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Existence of positive periodic solutions for a predator-prey model

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Abstract. In this paper, a class of nonlinear predator-prey models with three discrete delays is considered. By linearizing the system at the positive equilibrium point and analyzing the instability of the linearized system, two sufficient conditions to guarantee the existence of positive periodic solutions of the system are obtained. It is found that under suitable conditions on the parameters, time delay affects the stability of the system. The present method does not need to consider a bifurcating equation which is very complex for such a predator-prey model with three discrete delays. Some numerical simulations are provided to illustrate our theoretical prediction.

Keywords. Predator-prey model, delay, instability, positive periodic solution

1 Introduction

In 2001, Korobeinikov considered the global stability of a Leslie-Gower predator-prey model as the following [1]:

$$\begin{cases} H'(t) = (r_1 - a_1 P - b_1 H)H, \\ P'(t) = (r_2 - a_2 \frac{P}{H})P. \end{cases} \tag{1.1}$$

By introducing a Lyapunov function, the author was able to show that the unique coexisting point of system (1.1) is globally stable. In 2009, L. Chen and F. Chen introduced and studied the following Leslie-Gower predator-prey model with feedback controls [2]:

$$\begin{cases} N_1'(t) = (r_1 - a_1 N_2 - b_1 N_1 - c_1 u_1)N_1, \\ N_2'(t) = (r_2 - a_2 \frac{N_2}{N_1} - c_2 u_2)N_2, \\ u_1'(t) = -f_1 u_1 + g_1 N_1(t), \\ u_2'(t) = -f_2 u_2 + g_2 N_2(t), \end{cases} \tag{1.2}$$

where N_1 and N_2 denote the density of prey and predator populations, respectively; u_1 and u_2 are feedback control variables. The global stability of a unique interior equilibrium for model (1.2) was investigated. The main result together with its numerical simulations indicated that feedback control variables only change the position of the unique interior equilibrium and retain its global

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stability. In 2015, Sharma and Samanta investigated the following Leslie-Gower predator-prey model with disease in prey incorporating a prey refuge:

$$\begin{cases} X'(t) = r_1 X - bX(X+Y) - \beta XY, \\ Y'(t) = \beta XY - \frac{a(1-m)YZ}{a+(1-m)Y} - \delta Y, \\ Z'(t) = [r_2 - \frac{\gamma Z}{a+(1-m)Y}]Z. \end{cases} \tag{1.3}$$

The authors studied the positivity and boundedness of the solutions for model (1.3) and analyzed the existence of various equilibrium points and stability of the system at those equilibrium points. The influence of the infected prey refuge on each population density also discussed. By using the iterative technique and further precise analysis, sufficient conditions on the global attractivity of a positive equilibrium for a modified Leslie-Gower predator-prey model with Holling-type II schemes and a prey refuge were obtained [4]. On the other hand, many researchers have considered delayed prey and predator models [5-23]. For example, Nindjin et al. have discussed the following delayed predator-prey model [5]:

$$\begin{cases} x'(t) = (a_1 - bx(t) - \frac{c_1 y(t)}{x(t)+k_1})x(t), \\ y'(t) = (a_2 - \frac{c_2 y(t-\tau)}{x(t-\tau)+k_2})y(t). \end{cases} \tag{1.4}$$

By constructing a suitable Liapunov function, a sufficient condition for global stability of the positive equilibrium was obtained. Adak et al. have investigated the following Leslie-Gower prey-predator-parasite model [6]:

$$\begin{cases} S'(t) = rS(t)(1 - \frac{S+I}{K}) - \beta SI - q_1 ES, \\ I'(t) = \beta S(t-\tau)I(t-\tau) - cI - \frac{c_1 Iy}{I+K_1} - q_2 EI, \\ y'(t) = y(a_2 - \frac{c_2 y}{I+K_2}). \end{cases} \tag{1.5}$$

It was proved that the delay has no influence on the stability of different equilibrium points except the interior one. Delay may cause instability in an otherwise stable interior equilibrium point of the system and larger delay may even produce chaos if the infection rate is also high. Guo et al. have considered the complex dynamics in the Leslie-Gower type of the food chain system with multiple delays as follows [7]:

$$\begin{cases} x'(t) = x(t)(1 - x(t)) - y(t), \\ y'(t) = y(t)(a_1 - \frac{a_2 y(t)}{x(t-\tau_1)}) - z(t), \\ z'(t) = z(t)(a_3 - \frac{a_4 z(t)}{y(t-\tau_2)}). \end{cases} \tag{1.6}$$

