Normalized null hypersurfaces in the Lorentz-Minkowski space satisfying $L_r x = Ux + b$

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Abstract.   In the present paper, we classify all normalized null hypersurfaces $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ endowed with UCC-normalization with vanishing $1-$form τ, satisfying $L_r x = Ux + b$ for some (field of) screen constant matrix $U \in \mathbb{R}^{(n+2)\times(n+2)}$ and vector $b \in \mathbb{R}_1^{n+2}$, where L_r is the linearized operator of the $(r+1)th-$mean curvature of the normalized null hypersurface for $r = 0, ..., n$. For $r = 0$, $L_0 = \Delta^\eta$ is nothing but the (pseudo-)Laplacian operator on (M, g, N). We prove that the lightcone Λ_0^{n+1}, lightcone cylinders $\Lambda_0^{m+1} \times \mathbb{R}^{n-m}$, $1 \le m \le n-1$ and $(r+1)-$maximal Monge null hypersurfaces are the only UCC-normalized Monge null hypersurface with vanishing normalization $1-$form τ satisfying the above equation. In case U is the (field of) scalar matrix λI, $\lambda \in \mathbb{R}$ and hence is constant on the whole M, we show that the only normalized Monge null hypersurfaces $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ satisfying $\Delta^\eta x = \lambda x + b$, are open pieces of hyperplanes.

Keywords. Normalized null hypersurface, Second order operator, Newton transformation, Higher order mean curvature

1   Introduction

Isometric immersions in Euclidean spaces satisfying $\Delta x = Ax + b$ where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a constant matrix, $b \in \mathbb{R}^{n+1}$ is a constant vector and Δ the Laplacian operator with respect to the induced metric have always been subject to many investigations : Tahahashi ($A = \lambda Id$ and $b = 0$ [35]), Garay ([24] for hypersurfaces), Dillen, Pas and Verstraelen (surfaces in the specific case $n + 1 = 3$ [17]), Hasanis and Vlachos [25], and Chen and Petrovic [16].

In the work by Alías and Ferrández [3] the Euclidean target space is replaced by a pseudo-Euclidean one. They considered pseudo-Riemannian submanifolds M_s^n in pseudo-Euclidean spaces $\mathbb{R}_t^{n+m}$ satisfying the condition $\Delta x = Ax + B$, where A is a constant endomorphism of $\mathbb{R}_t^{n+m}$ and B is a constant vector in $\mathbb{R}_t^{n+m}$, and gave a characterization theorem. For hypersurfaces they show that M_s^n must be an open piece of a minimal hypersurface, a totally umbilical hypersurface or a pseudo-Riemannian product of a totally umbilical and a totally geodesic submanifold.

The Laplacian operator Δ can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, ..., L_{n-1}$, where L_k stands for the linearized operator of the first variation of the $(k+1)-$th

mean curvature arising from normal variations of the hypersurface. These operators are given for their action on smooth function f on M by $L_k(f) = tr(P_k \circ \nabla^2 f)$, where P_k denotes the $k-$th Newton transformation associated with the second fundamental form of the hypersurface and $\nabla^2 f$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f. Alías and Gürbüz [1] initiated the study of hypersurfaces in Euclidean space satisfying the general condition $L_k x = Ax + b$, where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a constant matrix and $b \in \mathbb{R}^{n+1}$ is a constant vector. A first attempt to solve this question have been made by Yang and Liu [36].

Pascual Lucas and Fabián Ramírez [28] studied pseudo-Riemannian hypersurfaces in Lorentz-Minkowski space satisfying the condition $L_k x = Ax + b$, where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a constant matrix and $b \in \mathbb{L}^{n+1}$ is a constant vector. These authors showed that a pseudo-Riemannian hypersurface $x : M \to \mathbb{L}^{n+1}$ satisfies this condition if and only if M is an $r-$maximal hypersurface, an open piece of the totally umbilical hypersurface $\mathbb{S}_1^n(r)$ or $\mathbb{H}^n(-r)$, or an open piece of generalized cylinder $\mathbb{S}_1^m(r) \times \mathbb{R}^{n-m}$ or $\mathbb{H}^m(-r) \times \mathbb{R}^{n-m}$, with $k+1 \le m \le n-1$, $r > 0$ is a constant, or $\mathbb{L}^m \times \mathbb{S}^{n-m}(r)$ with $r+1 \le n-m \le n-1$. Pascual Lucas and Fabián Ramírez also studied this problem in another space [29, 30]. As it can be remarkable, the case where the hypersurface is null (lightlike) has been avoided in all of the above discussions. It is the purpose of this paper to start filling this gap, relying on the Newton transformations that the first and the second authors have previously introduced in [6] and on pseudo-Laplacian operator as defined in [9].

In Section 2 we set notations and recall necessary materials on null hypersurfaces. Section 3 focuses on almost isoparametric null hypersurfaces and establishes sufficient conditions under which there is at most two screen principal curvatures. In Section 4 we consider solving the problem $\Delta^\eta x = \lambda x + b$ in Minkowski spaces $\mathbb{R}_1^{n+2}$ (Theorem 4.1). In Section 5 we introduce the second-order linear differential operators L_r ($0 \le r \le n$) and study some of their properties. Section 6 of the present paper focuses on Monge null hypersurfaces endowed with normalization (2.2) with specific properties. We show that the only Monge null hypersurfaces endowed with the normalization (2.2), satisfying $L_r x = Ux + b$ are the $(r+1)-$maximal ones.

2 Background materials on null hypersurfaces

Throughout this work, $(\overline{M}, \overline{g})$ is an $(n+2)-$dimensional Lorentzian manifold, $\overline{\nabla}$ and $\overline{R}$ will denote respectively the Levi-Civita connection and the Riemannian curvature of $\overline{g}$. (Tools of the metric $\overline{g}$ will be surmounted with a line.) All manifolds are taken smooth and connected. Let $x : (M, g) \longrightarrow (\overline{M}, \overline{g})$ be a null hypersurface isometrically immersed. At each $p \in M$, the restriction $\overline{g}_{p|T_pM}$ is degenerate, that is there exists a non-zero vector $U \in T_pM$ such that $\overline{g}(U, X) = 0$ for all $X \in T_pM$. By *screen distribution* on M^{n+1}, we mean a complementary bundle of $TM^\perp$ in TM. It is then a rank n nondegenerate distribution over M. For reasons that will become obvious in few lines below, let denote such a distribution by $\mathscr{S}(N)$. We then have,

$$TM \quad = \quad \mathscr{S}(N) \oplus_{Orth} TM^\perp, \tag{2.1}$$

where $\oplus_{Orth}$ denotes the orthogonal direct sum. From [18], it is known that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^\perp$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathscr{U}$ satisfying

$$\overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \qquad \forall \, W \in \mathscr{S}(N)|_{\mathscr{U}}. \tag{2.2}$$

Then $T\overline{M}$ is decomposed as follows:

$$T\overline{M}|_M \;=\; TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{Orth} \mathscr{S}(N). \tag{2.3}$$

We call $tr(TM)$ a *(null) transversal vector bundle* along M. In fact, from (2.2) and (2.3) one shows that, conversely, a choice of a transversal bundle $tr(TM)$ determines uniquely the screen distribution $\mathscr{S}(N)$. A vector field N as in (2.2) is called a *null transversal vector field* of M. It is then noteworthy that the choice of a null transversal vector field N along M determines both the null transversal vector bundle, the screen distribution $\mathscr{S}(N)$ and a unique radical vector field, say ξ, satisfying (2.2). Tangent vector fields to $\mathscr{S}(N)$ (resp. to $TM^{\perp}$) are called horizontal (resp. vertical). Now, to continue our discussion, we need to clarify the concept of rigging for our null hypersurface.

Definition 1. Let M be a null hypersurface of a Lorentzian manifold. A rigging for M is a vector field L defined on some open set containing M such that $L_p \notin T_pM$ for each $p \in M$.

Let N be a null rigging for M (that means the restriction of N on M is a null vector field) and $\theta = \overline{g}(N, \cdot)$ the $1-$form metrically equivalent to N defined on $\overline{M}$. Then, take $\eta = x^{\star}\theta$ to be its restriction to M. The normalization (M, g, N) will be said to be *closed* if the $1-$form θ is closed which implies that the same is for η on M. It is easy to check that $\mathscr{S}(N) = \ker(\eta)$ and that the screen distribution $\mathscr{S}(N)$ is integrable whenever η is closed. On a normalized null hypersurface (M, g, N), the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y \;=\; \nabla_X Y + B^N(X,Y)N, \tag{2.4}$$
$$\overline{\nabla}_X N \;=\; -A_N X + \tau^N(X)N, \tag{2.5}$$
$$\nabla_X PY \;=\; \overset{\star}{\nabla}_X PY + C^N(X, PY)\xi, \tag{2.6}$$
$$\nabla_X \xi \;=\; -\overset{\star}{A}_{\xi} X - \tau^N(X)\xi, \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $(\overline{M}, \overline{g})$, ∇ denotes the connection on M induced from $\overline{\nabla}$ through the projection along the rigging N and $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathscr{S}(N)$ induced from ∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma(\mathscr{S}(N))$ with respect to the decomposition (2.1). Now the $(0,2)$ tensors B^N and C^N are the second fundamental forms on TM and $\mathscr{S}(N)$ respectively, A_N and $\overset{\star}{A}_{\xi}$ are the shape operators on TM and $\mathscr{S}(N)$ respectively and τ^N a $1-$form on TM defined by $\tau^N(X) = \overline{g}(\overline{\nabla}_X N, \xi)$. For the second fundamental forms B^N and C^N the following holds

$$B^N(X,Y) = g(\overset{\star}{A}_{\xi} X, Y), \quad C^N(X, PY) = g(A_N X, Y) \quad \forall X, Y \in \Gamma(TM), \tag{2.8}$$

and

$$B^N(X, \xi) = 0, \quad \overset{\star}{A}_{\xi}\xi = 0 \qquad \forall X \in \Gamma(TM). \tag{2.9}$$

A null hypersurface M is said to be *totally umbilical* (resp. *totally geodesic*) if there exists a smooth function ρ on M such that at each $p \in M$ and for all $u, v \in T_pM$, $B^N(p)(u,v) = \rho(p)g(u,v)$ or equivalently $\overset{\star}{A}_{\xi} = \rho P$ (resp. B^N vanishes or equivalently $\overset{\star}{A}_{\xi} = 0$). These are intrinsic notions on any null hypersurface in the sense that they don't depend on the chosen null rigging. Also, the screen distribution $\mathscr{S}(N)$ is *totally umbilical* (resp. *totally geodesic*) if $C^N(X, PY) = \lambda g(X, Y)$ for all $X, Y \in \Gamma(TM)$ (resp. $C^N = 0$), which is equivalent to $A_N = \lambda P$ (resp. $A_N = 0$).

