2) = 3, f_7(v_2^2) = 7, f_7(v_3^2) = 9$. Then the weights are $w_7(u_i^j) = 19, i, j = 1, 2,$ $w_7(v_i^1) = 8, i = 1, 2, 3$ and $w_7(v_i^2) = 9, i = 1, 2, 3$. Thus $\chi_{lda}(2K_{2,3}) \leq 3$. Hence $\chi_{lda}(2K_{2,3}) = 3$. □

The above Theorem 3.5 and Theorem 3.6 leads to the following open problem.

Problem 3.7. *Determine the local distance antimagic labeling with exact 2-colors for $mK_{r,s}$, where $r+s$ is odd and m is even.*

Weight	**Partitions**	**Others**	**Weight**	**Partitions**	**Others**
11	(1,3,7)(2,4,5)	6,8,9,10	12	(1,2,9)(3,4,5)	6,7,8,10
13	(1,2,10)(3,4,6)	5,7,8,9		(1,3,8)(2,4,6)	5,7,9,10
	(1,3,9)(2,4,7)	5,6,8,10		(1,5,6)(2,3,7)	4,8,9,10
	(1,3,9)(2,5,6)	4,7,8,10	15	(1,4,10)(2,5,8)	3,6,7,9
	(1,4,8)(2,5,6)	3,7,9,10		(1,4,10)(2,6,7)	3,5,8,9
	(1,5,7)(2,3,8)	4,6,9,10		(1,4,10)(3,5,7)	2,6,8,9
	(1,5,7)(3,4,6)	2,8,9,10		(1,5,9)(2,3,10)	4,6,7,8
14	(1,3,10)(2,4,8)	5,6,7,9		(1,5,9)(2,6,7)	3,4,8,10
	(1,3,10)(2,5,7)	4,6,8,9		(1,5,9)(3,4,8)	2,6,7,10
	(1,4,9)(2,5,7)	3,6,8,10		(1,6,8)(2,3,10)	4,5,7,9
	(1,4,9)(3,5,6)	2,7,8,10		(1,6,8)(2,4,9)	3,5,7,10
	(1,5,8)(2,3,9)	4,6,7,10		(1,6,8)(3,5,7)	2,4,9,10
	(1,6,7)(2,3,9)	4,5,8,10		(2,3,10)(4,5,6)	1,7,8,9
	(1,6,7)(2,4,8)	3,5,9,10		(2,4,9)(3,5,7)	1,6,8,10
	(2,8,4)(3,5,6)	1,9,7,10		(2,6,7)(3,4,8)	1,5,9,10

Table 1: Possible partitions of labels with weights 11,12,13,14 and 15.

Weight	**Partitions**	**Others**	**Weight**	**Partitions**	**Others**
18	(1,7,10)(3,6,9)	2,4,5,8	19	(1,8,10)(3,7,9)	2,4,5,6
	(1,7,10)(4,5,9)	2,3,6,8		(1,8,10)(4,6,9)	2,3,5,7
	(1,7,10)(4,6,8)	2,3,5,9		(2,7,10)(4,6,9)	1,3,5,8
	(1,8,9)(2,6,10)	3,4,5,7		(2,7,10)(5,6,8)	1,3,4,9
	(1,8,9)(3,5,10)	2,4,6,7		(2,8,9)(3,6,10)	1,4,5,7
	(1,8,9)(5,6,7)	2,3,4,10		(2,8,9)(4,5,10)	1,3,6,7
	(2,6,10)(3,7,8)	1,4,5,9		(3,6,10)(4,7,8)	1,2,5,9
	(2,6,10)(4,5,9)	1,3,7,8	20	(1,9,10)(5,7,8)	2,3,4,6
	(2,7,9)(3,5,10)	1,4,6,8		(2,8,10)(4,7,9)	1,3,5,6
	(2,7,9)(4,6,8)	1,3,5,10		(2,8,10)(5,6,9)	1,3,4,7
	(3,5,10)(4,6,8)	1,2,7,9		(3,7,10)(5,6,9)	1,2,4,8
	(3,7,8)(4,5,9)	1,2,6,10		(3,8,9)(4,6,10)	1,2,5,7
22	(4,8,10)(6,7,9)	1,2,3,5		(4,6,10)(5,7,8)	1,2,3,9

Table 2: Possible partitions of labels with weight 18,19,20 and 22.

Weight	**Partitions**	**Others**
21	(2,9,10)(6,7,8)	1,3,4,5
	(3,8,10)(5,7,9)	1,2,4,6
	(4,8,9)(5,6,10)	1,2,3,7

Table 3: Possible partitions of labels with weight is 21.

Weight	**Partitions**	**Others**	**Weight**	**Partitions**	**Others**
16	(1,5,10)(2,6,8)	3,4,7,9	17	(1,6,10)(2,7,8)	3,4,5,7
	(1,5,10)(3,4,9)	2,6,7,8		(1,6,10)(3,5,9)	2,4,7,8
	(1,5,10)(3,6,7)	2,4,8,9		(1,6,10)(4,5,8)	2,3,7,9
	(1,6,9)(2,4,10)	3,5,7,8		(1,7,9)(2,5,8)	3,4,5,10
	(1,6,9)(3,5,8)	2,4,7,10		(1,7,9)(3,4,10)	2,5,6,8
	(1,6,9)(4,5,7)	2,3,8,10		(1,7,9)(3,6,8)	2,4,5,10
	(1,7,8)(2,4,10)	3,5,6,9		(1,7,9)(4,5,8)	2,3,6,10
	(1,7,8)(2,5,9)	3,4,6,10		(2,5,10)(3,6,8)	1,4,7,9
	(1,7,8)(3,4,9)	2,5,6,10		(2,5,10)(4,6,7)	1,3,8,9
	(2,5,9)(3,6,7)	1,4,8,10		(2,6,9)(3,4,10)	1,5,7,8
	(2,6,8)(3,4,9)	1,5,7,10		(2,6,9)(4,5,8)	1,3,7,10
	(2,6,8)(4,5,7)	1,3,9,10		(2,7,8)(3,4,10)	1,5,6,9
17	(2,7,8)(3,5,9)	1,4,6,10		(3,5,9)(4,6,7)	1,2,8,10

Table 4: Possible partitions of labels with weights 16 and 17.

4 Conclusion

In this paper, we proved that the local distance antimagic chromatic number for a connected graph G is 2 only when $G \cong K_{m,n}$. We also established that the local distance antimagic chromatic number for the union of m copies of complete bipartite graph $mK_{r,s}$ is 2 in all possible cases except for the case $r+s$ is odd and m is even. When $r+s$ is odd and m is even we established that $\chi_{lda}(mK_{r,s}) \leq 3$. In particular, for $r=2, s=3$ and $m=2$ we proved that $\chi_{lda}(2K_{2,3}) = 3$. The problem of generalizing the result and establishing the exact bounds for $\chi_{lda}(mK_{r,s})$ when $r+s$ is odd and m is even remains open.

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V. Priyadharshini Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology, Vellore-632 014,Tamil Nadu, India.

E-mail: priyavenkat1310@gmail.com

M. Nalliah Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology, Vellore-632 014,Tamil Nadu,India.

E-mail: nalliahklu@gmail.com