The Hopf bifurcation and periodic solution were investigated in detail in terms of the central manifold reduction and normal form method for model (1.6). Numerical simulations were also performed to display some complex dynamics. Motivated by the above models, in this paper we consider the existence of positive periodic solutions for the following Leslie-Gower prey-predator with multiple delays:

$$\begin{cases} x'(t) = rx(t)(1 - \frac{x(t-\tau_1)}{K}) - bxy - b_1 y(t-\tau_2), \\ y'(t) = y(t)(a_1 - \frac{a_2 y(t)}{x(t-\tau_1)}) - b_2 z(t-\tau_3), \\ z'(t) = z(t)(a_3 - \frac{a_4 z(t)}{y(t-\tau_2)}), \end{cases} \tag{1.7}$$

where a_i, b_i, b and r all are positive parameters. The initial conditions are $x(t) = \phi(t) \geq 0, \phi(0) > 0, y(t) = \varphi(t) \geq 0, \varphi(0) > 0$, and $z(t) = \psi(t) \geq 0, \psi(0) > 0, t \in [-\tau, 0]$, where $\tau = \max\{\tau_1, \tau_2, \tau_3\}$.

Noting that in the system (1.7), there are three time delays: τ_1 means that the saturation of the environment would fully affect the species populations after some delay, for instead some of the species or resources could be exhausted, but not immediately. The delay τ_2 could mean the first predator species has to mature before it takes a large number of prey, and the same could be true for τ_3. If τ_1, τ_2 and τ_3 are different positive numbers, the bifurcation method is hard to deal with the model (1.7). Because the bifurcation equation of the model (1.7) will be

$$p_1(\lambda) + p_2(\lambda)e^{-\lambda\tau_1} + p_3(\lambda)e^{-\lambda\tau_2} + p_4(\lambda)e^{-\lambda\tau_3} = 0. \tag{1.8}$$

Noting that the equation (1.8) is a transcendental equation with three different parameters τ_1, τ_2, and τ_3. It is too hard to discuss equation (1.8). To my best knowledge, even if for a two delays bifurcation equation, it was considered three cases: (i) $\tau_1 = 0$, (ii) $\tau_1 = \tau_2$, (iii) $\tau_1 \in (0, \tau_0)$ was fixed, those cases indicate that there is one delay in the bifurcation equation (see [7-13]). By means of the extended Chafee's criterion, the present paper investigates the existence of periodic solutions for the model (1.7).

2 Preliminaries

For the model (1.7) we have the following lemma:

Lemma 1. Assume that $a_1a_4 - b_2a_3 > 0$, and $b_1b_2a_3 + ra_2a_4 - b_1a_1a_4 > 0$, then the system (1.7) has a positive equilibrium point and all solutions are bounded.

Proof. Obviously, under the above hypotheses, $x^* = \frac{K(b_1b_2a_3+ra_2a_4-b_1a_1a_4)}{ra_2a_4+Kb(a_1a_4-b_2a_3)}$, $y^* = \frac{a_1a_4-b_2a_3}{a_2a_4}x^*$, and $z^* = \frac{a_3}{a_4}y^*$ is a positive equilibrium point of the system (1.7). From (1.7) we have

$$\begin{cases} x(t) = x(0)\exp(\int_0^t [r - \frac{rx(s-\tau_1)}{K} - by(s) - \frac{b_1y(s-\tau_2)}{x(s)}]ds) > 0, \forall t \geq 0 \\ y(t) = y(0)\exp(\int_0^t [a_1 - \frac{a_2y(s)}{x(s-\tau_1)} - \frac{b_2z(s-\tau_3)}{y(s)}]ds) > 0, \\ z(t) = z(0)\exp(\int_0^t [a_3 - \frac{a_4z(s)}{y(s-\tau_2)}]ds) > 0. \end{cases} \tag{2.1}$$

Noting that $x(0) > 0, y(0) > 0$ and $z(0) > 0$. From the above expressions of $x(t), y(t)$, and $z(t)$, it is clear that all $x(t), y(t)$, and $z(t)$ remain nonnegative for all finite time. On the other hand, from the first equation of the system (1.7), we get

$$x'(t) = rx(t)(1 - \frac{x(t-\tau_1)}{K}) - bxy - b_1y(t-\tau_2) \leq rx(t)(1 - \frac{x(t-\tau_1)}{K}). \tag{2.2}$$

Since equation (2.2) is a Bernoulli's equation, we easily get $x(t) < K(\forall t \geq 0)$. Thus, we have $y(t) \leq \frac{a_2}{Ka_1}$ and $z(t) \leq \frac{Ka_1a_4}{a_2a_3}$. Therefore, all solutions of the system (1.7) are bounded. The proof is completed.