The induced connection ∇ is torsion-free, but not necessarily $g-$metric unless M be totally geodesic. In fact we have for all tangent vector fields X, Y and Z in TM,

$$(\nabla_X g)(Y, Z) = B^N(X, Y)\eta(Z) + B^N(X, Z)\eta(Y). \tag{2.10}$$

Also, due to the degeneracy of the induced metric g on the null hypersurface M, it is not possible to define the natural dual (musical) isomorphisms $\flat$ and $\sharp$ between the tangent vector bundle TM and its dual $T^\star M$ following the usual Riemannian way. However, this construction is made possible by setting a rigging (normalization) N. Consider a normalized null hypersurface (M, g, N) and define

$$\flat_\eta : \Gamma(TM) \longrightarrow \Gamma(T^*M), \quad X \longmapsto X^{\flat_\eta} = g(\ X\ ,\ \bullet\ ) + \eta(X)\eta. \tag{2.11}$$

Clearly, such a $\flat_\eta$ is an isomorphism of $\Gamma(TM)$ onto $\Gamma(T^*M)$, and can be used to generalize the usual nondegenerate theory. Define a $(0, 2)-$tensor by $g_\eta(X, Y) = X^{\flat_\eta}(Y)$, for all $X, Y \in \Gamma(TM)$, i.e

$$g_\eta = g + \eta \otimes \eta. \tag{2.12}$$

Clearly, g_η defines a nondegenerate metric on M which plays an important role in defining the usual differential operators *gradient*, *divergence*, *Laplacian* with respect to the degenerate metric g on null hypersurfaces (see [10] for details). It is called the *associated metric* to g on the rigged null hypersurface (M, g, N). The following verifications are straightforward,

$$g_\eta(\xi, X) = \eta(X), \quad g_\eta(X, Y) = g(X, Y)\ \ \forall X \in \Gamma(TM)),\ \forall Y \in \Gamma(\mathscr{S}(N)). \tag{2.13}$$

In particular $g_\eta(\xi, \xi) = 1$ and last equality in (2.13) is telling us that restrict to $\mathscr{S}(N)$ the metrics g_η and g coincide. We will use the following member of the Gauss-Codazzi equations [18, p. 93]

$$\left\langle \overline{R}(X, Y)Z, \xi \right\rangle = (\nabla_X B^N)(Y, Z) - (\nabla_Y B^N)(X, Z) + \tau^N(X)B^N(Y, Z) - \tau^N(Y)B^N(X, Z). \tag{2.14}$$

We conclude this section by recalling some results of Atindogbé et al. (2015). A proof of the following Lemma can be found in [6].

Lemma 2.1. *For all $X, Y \in \Gamma(TM)$,*

$$\left\langle A_N X, Y \right\rangle - \left\langle X, A_N Y \right\rangle = \tau^N(X)\eta(Y) - \tau^N(Y)\eta(X) - 2d\eta(X, Y), \tag{2.15}$$

where (throughout) $\left\langle\ ,\ \right\rangle = \overline{g}$ stands for the ambient Lorentzian metric.

In case the normalization is closed the (connection) $1-$form τ^N is related to the shape operator of M as follows.

Lemma 2.2. *Let (M, g, N) be a closed normalization of a null hypersurface M in a Lorentzian manifold.*

(a) *$\tau^N|_{\mathscr{S}(N)} = 0$ iff $A_N \xi = 0$ (or equivalently $C^N(\xi, \cdot) = 0$).*

(b) *The dual vector field of the connection one-form τ^N with respect to the rigging N is $-A_N\xi$, in particular $\tau^N(\xi) = 0$ iff*

$$\tau^N = -\left\langle A_N\xi\ ,\ \cdot\ \right\rangle. \tag{2.16}$$

Proof. Assume η is closed and let X, Y be tangent vector fields to M. The condition $X \cdot \eta(Y) - Y \cdot \eta(X) - \eta([X,Y]) = 0$ is equivalent to $\left\langle \overline{\nabla}_X N, Y \right\rangle = \left\langle \overline{\nabla}_Y N, X \right\rangle$. Then by the Weingarten formula, we get $\left\langle -A_N X, Y \right\rangle + \tau^N(X)\eta(Y) = \left\langle -A_N Y, X \right\rangle + \tau^N(Y)\eta(X)$. In this relation, take $Y = \xi$ to get $\tau^N(X) = -\left\langle A_N\xi, X \right\rangle + \tau^N(\xi)\eta(X)$ and the claims follow. $\qquad\square$

As $\overset{\star}{A}_\xi$ is self-adjoint linear operator on each fiber T_pM with $\overset{\star}{A}_\xi \, \xi = 0$ then $\overset{\star}{A}_\xi$ is diagonalizable and have $(n+1)$ real-valued eigenfunctions $\overset{\star}{k}_0 = 0, \overset{\star}{k}_1, ..., \overset{\star}{k}_n$. We denote by $(\overset{\star}{E}_0 = \xi, \ \overset{\star}{E}_1, \ ..., \ \overset{\star}{E}_n)$ the corresponding quasi-orthonormal basis of eigenvectors fields. The $r - th$ mean curvature of the null hypersurface with respect to the shape operator $\overset{\star}{A}_\xi$ is given by

$$\overset{\star}{H}_r = \binom{n+1}{r}^{-1} \sigma_r(\overset{\star}{k}_0, ..., \overset{\star}{k}_n) \ \text{ and } \ \overset{\star}{H}_0 = 1 \text{ (constant function 1)},$$

where σ_r is the $r - th$ elementary symmetric polynomial. We set $\overset{\star}{S}_r = \sigma_r(\overset{\star}{k}_0, ..., \overset{\star}{k}_n)$ and $\overset{\star}{S}_r^{\alpha} = \sigma_r(\overset{\star}{k}_0, ..., \overset{\star}{k}_{\alpha-1}, \overset{\star}{k}_{\alpha+1}, ..., \overset{\star}{k}_n)$.

Definition 2 ($r-$maximality). Let $1 \leq r \leq n+1$ be an integer. A null hypersurface M with $\overset{\star}{H}_r = 0$ is said to be $r-$maximal.

For $0 \leq r \leq n+1$, the $r - th$ Newton transformation $\overset{\star}{T}_r$ with respect to the shape operator $\overset{\star}{A}_\xi$ is the $\text{End}(\Gamma(TM))$ element given by

$$\overset{\star}{T}_r = \sum_{a=0}^{r}(-1)^a \, \overset{\star}{S}_a \overset{\star}{A}_\xi^{r-a}.$$

Inductively,

$$\overset{\star}{T}_0 = I \ \text{ and } \ \overset{\star}{T}_r = (-1)^r \overset{\star}{S}_r I + \overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1},$$

where I denotes the identity of $\Gamma(TM)$ and $\overset{\star}{T}_{n+1} = 0$ (from Cayley-Hamilton theorem). By algebraic computations, one shows the following.

Proposition 2.1 ([6]). *1. $\overset{\star}{T}_r$ is self-adjoint and commute with $\overset{\star}{A}_\xi$;*

2. $\overset{\star}{T}_r \, \overset{\star}{E}_\alpha = (-1)^r \, \overset{\star}{S}_r^{\alpha} \, \overset{\star}{E}_\alpha$;

3. $tr(\overset{\star}{T}_r) = (-1)^r(n+1-r) \, \overset{\star}{S}_r$;

4. $tr\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1} \right) = (-1)^{r-1}r \, \overset{\star}{S}_r$;

5. $tr\left(\overset{\star}{A}_\xi^2 \circ \overset{\star}{T}_{r-1} \right) = (-1)^{r-1}\left(\overset{\star}{S}_1 \overset{\star}{S}_r - (r+1) \, \overset{\star}{S}_{r+1}\right)$;

6. $tr(\overset{\star}{T}_{r-1} \circ \nabla_X \overset{\star}{A}_\xi) = (-1)^{r-1}X(\overset{\star}{S}_r)$.

We also proved the following two lemmas in [6], the second being derived from the first one by taking $r = 1$.

Lemma 2.3 ([6]). *Let* $x : (M, g) \hookrightarrow \mathbb{R}_1^{n+2}$ *be a null hypersurface of the Euclidean space* $\mathbb{R}_1^{n+2}$. *Then for* $r = 1, ..., n+1$,

$$(-1)^{r-1}\xi(\overset{\star}{S}_r) + \tau(\xi)tr(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}) - tr(\overset{\star}{A}_\xi^2 \circ \overset{\star}{T}_{r-1}) = 0. \tag{2.17}$$

Lemma 2.4 ([6]). *Let* $x : (M, g) \hookrightarrow \mathbb{R}_1^{n+2}$ *be a null hypersurface of the pseudo-Euclidean space* $\mathbb{R}_1^{n+2}$. *Then* M *is maximal if and only if* M *is totally geodesic.*

Let $(X_0 = \xi, X_1, ..., X_n)$ be a g_η−orthonormal basis of $\Gamma(TM)$ with $\text{span}\{X_1, ..., X_n\} = \mathscr{S}(N)$. The divergence of the operator $\overset{\star}{T}_r$ is the vector field $div^\nabla \left(\overset{\star}{T}_r\right) \in \Gamma(TM)$ define as the trace of the $End(TM)$−valued operator $\nabla\overset{\star}{T}_r$ and given by

$$div^\nabla \left(\overset{\star}{T}_r\right) = tr(\nabla\overset{\star}{T}_r) = \sum_{a,b=0}^{n} g_\eta^{ab}(\nabla\overset{\star}{T}_r)(X_a, X_b) = \sum_{a=0}^{n}(\nabla_{X_a}\overset{\star}{T}_r)X_a. \tag{2.18}$$

3   (Almost) Isoparametric normalized null hypersurfaces

A nondegenerate hypersurface M in a real space-form $Q(c)$ of constant sectional curvature c is said to be isoparametric if it has constant principal curvatures. An isoparametric hypersurface M in $\mathbb{R}^n$ can have at most two different principal curvatures, and M must be an open subset of a hyperplane, hypersphere or a spherical cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k-1}$. This was shown by Levi-Civita [26] for $n = 3$ and by B. Segre [33] for arbitrary n. Similarly, E. Cartan [12] proved that an isoparametric hypersurface M in a hyperbolic space H^n can have at most two different principal curvatures, and M must be either totally umbilic or else be an open subset of a standard product $\mathbb{S}^k \times \mathbb{H}^{n-k-1}$ in H^n.

Definition 3.   • A normalized null hypersurface $x : (M, g, N) \to (\overline{M}^{n+2}, \overline{g})$ isometrically immersed into a Lorentzian manifold, is said to be **isoparametric** if the screen principal curvatures (eigenfunctions of $\overset{\star}{A}_\xi$) are constants.

• A normalized null hypersurface $x : (M, g, N) \to (\overline{M}^{n+2}, \overline{g})$ isometrically immersed into Lorentzian manifold, is said to be **almost isoparametric** if the screen distribution $\mathscr{S}(N)$ is integrable and all the screen principal curvatures are constant on each leaf of $\mathscr{S}(N)$.

Example 1. Every totally geodesic null hypersurface is isoparametric.

In [3] it is shown that $M = \{(x^0, ..., x^5) \in \mathbb{R}^6 ; \ x^0 + x^1 = 0\}$ is a null hypersurface of the 6−dimensional real space $\overline{M} = \mathbb{R}^6$ endowed with the Lorentzian metric

$$\overline{g} = -(dx^0)^2 + (dx^1)^2 + \exp 2x^0[(dx^2)^2 + (dx^3)^2] + \exp 2x^1[(dx^4)^2 + (dx^5)^2],$$

and for the null rigging $N = -\frac{1}{2}\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right)$ with corresponding rigged vector field $\xi = \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}$, the screen distribution is $\mathscr{S}(N) = span\{\overset{\star}{E}_1, \overset{\star}{E}_2, \overset{\star}{E}_3, \overset{\star}{E}_4\}$ with

$$\overset{\star}{E}_1 = e^{-2x^0}\frac{\partial}{\partial x^2}, \quad \overset{\star}{E}_2 = e^{-2x^0}\frac{\partial}{\partial x^3}, \quad \overset{\star}{E}_3 = e^{-2x^1}\frac{\partial}{\partial x^4}, \quad \overset{\star}{E}_4 = e^{-2x^1}\frac{\partial}{\partial x^5},$$

and corresponding principal curvatures are $\overset{\star}{k}_1 = -1 = \overset{\star}{k}_2$, $\overset{\star}{k}_3 = 1 = \overset{\star}{k}_4$ are all constant. Hence, (M, g, N) is an (almost) isoparametric null hypersurface.

Let us recall the following result which gives Cartan's identity for null hypersurfaces.

Theorem 3.1. *[7] Let (M, g, N) be a lightlike hypersurface of an $(n+2)-$dimensional Lorentzian space-form $(\overline{M}(c), \overline{g})$ with $\tau = 0$. Assume that $E_0 = \xi, E_1, ..., E_n$ are eigenfunctions of $\overset{\star}{A}_\xi$ satisfying $\overset{\star}{A}_\xi E_i = \lambda_i E_i$ $(i \geq 1)$ and λ_i is constant along $\mathscr{S}(N)$. Then for every $i \geq 1$,*

$$\sum_{j=1, \lambda_j \neq \lambda_i}^{n} \frac{c + \lambda_j g(A_N E_i, E_i) + \lambda_i g(A_N E_j, E_j)}{\lambda_i - \lambda_j} = 0.$$

In [32, 34] it can be seen some generalizations of these Cartan identities for null hypersurfaces. We use Cartan's identity to prove the following Lemma.

Lemma 3.1. *Let $x : (M, g, N) \to \overline{M}_1^{n+2}(c)$ be an almost isoparametric normalized null hypersurface isometrically immersed into a Lorentzian manifold with constant sectional curvature $c \leq 0$. If there exists a conformal screen (re-)normalization with vanishing $1-$form τ, then M has at most two different screen principal curvatures. In particular when $c = 0$, M has at most one non-zero screen principal curvature and when $c < 0$, M has exactly two or no non-zero screen principal curvatures.*

Proof. Let $x : (M, g, N) \to Q_1^{n+2}(c)$ be a almost isoparametric normalized null hypersurface with conformal screen distribution $(A_N = \phi \overset{\star}{A}_\xi)$ and $\tau = 0$. Let $\lambda_1, ..., \lambda_p$ be all distinct screen principal curvatures of the sharpe operator $\overset{\star}{A}_\xi$, with algebraic multiplicities $\nu_1, ..., \nu_p$. By the previous Theorem for any $i = 1, ..., p$ Cartan identity can be write as

$$\sum_{j=1, j \neq i}^{p} \nu_j \frac{c + 2\phi \lambda_j \lambda_i}{\lambda_i - \lambda_j} = 0. \tag{3.1}$$

Without loss of generalities, we may assume $\lambda_1 < \lambda_2 < \cdots < \lambda_p$, and $\lambda_p \geq 0$. Choose the largest nonnegative λ_i such that $2\phi \lambda_i \lambda_{i-1} \leq c$. Then

$$\frac{c + 2\phi \lambda_j \lambda_i}{\lambda_i - \lambda_j} \leq 0$$

for any $j \neq i$. Hence $2\phi \lambda_i \lambda_j = c$ if $i \neq j$. Therefore $p \leq 2$. $\qquad\square$

The following theorem due to [7], classifies almost isoparametric normalized null hypersurface endowed with a screen conformal normalization.

Theorem 3.2. *[7] Let $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ be a almost isoparametric normalized null hypersurface endowed with a screen conformal normalization. Then, M is either a proper totally umbilical or totally geodesic null hyersurface or an open piece of a null triplet product $C \times M_\lambda^{n-r} \times \mathbb{R}^r$, where C is a null curve and M_λ^{n-r} is a totally umbilical spacelike submanifold of $\mathbb{R}_1^{n+2}$.*

4   Normalized null hypersurfaces $x : (M, g) \to \mathbb{R}_1^{n+2}$ satisfying $\Delta^\eta x = \lambda x + b,\;\; \lambda \in \mathbb{R}, b \in \mathbb{R}_1^{n+2}$

From now on, we set $\overline{M} = \mathbb{R}_1^{n+2}$ and

$$\overline{g} = \langle \cdot, \cdot \rangle := -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^{n+1})^2, \tag{4.1}$$

denotes the Minkowski metric, with $(x^0, ..., x^{n+1})$ the usual Cartesian coordinates system of $\mathbb{R}_1^{n+2}$.

Let $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ be a normalized null hypersurface of $\mathbb{R}_1^{n+2}$ and f be a smooth function on M. We define its *pseudo-gradient* to be its gradient with respect to the associated (nondegenerate) metric g_η and denote it $\nabla^\eta f$. We also define its *pseudo-Hessian* to be the linear operator $\nabla_\eta^2 f : \Gamma(TM) \to \Gamma(TM)$ defined by

$$\nabla_\eta^2 f(X) = \nabla_X \nabla^\eta f, \quad \forall X \in \Gamma(TM).$$

For a linear connection D on the vector bundle TM and $X \in \Gamma(TM)$, the trace of the $\Gamma(TM)-$endomorphism $DX : Y \to D_Y X$ gives the divergence of X with respect to D, i.e

$$div^D(X) := trace(DX). \tag{4.2}$$

Definition 4. Let f be a smooth function on the normalized null hypersurface (M, g, N). The pseudo-Laplacian of the first kind $\Delta_1^\eta f$ and second kind $\Delta_2^\eta f$ are defined respectively by

$$\Delta_1^\eta f := div^\nabla(\nabla^\eta f), \tag{4.3}$$
$$\Delta_2^\eta f := div^{\nabla^\eta}(\nabla^\eta f). \tag{4.4}$$

Remark 1. The Laplacian of second kind of f is just its Laplacian with respect to the associated (Riemannian) metric g_η. The pseudo-Laplacian of first kind will be simply denoted Δ^η.

Let $a \in \mathbb{R}^{n+2}$ be a fixed vector. Then, $\langle x, a \rangle \in \mathcal{C}^\infty(M)$ and $\forall X \in \Gamma(TM)$,

$$g_\eta(\nabla^\eta \langle x, a \rangle, X) = X \cdot \langle x, a \rangle = \langle X, a \rangle = g_\eta(Pa^\top + \langle \xi, a \rangle \xi, X),$$

where $a^\top$ is the projection of a onto TM with respect to the decomposition (2.1). Thus,

$$\nabla^\eta \langle x, a \rangle = Pa^\top + \langle \xi, a \rangle \xi = a - \langle N, a \rangle \xi - \langle \xi, a \rangle N + \langle \xi, a \rangle \xi, \tag{4.5}$$

and for all $X \in \Gamma(TM)$, by using (2.6), (2.7), we get

$$\begin{aligned}
\nabla_X \nabla^\eta \langle x, a \rangle &= \langle A_N(X) - \overset{\star}{A}_\xi (X) - 2\tau(X)\xi, a \rangle \xi \\
&+ \langle N, a \rangle \overset{\star}{A}_\xi (X) + (A_N(X) - \langle \xi, a \rangle \overset{\star}{A}_\xi (X)).
\end{aligned} \tag{4.6}$$

It follows that

$$\Delta^\eta \langle x, a \rangle = \langle A_N(\xi), a \rangle + (S_1 - \overset{\star}{S}_1 - 2\tau(\xi))\langle \xi, a \rangle + \overset{\star}{S}_1 \langle N, a \rangle. \tag{4.7}$$

We extend Δ^η on $\otimes^{n+2}\mathcal{C}^\infty(M)$ by

$$\Delta^\eta(f_0, ..., f_{n+1}) = (\Delta^\eta f_0, ..., \Delta^\eta f_{n+1}), \tag{4.8}$$

for all $(f_0, ..., f_{n+1}) \in \otimes^{n+2}\mathcal{C}^\infty(M)$. Let $(e_0, ..., e_{n+1})$ be the standard orthonormal basis of $\mathbb{R}_1^{n+2}$. Then by using (4.7) and (4.8),

$$\Delta^\eta x = (\epsilon_0 \Delta^\eta \langle x, e_0 \rangle, ..., \epsilon_{n+1} \Delta^\eta \langle x, e_{n+1} \rangle) = A_N(\xi) + (S_1 - \overset{\star}{S}_1 - 2\tau(\xi))\xi + \overset{\star}{S}_1 N, \tag{4.9}$$

where $\epsilon_A = \langle e_A, e_A \rangle = \pm 1$. We say that the normalized null hypersurface (M, g, N) is pseudo-harmonic (of the first kind) if $\Delta^\eta x = 0$. The following Lemma shows that to find a normalization N such that (M, g, N) is pseudo-harmonic, it is necessary for M to be totally geodesic.

Lemma 4.1. *Let $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ be a normalized null hypersurface of the pseudo-Euclidean space $\mathbb{R}_1^{n+2}$. Then M is pseudo-harmonic of first kind if and only if M is totally geodesic and the normalization satisfies $A_N(\xi) = 0$ and $trace(A_N) = 2\tau(\xi)$.*

Proof. As $A_N \xi \in \mathscr{S}(N)$ and ξ and N are linearly independent, equation (4.9) and Lemma 2.4 show that, M is pseudo-harmonic if and only if

$$\Delta^\eta x = 0 \Leftrightarrow \begin{cases} \overset{\star}{S_1} = 0 \\ A_N(\xi) = 0 \\ S_1 - \overset{\star}{S_1} - 2\tau(\xi) = 0 \end{cases} \Leftrightarrow \begin{cases} M \text{ is totally geodesic} \\ A_N(\xi) = 0 \\ trace(A_N) = 2\tau(\xi). \end{cases}$$

$\square$

In [10], the first author (jointly with J.-P. Ezin and J. Tossa) considered the problem $\Delta^\eta x = \lambda x$ (that is with $b = 0$) under the assumption that the normalized null hypersurface in $\mathbb{R}_1^{n+1}$ satisfies $A_N = 0$. It is established that λ must be 0 (i.e the null hypersurface is pseudo-harmonic) and $trace\left(\overset{\star}{A_\xi} \right) = 0$ which leads to M is totally geodesic. The following result is a generalization of that fact.