Now let $f_1 = -\frac{a_2y^2(t)}{x(t-\tau_1)}$ and $f_2 = -\frac{a_4z^2(t)}{y(t-\tau_2)}$, make the change of variables $x(t) \to x(t) - x^*, y(t) \to y(t) - y^*, z(t) \to z(t) - z^*$, by using the Taylor's expansion, we have an equivalent system of the model (1.7) as follows:

$$\begin{cases} x'(t) = c_{11}x(t) - c_{12}y(t) - c_{13}x(t-\tau_1) - b_1y(t-\tau_2) - \frac{r}{K}x(t)x(t-\tau_1) - \beta x(t)y(t), \\ y'(t) = c_{21}y(t) + c_{22}x(t-\tau_1) - b_2z(t-\tau_3) - c_{23}y^2(t) + c_{24}y(t)x(t-\tau_1) - c_{25}x^2(t-\tau_1) \\ \qquad + \sum_{i+j>2} \frac{x^i}{i!}\frac{y^j}{j!} \cdot \frac{\partial^{i+j}f_1}{\partial x^i \partial y^j}|_{(x^*,y^*)}, \\ z'(t) = c_{31}z(t) + c_{32}y(t-\tau_2) - c_{33}z^2(t) + c_{34}z(t)y(t-\tau_2) - c_{35}y^2(t-\tau_2) \\ \qquad + \sum_{j+k>2} \frac{y^j}{j!}\frac{z^k}{k!} \cdot \frac{\partial^{j+k}f_2}{\partial y^j \partial z^k}|_{(y^*,z^*)}, \end{cases} \tag{2.3}$$

where $c_{11} = r - \frac{r}{K} - by^*, c_{12} = bx^*, c_{13} = \frac{rx^*}{K}, c_{21} = \frac{a_1x^* - 2a_2y^*}{x^*}, c_{22} = \frac{a_2(y^*)^2}{(x^*)^2}, c_{23} = \frac{a_2}{x^*}, c_{24} = \frac{2a_2y^*}{(x^*)^2}, c_{25} = \frac{a_2(y^*)^2}{(x^*)^2}, c_{31} = \frac{a_3y^* - 2a_4z^*}{y^*}, c_{32} = \frac{a_4(z^*)^2}{(y^*)^2}, c_{33} = \frac{a_4}{y^*}, c_{34} = \frac{2a_4z^*}{(y^*)^2}, c_{35} = \frac{a_4(z^*)^2}{(y^*)^2}$. System (2.3) can be written as a matrix form

$$u'(t) = Au(t) + Bu(t-\tau) + \Phi(u(t), u(t-\tau)), \tag{2.4}$$

where $u(t) = [x(t), y(t), z(t)]^T, u(t-\tau) = [x(t-\tau_1), y(t-\tau_2), z(t-\tau_3)]^T$, $\Phi(u(t), u(t-\tau)) = [-\frac{r}{K}x(t)x(t-\tau_1) - bx(t)y(t), -c_{23}y^2(t) + c_{24}y(t)x(t-\tau_1) - c_{25}x^2(t-\tau_1)$
$+ \sum_{i+j>2} \frac{x^i}{i!}\frac{y^j}{j!} \cdot \frac{\partial^{i+j}f_1}{\partial x^i \partial y^j}|_{(x^*,y^*)}, -c_{33}z^2(t) + c_{34}z(t)y(t-\tau_2) - c_{35}y^2(t-\tau_2)$
$+ \sum_{j+k>2} \frac{y^j}{j!}\frac{z^k}{k!} \cdot \frac{\partial^{j+k}f_2}{\partial y^j \partial z^k}|_{(y^*,z^*)}]^T$, and

$$A = (a_{ij})_{3\times 3} = \begin{pmatrix} c_{11} & -c_{12} & 0 \\ 0 & c_{21} & 0 \\ 0 & 0 & c_{31} \end{pmatrix},$$

$$B = (b_{ij})_{3\times 3} = \begin{pmatrix} -c_{13} & -b_1 & 0 \\ c_{22} & 0 & -b_2 \\ 0 & c_{32} & 0 \end{pmatrix}.$$

The linearized system of (2.4) is the following:

$$u'(t) = Au(t) + Bu(t-\tau). \tag{2.5}$$

It is known that the zero equilibrium point of the system (2.3) (or (2.4)) corresponds the positive equilibrium point (x^*, y^*, z^*) of the equivalent system (1.7). In what follows, we only consider the instability of the zero equilibrium point of the system (2.3).