Theorem 4.1. *Let $\lambda \in \mathbb{R}$ and $b \in \mathbb{R}_1^{n+2}$. If a normalized null hypersurface $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ satisfies $\Delta^\eta x = \lambda x + b$ then M is totally geodesic and $\lambda = \langle \overline{\nabla}_N A_N(\xi), \xi \rangle$.*

Proof. Suppose that x satisfies $\Delta^\eta x = \lambda x + b$ then,

$$A_N(\xi) + (S_1 - \overset{\star}{S_1} - 2\tau(\xi))\xi + \overset{\star}{S_1} N = \lambda x + b. \tag{4.10}$$

Taking covariant derivative of (4.10) by ξ we obtain

$$\begin{aligned} \lambda \xi &= \overset{\star}{\nabla}_\xi A_N(\xi) + (\xi \cdot \overset{\star}{S_1} + \tau(\xi) \overset{\star}{S_1})N - \overset{\star}{S_1} A_N(\xi) \\ &+ \xi\left[C(\xi, A_N(\xi)) + \xi S_1 - \xi \overset{\star}{S_1} - 2\xi\tau(\xi) - S_1\tau(\xi) + \overset{\star}{S_1} \tau(\xi) + 2\tau(\xi)^2 \right] \end{aligned} \tag{4.11}$$

Hence,

$$\begin{cases} \overset{\star}{\nabla}_\xi A_N(\xi) = \overset{\star}{S_1} A_N(\xi) \\ \xi \cdot \overset{\star}{S_1} + \tau(\xi) \overset{\star}{S_1} = 0 \overset{(2.17)}{\Longrightarrow} tr\left(\overset{\star}{A_\xi}^2 \right) = 0 \\ C(\xi, A_N(\xi)) + \xi \cdot S_1 - \xi \cdot \overset{\star}{S_1} - 2\xi \cdot \tau(\xi) - S_1\tau(\xi) + \overset{\star}{S_1} \tau(\xi) + 2\tau(\xi)^2 = \lambda \end{cases}$$

and M is totally geodesic. Taking covariant derivative of (4.10) by N we obtain

$$\begin{aligned} \lambda N = \overline{\nabla}_N A_N(\xi) &- (N \cdot \overset{\star}{S_1} + 2N \cdot \tau(\xi) - N \cdot S_1)\xi \\ &+ (S_1 - \overset{\star}{S_1} - 2\tau(\xi))\overline{\nabla}_N \xi + N \cdot \overset{\star}{S_1} N + \overset{\star}{S_1} \overline{\nabla}_N N. \end{aligned}$$

Contracting with ξ leads to $\lambda = \langle \overline{\nabla}_N A_N(\xi), \xi \rangle$.

$\square$

5   The second-order linear differential operators L_r $(0 \leq r \leq n)$

Let $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ be a null hypersurface furnished with a closed normalization with vanishing τ^N, in the pseudo-Euclidean space $\mathbb{R}_1^{n+2}$. By Lemma 2.2, $A_N \xi = 0$ and

$$C(\xi, PY) = 0, \ \forall Y \in \Gamma(TM). \tag{5.1}$$

Let $(X_0 = \xi, X_1, ..., X_n)$ be a g_η–orthonormal basis of $\Gamma(TM)$ with $\mathrm{span}\{X_1, ..., X_n\} = \mathscr{S}(N)$. By direct calculating,

$$\left(\nabla_\xi \overset{\star}{T}_r\right) \xi = (-1)^r \xi \left(\overset{\star}{S}_r\right) \xi. \tag{5.2}$$

By defintion,

$$
\begin{aligned}
div^\nabla(\overset{\star}{T}_r \nabla^\eta f) &= trace(\nabla \overset{\star}{T}_r \nabla^\eta f) \\
&= \sum_{i=1}^{n} \left\{ \left\langle \left(\nabla_{X_i} \overset{\star}{T}_r\right) \nabla^\eta f, X_i \right\rangle + \left\langle \overset{\star}{T}_r \nabla_{X_i} \nabla^\eta f, X_i \right\rangle \right\} \\
&\quad + \ \eta(\nabla_\xi \overset{\star}{T}_r \nabla^\eta f).
\end{aligned}
\tag{5.3}
$$

For each i,

$$
\begin{aligned}
&\left\langle \left(\nabla_{X_i} \overset{\star}{T}_r\right) \nabla^\eta f, X_i \right\rangle \\
&= \left\langle \left(\nabla_{X_i} \overset{\star}{T}_r\right) X_i, \nabla^\eta f \right\rangle - \eta(\overset{\star}{T}_r \nabla^\eta f) B(X_i, X_i) + \eta(\nabla^\eta f) B(\overset{\star}{T}_r X_i, X_i) \\
&= \left\langle \left(\nabla_{X_i} \overset{\star}{T}_r\right) X_i, \nabla^\eta f \right\rangle + \eta(\nabla^\eta f) \left\langle \overset{\star}{A}_\xi^2 \circ \overset{\star}{T}_r X_i, X_i \right\rangle.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
div^\nabla(\overset{\star}{T}_r \nabla^\eta f) &= \left\langle \nabla^\eta f, div^\nabla(\overset{\star}{T}_r) \right\rangle + \eta(\nabla^\eta f) tr(\overset{\star}{A}^2 \circ \overset{\star}{T}_{r-1}) + tr(\overset{\star}{T}_r \circ \nabla_\eta^2 f) \\
&\quad - \ \eta(\overset{\star}{T}_r \nabla_\xi \nabla^\eta f) + \eta(\nabla_\xi \overset{\star}{T}_r \nabla^\eta f)
\end{aligned}
\tag{5.4}
$$

By using (2.6) and (5.1) one finds

$$\eta(\nabla_\xi \overset{\star}{T}_r \nabla^\eta f) = (-1)^r \eta(\nabla^\eta f) \xi(\overset{\star}{S}_r) + (-1)^r \overset{\star}{S}_r \eta(\nabla_\xi \nabla^\eta f), \tag{5.5}$$

$$\eta(\overset{\star}{T}_r \nabla_\xi \nabla^\eta f) = (-1)^r \overset{\star}{S}_r \eta(\nabla_\xi \nabla^\eta f). \tag{5.6}$$

Replace (5.5) and (5.6) in (5.4) we obtain

$$div^\nabla(\overset{\star}{T}_r \nabla^\eta f) = \left\langle \nabla^\eta f, div^\nabla(\overset{\star}{T}_r) \right\rangle + tr(\overset{\star}{T}_r \circ \nabla_\eta^2 f) + \eta(\nabla^\eta f) \left((-1)^r \xi(\overset{\star}{S}_r) + tr(\overset{\star}{A}^2 \circ \overset{\star}{T}_{r-1}) \right). \tag{5.7}$$

Thanks to [6], since the ambient manifold is the pseudo-Euclidean space form $\mathbb{R}_1^{n+2}$ and τ identically vanishes, the divergence $div^\nabla(\overset{\star}{T}_r)$ is $TM^\perp$–valued and $(-1)^r \xi(\overset{\star}{S}_r) + tr(\overset{\star}{A}^2 \circ \overset{\star}{T}_{r-1}) = 0$. Hence (5.7) becomes

$$div^\nabla(\overset{\star}{T}_r \nabla^\eta f) = tr(\overset{\star}{T}_r \circ \nabla_\eta^2 f). \tag{5.8}$$

Let r be an integer with $0 \le r \le n$. We define de second-order linear differential operator $L_r : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ by

$$L_r f = tr(\overset{\star}{T}_r \circ \nabla_\eta^2 f), \ \ \forall f \in \mathcal{C}^\infty(M). \tag{5.9}$$

It is easy to check that L_o is nothing but the first kind pseudo-Laplacian operator Δ^η and that L_r satisfies for $f, g \in \mathcal{C}^\infty(M)$,

$$L_r(fg) = fL_r g + gL_r f + 2\left\langle \overset{\star}{T}_r \nabla^\eta f, \nabla^\eta g \right\rangle + 2\eta(\overset{\star}{T}_r \nabla^\eta f)\eta(\nabla^\eta g). \tag{5.10}$$

Let $a \in \mathbb{R}_1^{n+2}$ be a fixed vector and $X \in \Gamma(TM)$. We know that $\langle x, a \rangle \in \mathcal{C}^\infty(M)$ and since τ identically vanishes (4.6) becomes

$$\nabla_X \nabla^\eta \langle x, a \rangle = \langle A_N(X) - \overset{\star}{A}_\xi(X), a \rangle \xi + \langle N, a \rangle \overset{\star}{A}_\xi(X) + \langle \xi, a \rangle (A_N(X) - \overset{\star}{A}_\xi(X)). \tag{5.11}$$

Using the definition of the second-order linear operator and above relation,

$$L_r \langle x, a \rangle = tr(\overset{\star}{T}_r \circ \langle A_N - \overset{\star}{A}_\xi, a \rangle \xi) + \langle N, a \rangle tr(\overset{\star}{T}_r \circ \overset{\star}{A}_\xi) + \langle \xi, a \rangle tr(\overset{\star}{T}_r \circ (\overset{\star}{A}_\xi - A_N)).$$

By using Proposition 2.1 and the fact that $tr(\overset{\star}{T}_r \circ \langle A_N - \overset{\star}{A}_\xi, a \rangle \xi) = 0$ we obtain

$$L_r \langle x, a \rangle = (-1)^r(r+1) \overset{\star}{S}_{r+1} \langle N, a \rangle + tr(\overset{\star}{T}_r \circ (\overset{\star}{A}_\xi - A_N))\langle \xi, a \rangle. \tag{5.12}$$

We extend L_r on $\otimes^{n+2}\mathcal{C}^\infty(M)$ by

$$L_r(f_0, ..., f_{n+1}) = (L_r f_0, ..., L_r f_{n+1}), \tag{5.13}$$

for all $(f_0, ..., f_{n+1}) \in \otimes^{n+2}\mathcal{C}^\infty(M)$. Then (5.12) gives

$$L_r x = (-1)^r(r+1) \overset{\star}{S}_{r+1} N + tr(\overset{\star}{T}_r \circ (\overset{\star}{A}_\xi - A_N))\xi. \tag{5.14}$$

In case the normalization has conformal screen with (conformal) factor ϕ, i.e

$$A_N = \phi \overset{\star}{A}_\xi,$$

it follows that

$$L_r x = (-1)^r(r+1) \overset{\star}{S}_{r+1} \left(N + (1-\phi)\xi \right). \tag{5.15}$$

The normalized null hypersurface will said to be L_r-harmonic if $L_r x = 0$ (the case $r = 0$ represents the pseudo-harmonicity of first kind). Thus, (5.15) leads to the following.

Lemma 5.1. *Let $x : (M, g, N) \hookrightarrow \mathbb{R}_1^{n+2}$ be a normalized null hypersurface with conformal screen $\mathscr{S}(N)$ and vanishing normalization 1-form τ. Then M is L_r-harmonic if and only if M is $(r+1)$-maximal.*

Definition 5. A unitary conformal closed $(UCC-)$ normalized null hypersurface is one for which the normalization is closed and (the integrable) screen distribution is conformal with constant conformal factor 1.

Below, only such normalizations will be in use and (5.15) takes the form

$$L_r x = (-1)^r(r+1) \overset{\star}{S}_{r+1} N. \tag{5.16}$$

The main purpose of this paper is then to solve the unknown x equation

$$L_r x = (-1)^r(r+1) \overset{\star}{S}_{r+1} N = Ux + b, \tag{5.17}$$

where U is some constant matrix along leaves of the associated integrable screen distribution and $b \in \mathbb{R}_1^{n+2}$ some constant vector.

6   Monge null hypersurfaces $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ satisfying $L_r x = U x + b$

Our goal in this section is to characterize normalized Monge null hypersurfaces $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ satisfying equation (5.17).

Let $x : (M, g) \to \mathbb{R}_1^{n+2}$ be the Monge hypersurface

$$M = \Big\{ x = (x_0, ..., x_{n+1}) \in \mathbb{R} \times D, \ \ x_0 = F(x_1, ..., x_{n+1}) \Big\}, \tag{6.1}$$

in the Minkowski space $\mathbb{R}_1^{n+2}$ where $F : D \longrightarrow \mathbb{R}$ is a smooth function defined on an open subset D of $\mathbb{R}^{n+1}$. For a vector field $X = X^A \dfrac{\partial}{\partial x^A} \in \mathbb{R}_1^{n+2}$ a necessary and sufficient condition to be tangent to M is that $X^0 = X^1 F'_{x^1} + \cdots + X^1 F'_{x^{n+1}}$. Then $\mathbf{n} = \dfrac{\partial}{\partial x^0} + \sum_{a=1}^{n+1} F'_{x^a} \dfrac{\partial}{\partial x^a}$ is normal to M. The later is a null hypersurface if and only if $\mathbf{n}$ is a null vector. This is equivalent to

$$\sum_{a=1}^{n+1} \left(F'_{x^a} \right)^2 = ||\nabla F||^2 = 1, \tag{6.2}$$

where ∇F is the gradient of F with respect to the Euclidean structure $||\cdot||$ of $\mathbb{R}^{n+1}$. Then, taking partial derivative of (6.2) with respect to x^b ($1 \le b \le n+1$) leads to

$$\sum_{a=1}^{n+1} F'_{x^a} F''_{x^a x^b} = 0. \tag{6.3}$$

In [19], Duggal and Bejancu proved the following

Theorem 6.1 ([19], page 122). *Let $x : (M, g) \to \mathbb{R}_1^{n+2}$ be a Monge hypersurface isometrically immersed into the Minkowski space $\mathbb{R}_1^{n+2}$ and defined as the graph of the smooth function $F : D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}^{n+1}$. Then M is totally geodesic if and if M is an open subset of a hyperplane of $\mathbb{R}^{n+2}$, thus*

$$F(x^1, ..., x^{n+1}) = \sum_{a=1}^{n+1} c_a x^a + c, \ \ \forall (x^1, ..., x^{n+1}) \in D,$$

where $\{c_1, ..., c_{n+1}, c\}$ are real numbers satisfying $\sum\limits_{a=1}^{n+1} (c_a)^2 = 1$.

6.1   Generic $UCC-$normalization on Monge null hypersurfaces

Throughout, the Monge null hypersurface will be endowed with the (physically and geometrically) relevant rigging

$$\mathscr{N}_F = \frac{1}{\sqrt{2}} \Big[-\frac{\partial}{\partial x^0} + \sum_{a=1}^{n+1} F'_{x^a} \frac{\partial}{\partial x^a} \Big] = \frac{1}{\sqrt{2}} (-1, \nabla F). \tag{6.4}$$

The corresponding rigged vector field is then given by

$$\xi_F = \frac{1}{\sqrt{2}} \mathbf{n} = \frac{1}{\sqrt{2}} \Big[\frac{\partial}{\partial x^0} + \sum_{a=1}^{n+1} F'_{x^a} \frac{\partial}{\partial x^a} \Big] = \frac{1}{\sqrt{2}} (1, \nabla F). \tag{6.5}$$

We show below that this is a closed normalization with vanishing normalizing 1−form $\tau^{\mathscr{N}_F}$ and conformal (hence integrable) screen distribution with unit conformal factor $\phi = 1$. It is a consequence of Theorem 4.1 in [10] that the induced connexion ∇ coincides with the Levi-Civita connexion ∇^η of the associated metric g_η ($\nabla^\eta = \nabla$). In fact, let us consider the natural (global) parametrization of M given by

$$
\begin{cases}
x^0 = F(u^1, ..., u^{n+1}) \\
x^a = u^a \\
a = 1, ..., n+1
\end{cases}
\qquad (u^1, ..., u^{n+1}) \in D \ .
\tag{6.6}
$$

Then $\Gamma(TM)$ is spanned by $\{\frac{\partial}{\partial u^a}\}_a$ with

$$
\frac{\partial}{\partial u^a} = F'_{u^a} \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^a}.
\tag{6.7}
$$

Now take covariant derivative by the flat connection $\overline{\nabla}$ and use (6.3) to get

$$
\overline{\nabla}_{\frac{\partial}{\partial u^a}} \xi_F = \frac{1}{\sqrt{2}} \sum_{b=1}^{n+1} \left(-F''_{u^a u^b} F'_{u^b} \frac{\partial}{\partial x^0} + F''_{u^a u^b} \frac{\partial}{\partial x^b} \right) \overline{\nabla}_{\frac{\partial}{\partial u^a}} \xi_F = \frac{1}{\sqrt{2}} \sum_{b=1}^{n+1} F''_{u^a u^b} \frac{\partial}{\partial u^b}.
\tag{6.8}
$$

Now we prove the following

Proposition 6.1. *Let $x : (M, g, \mathscr{N}_F) \longrightarrow \mathbb{R}_1^{n+2}$ be a Monge null hypersurface endowed with the rigging $\mathscr{N}_F$ as in (6.4). Then the following hold.*

1. *The 1−form $\tau^{\mathscr{N}_F}$ vanishes identically.*

2. *The screen distribution is conformal with $\phi = 1$ as conformal factor.*

3. *The screen distribution is integrable with leaves the level sets of the function F.*

4. *The induced connexion ∇^F coincides with the Levi-Civita connexion of the (Riemannian) associated metric g_η, i.e*
$$
\nabla^\eta = \nabla.
$$

5. *The immersion satisfies the Laplace-Beltrami equation*
$$
\Delta x = (n+1) \overset{\star}{H} \mathscr{N}_F,
$$
which is a particular case of the more general fact
$$
L_r x = (-1)^r (r+1) \binom{n+1}{r+1} \overset{\star}{H}_{r+1} \mathscr{N}_F.
\tag{6.9}
$$

6. *In the natural basis $\{\frac{\partial}{\partial u^a}\}_a$, the divergence (with respect to the induced connexion) of some vector field $X = X^a \frac{\partial}{\partial u^a}$ (as in usual Euclidean case) takes the form*
$$
div X = \frac{\partial X^a}{\partial u^a}
\tag{6.10}
$$

Proof. By (6.8) and (2.7), $\tau^{\mathcal{N}_F}$ identically vanishes and

$$\overset{\star}{A}_{\xi_F}\left(\frac{\partial}{\partial u^a}\right) = -\frac{1}{\sqrt{2}}\sum_{b=1}^{n+1} F''_{u^a u^b}\frac{\partial}{\partial u^b}. \tag{6.11}$$

Also one obtains,

$$A_{\mathcal{N}_F}\left(\frac{\partial}{\partial u^a}\right) = -\frac{1}{\sqrt{2}}\sum_{b=1}^{n+1} F''_{u^a u^b}\frac{\partial}{\partial u^b}. \tag{6.12}$$

Hence, $\overset{\star}{A}_{\xi_F} = A_{\mathcal{N}_F}$ which shows that the screen distribution is conformal with conformal factor $\phi = 1$. Let X, Y be two sections tangent to the screen structure. Then,

$$\left\langle [X,Y], \mathcal{N}_F \right\rangle = \langle X, \overline{\nabla}_Y \mathcal{N}_F \rangle - \langle Y, \overline{\nabla}_X \mathcal{N}_F \rangle = \langle Y, \overset{\star}{A}_{\xi_F} X \rangle - \langle X, \overset{\star}{A}_{\xi_F} Y \rangle = 0.$$

Hence, $[X,Y]$ is a section of the screen distribution. Thus, the screen distribution is involutible and by the Frobenüs theorem, it is integrable. We show later that leaves are really the level sets of F (subsection 6.3).

Since $\tau^{\mathcal{N}_F}$ identically vanishes and $\overset{\star}{A}_{\xi_F} = A_{\mathcal{N}_F}$, ∇ is the Levi-Civita connexion of the (Riemannian) associate metric (see theorem 4.1 in [10]). Hence, the two Laplacians of first and second kind coincide. By (5.17), the Laplace-Beltrami equation and (6.9) hold.

Let $X = X^a\frac{\partial}{\partial u^a}$ be some vector field.

$$X = X^a\frac{\partial}{\partial u^a} = X^0\frac{\partial}{\partial x^0} + X^a\frac{\partial}{\partial x^a},$$

with $X^0 = F'_{u^a}X^a$. We have, $\overline{\nabla}_{\partial_{u^b}}X = \partial_{u^b}(X^0)\partial x^0 + \partial_{u^b}(X^a)\partial_{x^a}$. In other side, one can write

$$\overline{\nabla}_{\partial_{u^b}}X = \nabla_{\partial_{u^b}}X + B(\partial_{u^b}, X)\mathcal{N}_F = f^a\partial_{u^a} + B(\partial_{u^b}, X)\mathcal{N}_F$$
$$= \left(F'_{u^a}f^a - \frac{1}{\sqrt{2}}B(\partial_{u^b}, X)\right)\partial_{x^0} + \left(f^a + \frac{1}{\sqrt{2}}B(\partial_{u^b}, X)\right)\partial_{x^a}.$$

After identification, we get

$$f^a = \partial_{u^b}(X^a) - \frac{1}{\sqrt{2}}F'_{u^a}B(\partial_{u^b}, X).$$

Hence,

$$\nabla_{\partial_{u^b}}X = \left(\partial_{u^b}(X^a) - \frac{1}{\sqrt{2}}F'_{u^a}B(\partial_{u^b}, X)\right)\partial_{u^a}.$$

The above relation together with the fact that $||\nabla F|| = 1$ gives (6.10). □

Hence on any Monge null hypersurface, the rigging $\mathcal{N}_F$ has some outstanding properties: the screen distribution is integrable, the 1−form $\tau^{\mathcal{N}_F}$ identically vanishes and

$$A_N\left(\frac{\partial}{\partial u^a}\right) = \overset{\star}{A}_\xi\left(\frac{\partial}{\partial u^a}\right) = -\frac{1}{\sqrt{2}}\sum_{b=1}^{n+1} F''_{u^a u^b}\frac{\partial}{\partial u^b}. \tag{6.13}$$

Then the matrix of $\overset{\star}{A}_\xi$ with respect to the basis $\{\frac{\partial}{\partial u^a}\}_a$ is given by

$$\overset{\star}{A}_\xi = -\frac{1}{\sqrt{2}} \begin{pmatrix} F''_{u^1 u^1} & \cdots & F''_{u^1 u^{n+1}} \\ \vdots & \ddots & \vdots \\ F''_{u^{n+1} u^1} & \cdots & F''_{u^{n+1} u^{n+1}} \end{pmatrix} = -\frac{1}{\sqrt{2}} Hess(F) \tag{6.14}$$

and,

$$\overset{\star}{S}_1 = tr(\overset{\star}{A}_\xi) = -\frac{1}{\sqrt{2}} \sum_{b=1}^{n+1} F''_{u^b u^b} = -\frac{1}{\sqrt{2}} \Delta F \tag{6.15}$$

It is then our goal to solve (5.17) with unknown x (or equivalently F taking into account (6.1)). Following are two basic examples: the lightcone $\wedge_0^{n+1}$ and the lightcone cylinder $\Lambda_0^{m+1} \times \mathbb{R}^{n-m}$.