3 The existence of periodic solution

Since system (2.5) is a linearized system of (2.4). One can see that system (2.4) is a disturbed system of (2.5). If the trivial solution of the system (2.5) is unstable, then the trivial solution of the system (2.4) is also unstable according to the theory of functional differential equations. So, we have the following theorem.

Theorem 1. Assume that the system (2.5) has a unique trivial solution, $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ are characteristic values of matrix A and matrix B, respectively. If there is a characteristic value, say $\alpha_1 > 0$, and $\alpha_1 > |\beta_1|$, or $Re(\alpha_1) > 0$, and $Re(\alpha_1) > |Re(\beta_1)| + |Im(\beta_1)|$. Then the trivial solution of System (2.5) (thus the system (2.3)) is unstable, implying that there exists a limit cycle in the system (1.7), namely, system (1.7) has a periodic solution.

Proof. We will show that the trivial solution of the system (2.5) is unstable. Since $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ are characteristic values of matrix A and matrix B, respectively, then the characteristic equation of the system (2.5) is the following:

$$\Pi_{i=1}^3 \lambda - \alpha_i - \beta_i e^{-\lambda\tau_i} = 0. \tag{3.1}$$

Under the assumption of theorem 1, we are led to an investigation of the nature of the roots for equation

$$\lambda - \alpha_1 - \beta_1 e^{-\lambda\tau_1} = 0. \tag{3.2}$$

If $\alpha_1 > 0$, and $\alpha_1 > |\beta_1|$, we claim that equation (3.2) has a real positive root. Let $g(\lambda) = \lambda - \alpha_1 - \beta_1 e^{-\lambda\tau_1}$. Then $g(\lambda)$ is a continuous function of λ. Noting that $g(0) = -\alpha_1 - \beta_1 \leq -\alpha_1 + |\beta_1| < 0$, there exists some suitably large positive number, say L such that $g(L) = L - \alpha_1 - \beta_1 e^{-L\tau_1} > 0$ since $e^{-L\tau_1} \to 0$ as L is sufficiently large. Based on the Intermediate Value Theorem, there exists a $\lambda_* \in (0, L)$ such that $g(\lambda_*) = 0$. In other words, there is a positive characteristic value of the equation (3.2). If $Re(\alpha_1) > 0$, and $Re(\alpha_1) > |Re(\beta_1)| + |Im(\beta_1)|$. We show that there is a positive real part characteristic value of the equation (3.2). Let $\lambda = \lambda_1 + i\lambda_2, \alpha_1 = \alpha_{11} + i\alpha_{12}, \beta_1 = \beta_{11} + i\beta_{12}$, where $\lambda_1 = Re(\lambda), \alpha_{11} = Re(\alpha_1), \beta_{11} = Re(\beta_1)$, and $\lambda_2 = Im(\lambda), \alpha_{12} = Im(\alpha_1), \beta_{12} = Im(\beta_1)$, respectively. Separating the real part and imaginary part of the equation (3.2) we have

$$\lambda_1 - \alpha_{11} - \beta_{11} e^{-\lambda_1\tau_1} \cos(\lambda_2\tau_1) - \beta_{12} e^{-\lambda_1\tau_1} \sin(\lambda_2\tau_1) = 0, \tag{3.3}$$

$$\lambda_2 - \alpha_{12} - \beta_{12} e^{-\lambda_1\tau_1} \cos(\lambda_2\tau_1) + \beta_{11} e^{-\lambda_1\tau_1} \sin(\lambda_2\tau_1) = 0. \tag{3.4}$$