6.2   Examples

6.2.1   The lightcone Λ_0^{n+1}

Let M be the future null cone in $\mathbb{R}_1^{n+2}$ which is the graph of the function

$$F = \left(\sum_{a=1}^{n+1} (x^a)^2 \right)^{1/2}.$$

This is a totally umbilical null hypersurface in $\mathbb{R}_1^{n+2}$ and the generic UCC-normalization (6.4) becomes

$$\mathcal{N}_F = -\frac{1}{\sqrt{2}} \frac{\partial}{\partial x^0} + \frac{1}{x^0 \sqrt{2}} \sum_{a=1}^{n+1} (x^a) \frac{\partial}{\partial x^a},$$

with corresponding rigged vector field

$$\xi_F = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^0} + \frac{1}{x^0 \sqrt{2}} \sum_{a=1}^{n+1} (x^a) \frac{\partial}{\partial x^a} = \sqrt{2} x^0 x.$$

All the principal curvatures are given by

$$\rho = -\frac{1}{x^0 \sqrt{2}}.$$

It follows from this equality that for $0 \leq r \leq n$, the $rth-$mean curvature is given by

$$\overset{\star}{H}_r = \binom{n+1}{r}^{-1} \overset{\star}{S}_r = (-1)^r \binom{n+1}{r}^{-1} \binom{n}{r} \left(\frac{1}{x^0 \sqrt{2}} \right)^r.$$

Hence,

$$L_r x = (r+1) \binom{n}{r+1} \left(\frac{1}{x^0 \sqrt{2}} \right)^{r+2} (x^0, -x^1, ..., -x^{n+1}). \tag{6.16}$$

For a matrix $U = (u_{AB}) \in \mathbb{R}^{(n+2) \times (n+2)}$ and constant vector $b = (b_0, ..., b_{n+1}) \in \mathbb{R}_1^{n+2}$,

$$Ux + b = \left(\sum_{A=0}^{n+1} u_{0A} x^A + b_0, ..., \sum_{A=0}^{n+1} u_{(n+1)A} x^A + b_{n+1} \right).$$

So, setting

$$
U = (r+1)\binom{n}{r+1}\left(\frac{1}{x^0\sqrt{2}}\right)^{r+2}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix} \text{ and } b = 0,
$$

we see that $L_r x = Ux + b$ where U is obviously constant on the leaves (of the integrable screen distribution given by) $x^0 = \text{constante}$.

6.2.2   The lightcone cylinder $\Lambda_0^{m+1} \times \mathbb{R}^{n-m}$, $1 \leq m \leq n$

The future lightcone cylinder $\Lambda_0^{m+1} \times \mathbb{R}^{n-m}$ is the Monge null hypersurface of $\mathbb{R}_1^{n+2}$, define as the graph of the function

$$
F = \left(\sum_{a=1}^{m+1}(x^a)^2\right)^{1/2}.
$$

The generic $UCC-$normalization (6.4) becomes

$$
\mathscr{N}_F = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x^0} + \frac{1}{x^0\sqrt{2}}\sum_{a=1}^{m+1}(x^a)\frac{\partial}{\partial x^a},
$$

with associated rigged vector field

$$
\xi_F = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^0} + \frac{1}{x^0\sqrt{2}}\sum_{a=1}^{m+1}(x^a)\frac{\partial}{\partial x^a}.
$$

The principal curvatures are given by

$$
k_0 = 0, k_1 = \cdots = k_m = -\frac{1}{x^0\sqrt{2}}, k_{m+1} = \cdots = k_n = 0.
$$

Then the $rth-$mean curvature is given by

$$
\overset{\star}{H}_r = \binom{n+1}{r}^{-1}\overset{\star}{S}_r = \begin{cases} (-1)^r\binom{n+1}{r}^{-1}\binom{m}{r}\left(\frac{1}{x^0\sqrt{2}}\right)^r & \text{for} \quad 0 \leq r \leq m \\ \\ 0 & \text{for} \quad m+1 \leq r \leq n \end{cases}
$$

It follows from (6.9) that

$$
L_r x = \begin{cases} (r+1)\binom{m}{r+1}\left(\frac{1}{x^0\sqrt{2}}\right)^{r+2}(x^0, -x^1, ..., -x^{n+1}) & \text{for} \quad 0 \leq r \leq m-1 \\ \\ 0 & \text{for} \quad m \leq r \leq n \end{cases} \tag{6.17}
$$

Hence, for $0 \leq r \leq m-1$, set

$$
U = (r+1)\binom{m}{r+1}\left(\frac{1}{x^0\sqrt{2}}\right)^{r+2}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix} \text{ and } b = 0.
$$

Then, $L_r x = Ux + b$, with U as required.

In view of Theorem 6.3 we establish the following.

Theorem 6.2. *Let D be an open connected subset of $\mathbb{R}^{n+1}$ ($n \geq 1$) and $f : D \to \mathbb{R}$ a non-constant smooth function. Then the following assertions are equivalent.*

1. *f is affine.*

2. *f is harmonic and its gradient in the real space $\mathbb{R}^{n+1}$ has constant non-zero norm.*

3. *The gradient of f in the real space $\mathbb{R}^{n+1}$ has constant non-zero norm and the Monge hypersurface graph of the function $F = f/||\nabla f||$ is null and maximal.*

4. *The gradient of f in the real space $\mathbb{R}^{n+1}$ has constant non-zero norm and the Monge hypersurface graph of the function $F = f/||\nabla f||$ is null and totally geodesic.*

Proof. By Lemma 2.4, the third and the last item are equivalent and by Theorem 6.1 the first and the last item are equivalent. If f is affine then f is harmonic and its gradient in the real space $\mathbb{R}^{n+1}$ has constant non-zero norm (since f is non-constant). If f is harmonic and its gradient in the real space $\mathbb{R}^{n+1}$ has constant non-zero norm, then $F = f/||\nabla f||$ has unitary norm and thus defines a Monge null hypersurface M. Endow M with the rigging (6.4). then, since F is harmonic, equation (6.15) shows that M is maximal. □

Thus, we get the following

Theorem 6.3. *Let $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ be a normalized Monge hypersurface isometrically immersed into the Minkowski space $\mathbb{R}_1^{n+2}$ defined as the graph of the smooth function $F : \mathbb{R}^{n+1} \supset D \to \mathbb{R}$, and endowed with the normalization (6.4). Then the following assertions are equivalent.*

1. *(M, g, N) is pseudo-harmonic.*

2. *(M, g, N) satisfies $\Delta^\eta x = \lambda x + b$, with $\lambda \in \mathbb{R}$ and $b \in \mathbb{R}^{n+2}$*

3. *F is harmonic.*

4. *M is maximal.*

5. *M is totally geodesic.*

6. *M is an open piece of a hyperplane.*

Proof. Since $||\nabla F|| = 1$, Theorem 6.2 applied with $f = F$ implies that the items 3 to 5 are equivalent and by Theorem 6.1, the items 5 and 6 are equivalent. Now, Lemma 5.1 with $r = 0$ implies that the items 1 and 4 are equivalent. Also, from Theorem 4.1, the item 2 implies the item 5 and from the item 5 of Proposition 6.1, the item 5 implies the item 2 with $\lambda = 0$ and $b = 0$. □

Observe that in the above theorem, only the first and the second items make use of the normalization.

6.3   Level sets of F

Let $c \in Im(F)$ be a regular value of F and $M_c = F^{-1}(c)$ the $c-$level set of F. Then, $x_c : M_c \to \mathbb{R}^{n+1}$ is a Riemanniann hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ and the Gauss map is the gradient ∇F of F. We take x_c to be the inclusion map and M_c is a subset of D. We then have the following diagram

$$M_c \overset{x_c}{\hookrightarrow} D \quad \overset{x}{\longrightarrow} \quad M \overset{i}{\hookrightarrow} \mathbb{R}_1^{n+2}$$
$$p(u^1, ..., u^{n+1}) \quad \mapsto \quad x(x^0 = F(u^1, ..., u^{n+1}), x^1 = u^1, ..., x^{n+1} = u^{n+1}).$$

We denote by ∇° and ∇_c the Levi-Civita connections of $\mathbb{R}^{n+1}$ and M_c respectively. For $0 \leq r \leq n$, we denote by s_r the $r-$th mean curvature of M_c in $\mathbb{R}^{n+1}$. Let $s : \Gamma(TM_c) \to \Gamma(TM_c)$ and b be the Weingarten operator and the second fondamental form respectively. $\forall X, Y \in \Gamma(TM_c)$,

$$b(X, Y) = \langle s(X), Y \rangle = -\langle \nabla_X^\circ \nabla F, Y \rangle = -\nabla^2 F(X, Y).$$

Hence,

$$b = -x_c^\star Hess(F). \tag{6.18}$$

It is easy to check that for all $X \in \Gamma(TM_c)$, $x_\star(x_{c\star}X) = x_\star(X) = (\langle X, \nabla F \rangle, X) = (0, X)$ and

$$\langle x_\star(X), N \rangle = \langle x_\star(X), \xi \rangle = \langle X, \nabla F \rangle = 0.$$

Thus the level sets $x(M_c)$ are leaves of the screen distribution $\mathscr{S}(N)$ of M (endowed with the normalization (6.4)). Thanks to (6.14) and (6.18), and the fact that ξ is not tangent to $x(M_c)$ and is a eigenvector of $Hess(F)$ associated to the eigenvalue 0, we then have that for all $0 \leq r \leq n$,

$$\overset{\star}{S}_r = (1/\sqrt{2})^r s_r. \tag{6.19}$$

Let $l_r : \mathcal{C}^\infty(M_c) \to \mathcal{C}^\infty(M_c)$ be the second-order linear differential operator given by.

$$l_r(f) = trace(P_r \circ \nabla^2 f) \tag{6.20}$$

for all $f \in \mathcal{C}^\infty(M_c)$, where P_r is the $r-$th Newton transformation with respect to the Weingarten operator s. Thanks to [1],

$$l_r x_c = s_{r+1} \nabla F. \tag{6.21}$$

Using (6.16), (6.4) and (6.21) we then have on $x(x_c(M_c))$,

$$\begin{aligned}
L_r x_{|x(M_c)} &= (-1)^r (r+1) \overset{\star}{S}_{r+1} N = (-1)^r (r+1)(1/\sqrt{2})^{r+2} s_{r+1}(-1, \nabla F) \\
L_r x_{|x(M_c)} &= (-1)^r (r+1)(1/\sqrt{2})^{r+2}(-s_{r+1}, l_r x_c)
\end{aligned} \tag{6.22}$$