Let $h(\lambda_1) = \lambda_1 - \alpha_{11} - \beta_{11} e^{-\lambda_1\tau_1} \cos(\lambda_2\tau_1) - \beta_{12} e^{-\lambda_1\tau_1} \sin(\lambda_2\tau_1)$. Then $h(\lambda_1)$ is a continuous function of λ_1. $h(0) = -\alpha_{11} - \beta_{11}\cos(\lambda_2\tau_1) - \beta_{12}\sin(\lambda_2\tau_1) < -\alpha_{11} + |\beta_{11}| + |\beta_{12}| < 0$ since $Re(\alpha_1) > |Re(\beta_1)| + |Im(\beta_1)|$. Obviously, there exists a positive number K such that $h(K) = K - \alpha_{11} - \beta_{11} e^{-K\tau_1} \cos(\lambda_2\tau_1) - \beta_{12} e^{-K\tau_1} \sin(\lambda_2\tau_1) > 0$. Again using the Intermediate Value Theorem, there exists a $\lambda^* \in (0, K)$ such that $h(\lambda^*) = 0$, implying that there is a positive real part characteristic value of the equation (3.2). This means that the trivial solution of the system (2.5) (also the disturbed system (2.4)) is unstable. Equivalently, the unique equilibrium point (x^*, y^*, z^*) of the system (1.7) is unstable. This instability of the unique equilibrium point together with the boundedness of the solutions will force system (1.7) to generate a limit cycle, namely, a periodic solution according to the extended Chafee's criterion [see 25] and the appendix of [26]. The proof is completed.

Let $\mu = \max\{c_{11}, c_{21} + |-c_{12}|, c_{31}\}, \sigma = \max\{|c_{22}| + |-c_{13}|, |c_{32}| + |-b_1|, |-b_2|\}, \tau_* = \min\{\tau_1, \tau_2, \tau_3\}$, then we have the following result.

Theorem 2. Assume that the system (2.5) has a unique trivial solution. If the following condition holds

$$\mu + \sigma > 0. \tag{3.5}$$

Then the trivial solution of the system (2.5) is unstable, implying that there exists a limit cycle of the system (1.7), namely, the system (1.7) has a periodic solution.

Proof. To prove the instability of the trivial solution of the system (2.5), consider a special case of (2.5):

$$u'(t) = Au(t) + Bu(t - \tau_*), \tag{3.6}$$

where $u(t-\tau_*) = [x(t-\tau_*), y(t-\tau_*), z(t-\tau_*)]^T$. Let $w(t) = x(t) + y(t) + z(t)$. Noting that all $x(t), y(t)$, and $z(t)$ are nonnegative for finite time. Therefore, $w(t) > 0$ for $t > 0$, and

$$w'(t) \leq \mu w(t) + \sigma w(t - \tau_*). \tag{3.7}$$

Specifically, consider a scalar equation

$$v'(t) = \mu v(t) + \sigma v(t - \tau_*). \tag{3.8}$$

According to the comparison theory of differential equation we have $w(t) \leq v(t)$. We claim that the trivial solution of the equation (3.8) is unstable. Indeed, the characteristic equation of (3.8) is the follows:

$$\lambda = \mu + \sigma e^{-\lambda\tau_*}. \tag{3.9}$$

Consider a function $\varphi(\lambda) = \lambda - \mu - \sigma e^{-\lambda\tau_*}$. Then $\varphi(\lambda)$ is a continuous function of λ. Noting that $\varphi(0) = -\mu - \sigma = -(\mu + \sigma) < 0$. Obviously, there exists a $M > 0$ such that $\varphi(M) = M - \mu - \sigma e^{-M\tau_*} > 0$. By the Intermediate Value Theorem, there exists a $\lambda_0 \in (0, M)$ such that $\varphi(\lambda_0) = 0$. In other words, there exists a positive characteristic root of the equation (3.9), this means that the trivial solution of the system (3.8) is unstable, implying that the trivial solution of the system (3.7), thus (3.6) is unstable. According to the basic theory of delayed differential equation: if the solution of the system is unstable for small time delay, then the instability of the solution will maintain as delay increases [24]. Therefore, the trivial solution of the system (2.5) is unstable, implying that the unique positive equilibrium point (x^*, y^*, z^*) of the system (1.7) is unstable. Similar to theorem 1, system (1.7) generates a limit cycle, namely, a periodic solution. The proof is completed.

4 Computer simulation result

In model (1.7), we select the parameters as follows: $r = 0.95, K = 25, b = 0.86, b_1 = 0.18, b_2 = 0.32, a_1 = 0.72, a_2 = 0.95, a_3 = 0.68, a_4 = 1.52$, then the unique positive equilibrium point $(x^*, y^*, z^*) = (1.5007, 0.9108, 0.3823)$. Thus, $c_{11} = 0.1287, c_{12} = 1.2906, c_{13} = 0.0570, c_{21} = -0.4335, c_{22} = 0.3503, c_{31} = 0.5548, c_{32} = 0.2710$. The characteristic values of matrix A and matrix B are $0.5548, -0.4335, 0.1287$ and $-0.0332, -0.0119 + 0.3858i, -0.0119 - 0.3858i$, respectively. Since $0.5548 > |-0.0332| > 0$, the condition of Theorem 1 is satisfied. There exist periodic solutions when time delays are selected as $\tau_1 = 0.65, \tau_2 = 0.58, \tau_3 = 0.62$, and $\tau_1 = 0.60, \tau_2 = 0.74, \tau_3 = 0.68$, respectively (see Fig.1). Then we select $r = 1.15, K = 45, b = 0.56, b_1 = 0.48, b_2 = 0.25, a_1 = 0.64, a_2 = 0.78, a_3 = 0.54, a_4 = 1.45$, we see that the unique positive equilibrium point $(x^*, y^*, z^*) = (1.9227, 1.3595, 0.5063)$. Thus, $c_{11} = 0.3628, c_{12} = 1.0767, c_{13} = 0.0493, c_{21} = -0.4875, c_{22} = 0.3873, c_{31} = -0.5262, c_{32} = 0.1960$. The characteristic values of matrix A and matrix B are $0.3628, -0.4875, -0.5262$ and $-0.0103, -0.0195 + 0.4839i, -0.0195 - 0.4839i$, respectively. Since $0.3628 > |-0.0103| > 0$, the condition of Theorem 1 is satisfied. when time delays are selected as $\tau_1 = 0.235, \tau_2 = 0.256, \tau_3 = 0.252$, and $\tau_1 = 0.232, \tau_2 = 0.260, \tau_3 = 0.285$, respectively, there exist periodic solutions (see Fig.2). Finally, we select $r = 1.35, K = 30, b = 0.26, b_1 = 0.45, b_2 = 0.35, a_1 = 0.58, a_2 = 1.18, a_3 = 0.95, a_4 = 1.65$, the unique positive equilibrium point $(x^*, y^*, z^*) = (9.3902, 3.0114, 1.7340)$. Thus, $c_{11} = 0.1287, c_{12} = 1.2906, c_{13} = 0.0570, c_{21} = -0.4335, c_{22} = 0.3503, c_{31} = 0.5548, c_{32} = 0.2710$, and $\mu = 0.8571, \sigma = 0.7210$. The condition of Theorem 2 is satisfied since $\mu + \sigma > 0$. When time delays are selected as $\tau_1 = 0.85, \tau_2 = 0.78, \tau_3 = 0.88$, and $\tau_1 = 0.90, \tau_2 = 0.80, \tau_3 = 0.85$, respectively, there exist positive periodic solutions (see Fig.3). From the figures, we see that the more the value of r, the greater the value of x^*. Time delays affect the oscillatory frequency not too much. Even if the parameter values of the first figure and the second figure are so different from each other, their oscillatory frequencies are almost the same, implying that the construction of the solutions is complex.

5 Conclusion

The present paper discusses a predator-prey model with three different time delays by means of the extended Chafee's criterion. Two sufficient conditions are provided to guarantee the existence of positive periodic solutions. We change the nonlinear predator-prey model to a equivalent system. The original positive equilibrium point changes to the zero equilibrium point of the

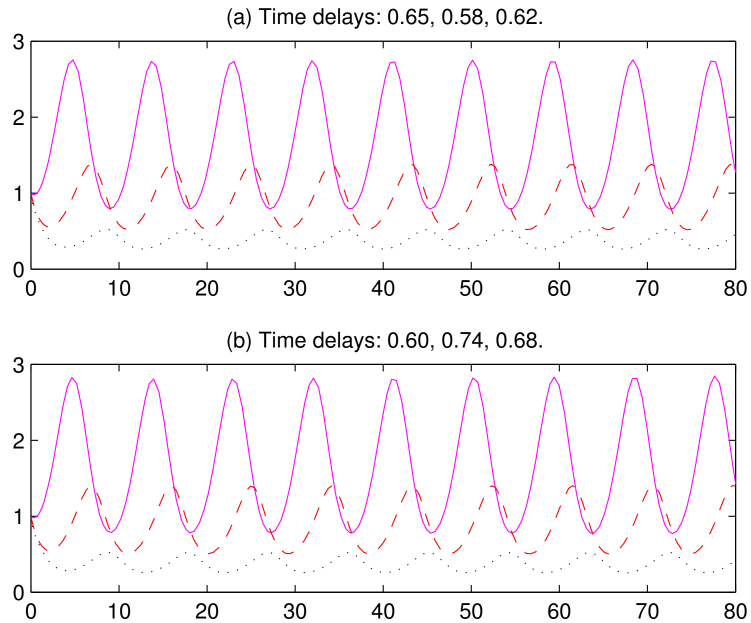

Fig. 1 There is a periodic solution, solid line: x(t), dashed line: y(t), dotted line: z(t).

equivalent system. The instability of the trivial solution of the equivalent system implies the instability of the positive equilibrium point of the original system. Our criterion is easy to check compared to the bifurcation method.