Let $U = (u_{AB})_{0 \leq A, B \leq n+1} \in \mathbb{R}^{(n+2) \times (n+2)}$ be (a screen constant) matrix and $b = (b_0, ..., b_{n+1}) \in \mathbb{R}_1^{n+2}$ a constant vector. We set $\tilde{U}_c = (u_{ab})_{1 \leq a, b \leq n+1}$, $U_h = (u_{00}, u_{01}, ..., u_{0n+1})$, $\tilde{U}_v = (u_{10}, ..., u_{n+10})$ and $\tilde{b}_v = (b_1, ..., b_{n+1})$, such that

$$U = \begin{pmatrix} & U_h \\ \tilde{U}_v & \tilde{U}_c \end{pmatrix} \text{ and } b = \begin{pmatrix} b_0 \\ \tilde{b}_v \end{pmatrix}.$$

Then,

$$Ux_{|x(M_c)} + b = ({}^tU \cdot x_{|x(M_c)} + b_0, \tilde{U}_c x_c + \tilde{b}_c), \tag{6.23}$$

with $\tilde{b}_c = c\tilde{U}_v + \tilde{b}_v$. Hence, $L_r x_{|x(M_c)} = Ux_{|x(M_c)} + b$ is equivalent to

$$(-1)^r (r+1) \left(\frac{1}{\sqrt{2}}\right)^{r+2} (-s_{r+1}, l_r x_c) \; = \; ({}^t U \cdot x_{|x(M_c)} + b_0, \tilde{U}_c x_c + \tilde{b}_c)$$

that is

$$\begin{cases} (-1)^{r+1}(r+1)\left(\frac{1}{\sqrt{2}}\right) \overset{\star}{S}_{r+1} = {}^t U \cdot x_{|x(M_c)} + b_0 \\ l_r x_c = U_c x_c + b_c \end{cases} , \tag{6.24}$$

with $U_c = (-1)^r (1/(r+1))(\sqrt{2})^{r+2} \tilde{U}_c x_c$ and $b_c = (-1)^r (1/(r+1))(\sqrt{2})^{r+2} \tilde{b}_c$. A well-known result by L. J. Alias and N. Gurbuz in [1] applied to the second set of (6.24) implies that the level set M_c is a Riemannian hypersurface with $s_{r+1} = 0$ or an open piece of a round hypersphere or an open piece of a generalized right spherical cylinder $\mathbb{S}^m(r) \times \mathbb{R}^{n-m}$, with $r + 1 \le m \le n - 1$. Let us examine each of the three cases.

1. If M_c is a hypersurface with $s_{r+1} = 0$ then, by use of (6.19) we see that $\overset{\star}{S}_{r+1} = 0$ thus M is $(r+1)$−maximal in $\mathbb{R}_1^{n+2}$.

2. Assume M_c is an open piece of a round hypersphere. Then, as $c = x_0$ is a regular value for F, there exists an interval $I \subset Im(F)$ such that for all c in I, M_c is an open piece of a round hypersphere of radius $r(x_0)$ and (locally) we have

$$M = \bigcup_{x_0 \in I} \{x_0\} \times S^n(r(x_0)).$$

An equation of the inclusion $M = \cup_{x_0 \in I}\{x_0\} \times S^n(r(x_0)) \hookrightarrow \mathbb{R}_1^{n+2}$ is

$$-[r(x_0)]^2 + x_1^2 + \cdots + x_{n+1}^2 = 0.$$

M is then the 0 level of the smooth function $f(x) = -[r(x_0)]^2 + x_1^2 + \cdots + x_{n+1}^2$. We have

$$df(x) = 2\Big(-r'(x_0)r(x_0)dx_0 + x_1 dx_1 + \cdots + x_{n+1}dx_{n+1}\Big),$$

and a normal vector field to M is given by $\xi = \Big(r'(x_0 r(x_0)), x_1, \cdots, x_{n+1}\Big)$. Then,

$$0 = \|\xi\|^2 = -r'(x_0)^2 r(x_0)^2 + x_1^2 + \cdots + x_{n+1}^2 = r(x_0)\Big(1 - r'(x_0^2)\Big),$$

which leads to $r(x_0) = \varepsilon$ with $\varepsilon = \pm 1$, that is $r(x_0) = \varepsilon x_0 + k$, $k \in \mathbb{R}$. Hence, M is given by

$$M : -(\pm x_0 + k)^2 + x_1^2 + \cdots + x_{n+1}^2 = 0,$$

which, up to a motion in $\mathbb{R}_1^{n+2}$, is the lightcone $\wedge_0^{n+1}$, see Figure 1.

3. If M_c is an open piece of a generalized right spherical cylinder $\mathbb{S}^m(r) \times \mathbb{R}^{n-m}$, then

$$M = \bigcup_{x_0 \in I} \{x_0\} \times S^m(r(x_0)) \times \mathbb{R}^{n-m}$$

and by a similar argument as in previous item, M is (up to a motion in $\mathbb{R}_1^{n+2}$) the lightcone cylinder $\Lambda_0^{m+1} \times \mathbb{R}^{n-m}$ as saw in Example 6.2.2.
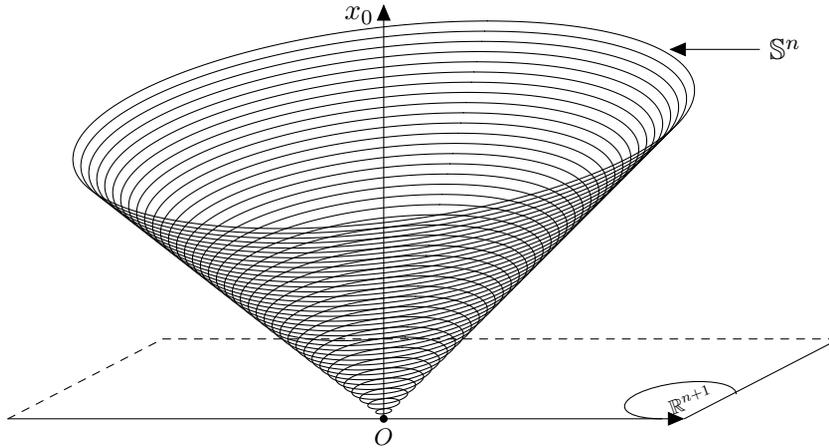
Figure 1: The lightcone of $\mathbb{R}_1^{n+2}$ is a stacking of hyperspheres $\mathbb{S}^n(x_0)$ of $\mathbb{R}^{n+1}$

So, the following is proved.

Theorem 6.4. *Let $x : (M, g) \to \mathbb{R}_1^{n+2}$ be a Monge null hypersurface isometrically immersed into the Minkowski space $\mathbb{R}_1^{n+2}$. Then endowed with the generic $UCC-$normalization N with vanishing τ, (M, g, N) satisfies the equation $L_r x = U x + b$, for some (field of) screen constant matrix $U \in \mathbb{R}^{(n+2)\times(n+2)}$, a constant vector $b \in \mathbb{R}_1^{n+2}$ and some integer $0 \le r \le n$, if and only if M is the lightcone Λ_0^{n+1} or the lightcone cylinder $\Lambda_0^{m+1} \times \mathbb{R}^{n-m}$ $(1 \le m \le n-1)$ or a $(r+1)-$maximal Monge null hypersurface.*

7   Normalized null hypersurfaces $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ satisfying $L_r x = \mathcal{U} x + b$

Let $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ be a normalized null hypersurface (not necessarily Monge), endowed with a Unitary Conformally Closed (UCC-)normalization with vanishing $1-$form τ. As we have seen, the $r-$th second order differential equation act on the position vector x by

$$L_r x = (-1)^r (r+1) \overset{\star}{S}_{r+1} N. \tag{7.1}$$

In the same manner we defined $L_r f$ in section 5, let us introduce $L_{r|\mathscr{S}} f$ to be the trace of the restriction of the endomorphism $\overset{\star}{T}_r \circ \nabla_\eta^2 f : \Gamma(TM) \to \Gamma(TM)$ on $\Gamma(\mathscr{S}(N))$. From now on, X, Y, Z are sections of the screen distribution $\mathscr{S}(N)$. Since g and g_η coincide on $\Gamma(\mathscr{S}(N))$, one has $\nabla^\eta = \overset{\star}{\nabla}$. Then, $L_{r|\mathscr{S}}$ can be viewed as the $r - th$ second order linear operator on the screen distribution and one has

$$L_{r|\mathscr{S}} f = trace \left(\overset{\star}{T}_r \circ \overset{\star}{\nabla}^2 f \right) \tag{7.2}$$

Now, we must compute $L_{r|\mathscr{S}}(x)$, $L_{r|\mathscr{S}}(N)$ and $L_{r|\mathscr{S}}(L_r x)$.

- $L_{r|\mathscr{S}}(x)$

$$\langle \overset{\star}{\nabla} \langle a, x \rangle, X \rangle = X \cdot \langle a, x \rangle = \langle P a^\top, x \rangle \Rightarrow \overset{\star}{\nabla} \langle a, x \rangle = P a^\top = a - \langle a, N \rangle \xi - \langle a, \xi \rangle N.$$

Take covariant derivative, use the fact that the normalization is unitary conformally and use Gauss formula, $\overset{\star}{\nabla}_X \overset{\star}{\nabla} \langle a, x \rangle = \langle a, N + \xi \rangle \overset{\star}{A}_\xi(X)$. And then,

$$L_{r|\mathscr{S}}(x) = (-1)^r (r+1) \overset{\star}{S}_{r+1} (N + \xi). \tag{7.3}$$

- $L_{r|\mathscr{S}}(N)$

$$\overset{\star}{\nabla} \langle a, N \rangle = \overset{\star}{A}_\xi(a^\top). \tag{7.4}$$

Take covariant derivative, use the fact that the normalization is unitary conformally, use Gauss formula and Gauss-Codazzi equation and Proposition 2.1 to get

$$L_{r|\mathscr{S}}(N) = (-1)^{r+1} \overset{\star}{\nabla} \overset{\star}{S}_{r+1} + (-1)^{r+1} \left(\overset{\star}{S}_1 \overset{\star}{S}_{r+1} - (r+2) \overset{\star}{S}_{r+2} \right). \tag{7.5}$$

- $L_{r|\mathscr{S}}(L_r x)$

Applying $L_{r|\mathscr{S}}$ to equality (7.1) and using (5.10) and (7.4) one gets

$$L_{r|\mathscr{S}}(L_r x) = (-1)^r L_{r|\mathscr{S}} \left(\overset{\star}{S}_{r+1} \right) N - (r+1) \overset{\star}{S}_{r+1} \left(\overset{\star}{S}_1 \overset{\star}{S}_{r+1} - (r+2) \overset{\star}{S}_{r+2} \right) \xi$$
$$- (r+1) \overset{\star}{S}_{r+1} \overset{\star}{\nabla} \overset{\star}{S}_{r+1} + 2(-1)^r (r+1) \left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r \right) \left(\overset{\star}{\nabla} \overset{\star}{S}_{r+1} \right) \tag{7.6}$$