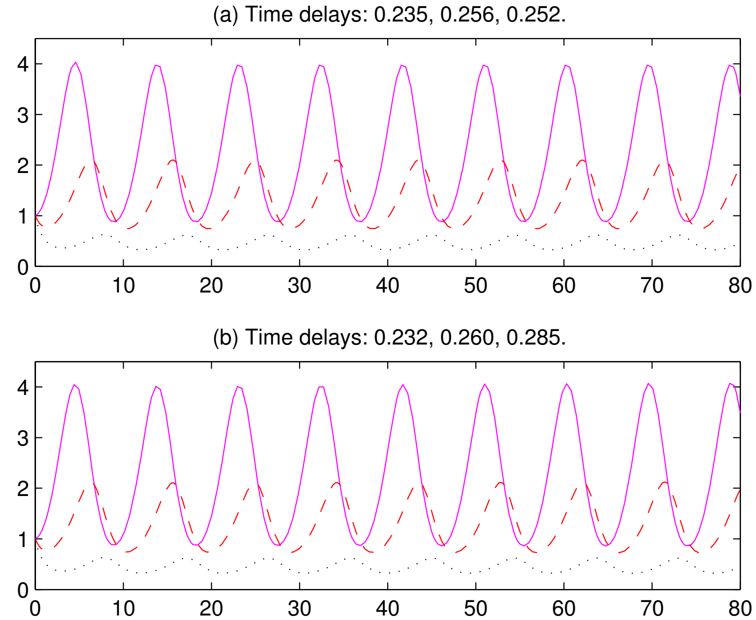

Fig. 2 There is a periodic solution, solid line: x(t), dashed line: y(t), dotted line: z(t).

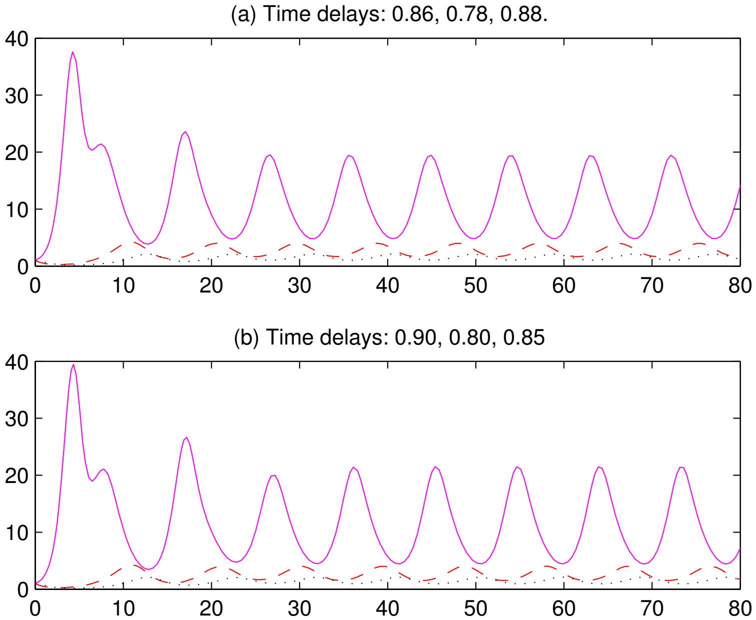

Fig. 3 There is a periodic solution, solid line: x(t), dashed line: y(t), dotted line: z(t).

References

[1] A. Korobeinikov, A Lyapunov function for Leslie-Gower predator-prey models, Applied Mathematics Letters, **14** (2001), 697-699.

[2] L. Chen and F. Chen, Global stability of a Leslie-Gower predator-prey model with feedback controls, Applied Mathematics Letters, **22** (2009), 1330-1334.

[3] S. Sharma and G. P. Samanta, A Leslie-Gower predator-prey model with disease in prey incorporating a prey refuge, Chaos, Solitons and Fractals, **70** (2015), 69-84.

[4] Q. Yue, Dynamics of a modified Leslie-Gower predator-prey model with Holling-type II schemes and a prey refuge, SpringerPlus, (2016), article ID: 14276504.

[5] A. F. Nindjin, M. A. Alaoui and M. Cadivel, Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay, Nonlinear Analysis: RWA, **7** (2005), 1104-1118.

[6] D. Adak, N. Bairagi and R. Hakl, Chaos in delay-induced Leslie-Gower prey-predator-parasite model and its control through prey harvesting, Nonlinear Analysis: RWA, (2020), article ID: 102998.

[7] L. Guo, Z. Song and J. Xu, Complex dynamics in the Leslie-Gower type of the food chain system with multiple delays, Commun Nonliear Sci Numer Simulat, **19** (2014), 2850-2865.

[8] R. K. Upadhyay, S. Mishra1, Y. Dong and Y. Takeuchi, Exploring the dynamics of a tritrophic food chain model with multiple gestation periods, Mathematical Biosciences and Engineering, **16** (2019), 4660-4691.

[9] Q. Liu, Y. Lin and J. Cao, Global Hopf bifurcation on two-Ddelays Leslie-Gower predator-prey system with a prey refuge, Computational and Mathematical Methods in Medicine, (2014), article ID: 619132.

[10] L. Li and Y. Zhang, Dynamic analysis and Hopf bifurcation of a Lengyel-Epstein system with two delays, Journal of Mathematics, (2021), article ID: 5554562.

[11] S. Wang, H. Tang and Z. Ma, Hopf bifurcation of a multiple-delayed predator-prey system with habitat complexity, Mathematics and Computers in Simulation, **180** (2021), 1-23.

[12] H. J. Alsakaji, S. Kundu and F. A. Rihan, Delay differential model of one-predator two-prey system with Monod-Haldane and holling type II functional responses, Applied Mathematics and Computation, **397** (2021), article ID: 125919.

[13] J. Banerjee, S. K. Sasmal and R. K. Layek, Supercritical and subcritical Hopf-bifurcations in a two-delayed prey-predator system with density-dependent mortality of predator and strong Allee effect in prey, BioSystems, **180** (2019), 19-37.

[14] A. F. Nindjin and M. A. Alaoui, Persistence and global stability in a delayed Leslie-Gower type three species food chain, Journal of Mathematical Analysis and Applications, **340** (2008), 340-357.

[15] H. F. Huo and W. T. Li, Periodic solutions of delayed Leslie-Gower predator-prey models, Applied Mathematics and Computation, **155** (2004), 591-605.

[16] L. Chen, J. Xu and Z. Li, Permanence and global attractivity of a delayed discrete predator-prey system with general Holling-type functional response and feedback controls, Discrete Dynamics in Nature and Society, (2008), article ID: 629620.

[17] J. Jiang, Y. Song, P. Yu, Delay induced triple-zero bifurcation in a delayed Leslie-type predator-prey model with additive Allee effect, International Journal of Bifurcation and Chaos, **26** (2016), article ID: 1650117.

[18] J. Jiao, R. Wang, H. Chang and X. Liu, Codimension bifurcation analysis of a modified Leslie-Gower predator-prey model with two delays, International Journal of Bifurcation and Chaos, **28** (2018), article ID: 1850060.

[19] X. Song and Y. Li, Dynamic behaviors of the periodic predator-prey model with modified Leslie-Gower Holling-type II schemes and impulsive effect, Nonlinear Analysis: RWA, **9** (2008), 64-79.

[20] L. Hu, L. Nie, Permanence and global stability for a nonautonomous predator-prey model with modified Leslie-Gower and Holling-type II Schemes with delays, Applied Mathematics, **2** (2011), 47-56.

[21] A. Mondal, A. K. Pal and G. P. Samanta, Stability and bifurcation analysis of a delayed three species food chain model with Crowley-Martin response function, Applications and Applied Mathematics, An International Journal, **13** (2018), 709-749.

[22] R. Yafia, F. E. Adnant and H. T. Alaoui, Stability of limit cycle in a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay, Applied Mathematics Sciences, **1** (2007), 119-131.

[23] Z. Guo, H. Huo, Q. Ren and H. Xiang, Bifurcation of a modified Leslie-Gower system with discrete and distributed delays, Journal of Nonlinear Modeling and Analysis, **1** (2019), 73-91.

[24] J. K. Hale and S. M. Verduyn, *Introduction to functional differential equations*, New York, Springer, 1993.

[25] N. Chafee, A bifurcation problem for a functional differential equation of finite retarded type, Journal of Mathematical Analysis and Applications, **35** (1971), 312-348.

[26] C. Feng and R. Plamondon, An oscillatory criterion for a time delayed neural network model, Neural Networks, **29** (2012), 70-79.

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