Since $\mathcal{U}$ and b are constants on the leaves of the screen distribution, taking covariant on (5.17) and using (7.1) one has

$$U X = (-1)^{r+1} (r+1) \overset{\star}{S}_{r+1} \overset{\star}{A}_\xi(X) + (-1)^r (r+1) \langle \overset{\star}{\nabla} \overset{\star}{S}_{r+1}, X \rangle N \tag{7.7}$$

Applying $L_{r|\mathscr{S}}$ to (5.17), and taking into account the fact that $\mathcal{U}$ and b are screen distribution constants, one has $L_{r|\mathscr{S}}(L_r x) = \mathcal{U} L_{r|\mathscr{S}}(x)$. And identify this with (7.6) and (7.3),

$$(-1)^r (r+1) \overset{\star}{S}_{r+1} (\mathcal{U} N + \mathcal{U} \xi) = (-1)^r L_{r|\mathscr{S}} \left(\overset{\star}{S}_{r+1} \right) N + 2(-1)^r (r+1) \left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r \right) \left(\overset{\star}{\nabla} \overset{\star}{S}_{r+1} \right)$$
$$- (r+1) \left[\overset{\star}{S}_{r+1} \overset{\star}{\nabla} \overset{\star}{S}_{r+1} + \overset{\star}{S}_{r+1} \left(\overset{\star}{S}_1 \overset{\star}{S}_{r+1} - (r+2) \overset{\star}{S}_{r+2} \right) \xi \right] \tag{7.8}$$

By using (7.7), it is easy to check that $\langle \mathcal{U} X, Y \rangle = \langle X, \mathcal{U} Y \rangle$. Taking covariant derivative of this by Z, and observe by (7.7) that $\langle \mathcal{U} X, N \rangle = 0$, one obtains

$$(\langle \mathcal{U} \xi + \mathcal{U} N, Y \rangle - \langle \mathcal{U} Y, \xi \rangle) \overset{\star}{A}_\xi(X) = (\langle \mathcal{U} \xi + \mathcal{U} N, X \rangle - \langle \mathcal{U} X, \xi \rangle) \overset{\star}{A}_\xi(Y). \tag{7.9}$$

Lemma 7.1. *Let $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ be a normalized null hypersurface furnished with a UCC-normalization N with vanishing $1-$form τ. If the immersion x satisfies the equation (5.17) for some $r = 1, ..., n$ then, the $(r+1)-$th mean curvature $\overset{\star}{H}_{r+1}$ is screen constant.*

Proof. Let L be a leaf of $\mathscr{S}(N)$. Let us consider the open set

$$\mathscr{U}_{r+1} = \{ p \in L; \ \overset{\star}{\nabla} \overset{\star}{H}_{r+1}^2 (p) = 0 \}.$$

We need to show that $\mathscr{U}_{r+1}$ is empty. If $\mathscr{U}_{r+1}$ is not empty then, from (7.8),

$$\langle \mathcal{U} N + \mathcal{U} \xi, X \rangle = -(-1)^r \left\langle \overset{\star}{S}_{r+1}, X \right\rangle + \frac{2}{\overset{\star}{S}_{r+1}} \left\langle \left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r \right) \left(\overset{\star}{\nabla} \overset{\star}{S}_{r+1} \right), X \right\rangle.$$

Related this with (7.7) and (7.9) one has,

$$\left\langle P_r\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right), Y\right\rangle \overset{\star}{A}_\xi(X) = \left\langle P_r\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right), X\right\rangle \overset{\star}{A}_\xi(Y), \tag{7.10}$$

with $P_r = (-1)^{r+1}(r+2)I + \frac{2}{\overset{\star}{S}_{r+1}}\overset{\star}{A}_\xi\circ\overset{\star}{T}_r$. We claim that $P_r\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right) = 0$ on $\mathscr{U}_{r+1}$. Otherwise, there exists a open set on which $P_r\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right) \neq 0$ and we can find a pseudo-orthonormal basis such that $\overset{\star}{E}_1$ is in the direction of $P_r\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right)$ and that give us $\overset{\star}{A}_\xi(\overset{\star}{E}_i) = 0$ for all $i \geq 2$ and this implies that $\overset{\star}{H}_{r+1}= 0$ since $r \geq 1$, which is a contradiction. Therefore, $P_r\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right) = 0$ on $\mathscr{U}_{r+1}$ which implies that

$$\left(\overset{\star}{A}_\xi\circ\overset{\star}{T}_r\right)\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right) = \frac{(-1)^r(r+2)}{2}\overset{\star}{S}_{r+1}\overset{\star}{\nabla}\overset{\star}{S}_{r+1} \;\; \text{on } \mathscr{U}_{r+1}.$$

By using inductive definition of Newton transformations,

$$\overset{\star}{T}_{r+1}\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right) = \frac{(-1)^r r}{2}\overset{\star}{S}_{r+1}\overset{\star}{\nabla}\overset{\star}{S}_{r+1} \;\; \text{on } \mathscr{U}_{r+1}. \tag{7.11}$$

Consider $\{\overset{\star}{E}_0= \xi, \overset{\star}{E}_1, ..., \overset{\star}{E}_n\}$ a local pseudo-orthonornal basis of principal direction of $\overset{\star}{A}_\xi$ One can write $\overset{\star}{\nabla}\overset{\star}{S}_{r+1}= \sum_{i=1}^n \left\langle\overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i\right\rangle\overset{\star}{E}_i$, and by using Proposition 2.1,

$$\overset{\star}{T}_{r+1}\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right) = (-1)^{r+1}\sum_{i=1}^n \left\langle\overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i\right\rangle\overset{\star}{S}^i_{r+1}\overset{\star}{E}_i . \tag{7.12}$$

Then, (7.11) is equivalent to

$$\left\langle\overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i\right\rangle\left(\overset{\star}{S}^i_{r+1} +(r/2)\overset{\star}{S}_{r+1}\right) = 0 \;\; \text{on } \mathscr{U}_{r+1},$$

for every $i = 1, ..., n$. Therefore, for every i such that $\langle\overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i\rangle \neq 0$, one gets

$$\overset{\star}{S}^i_{r+1}= -(r/2)\overset{\star}{S}_{r+1} \;\; \text{on } \mathscr{U}_{r+1}. \tag{7.13}$$

We claim that $\langle\overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i\rangle = 0$ for some i. Otherwise, (7.13) holds for every i, which implies that

$$tr\left(P\circ\overset{\star}{T}_{r+1}\right) = (-1)^{r+1}\sum_{i=1}^n \overset{\star}{S}^i_{r+1}= (-1)^r n(r/2)\overset{\star}{S}_{r+1} \;\; \text{on } \mathscr{U}_{r+1}. \tag{7.14}$$

Bearing in mind Proposition 2.1, the last equation yields $\overset{\star}{H}_{r+1}= 0$ on $\mathscr{U}_{r+1}$, which is a contradiction.

Now re-arranging the local pseudo-orthonormal basis if necessary or even taking another pseudo-orthonormal basis of principal directions, we may assume that there exists some $m \in \{1, ..., n-1\}$ such that

$$\begin{cases}\left\langle\overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i\right\rangle \neq 0 & \text{for } i = 1, ..., m, \text{ and } \overset{\star}{k}_1< \cdots <\overset{\star}{k}_m \\ \left\langle\overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i\right\rangle = 0 & \text{for } i = m+1, ..., 1\end{cases}. \tag{7.15}$$

By induction on the cardinality, we will prove that for every subset $J \subset \{1, ..., m\}$,

$$\overset{\star J}{S}_{r+1} = -(r/2) \overset{\star}{S}_{r+1} \quad \text{on } \mathscr{U}_{r+1}. \tag{7.16}$$

For $card(J) = 1$, (7.16) is nothing but (7.13). Let us assume that (7.16) holds for every set J with $card(J) = 1, 2, ..., p < m$ and take a set $J_0 = \{j_1, ..., j_{p+1}\} \subset \{1, ..., m\}$. Let J_1 and J_2 be the two sets of cardinality p such that

$$J_0 = \underbrace{\{j_1, j_3, ..., j_{p+1}\}}_{J_2} \cup \{j_2\} = \underbrace{\{j_2, j_3, ..., j_{p+1}\}}_{J_1} \cup \{j_1\}.$$

By using the induction hypothesis applied to J_1 and J_2 one has $\overset{\star J_1}{S}_{r+1} = \overset{\star J_2}{S}_{r+1} = -(r/2) \overset{\star}{S}_{r+1}$. Now, bearing in mind Proposition 2.1, from the first equality of last equation we obtain

$$k_{j_2} \overset{\star J_0}{S}_r + \overset{\star J_0}{S}_{r+1} = k_{j_2} \overset{\star J_0}{S}_r + \overset{\star J_0}{S}_{r+1},$$

and then $(k_{j_1} - k_{j_2}) \overset{\star J_0}{S}_r = 0$. From here and (7.15) one has $\overset{\star J_0}{S}_r = 0$, and then,

$$(2 + r/2) \overset{\star}{S}_{r+1} = \overset{\star J_1}{S}_{r+1} = \overset{\star J_2}{S}_{r+1} = \overset{\star J_0}{S}_{r+1}.$$

Thus (7.16) holds. From (7.7) and (7.15) one has $\mathcal{U} \overset{\star}{E}_i = \eta_i \overset{\star}{E}_i$ for $i = 1, \ldots, m$, where $\eta_i = (-1)^{r+1}(r+1) \overset{\star}{S}_{r+1} \overset{\star}{k}_i$ is screen constant as eigenvalue of the screen constant matrix $\mathcal{U}$. from (7.16) for the set $J = \{1, ..., m\}$ we get

$$-(r/2) \overset{\star}{S}_{r+1} = \sum_{m < i_1 < \cdots < i_{r+1} < n} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_{r+1}} = (-1)^{r+1} \frac{\displaystyle\sum_{m < i_1 < \cdots < i_{r+1} < n} \eta_{i_1} \cdots \eta_{i_{r+1}}}{(r+1)^{r+1} \overset{\star r+1}{S}_{r+1}},$$

showing that $\overset{\star}{H}_{r+1}$ is locally constant on $\mathscr{U}_{r+1}$, which is a contradiction. This finishes the proof. $\square$

7.1 The classification theorem

Theorem 7.1. *Let* $x : (M, g, N) \to \mathbb{R}^{n+2}_1$ *be a normalized null hypersurface which carries a UCC-normalization* N *with vanishing* $1-$*form* τ, *and let* L_r *be the linearized operator of the* $(r+1)-$*mean curvature of* M, *for some fixed* $r = 1, ..., n$. *Then the immersion* x *satisfies the equation* $L_r x = \mathcal{U} x + b$, *for some field of screen constant matrix* $\mathcal{U} \in \mathbb{R}^{(n+2) \times (n+2)}$ *and field of screen constant vector* $b \in \mathbb{R}^{n+2}$, *if and only if* (M, g, N) *is either an* $(r+1)-$*maximal null hypersurface or an almost isoparametric normalized null hypersurface with* $N = \mathcal{U}_0 x + b_0$, *for some field of screen constant matrix* $\mathcal{U}_0 \in \mathbb{R}^{(n+2) \times (n+2)}$ *and field of screen constant vector* $b_0 \in \mathbb{R}^{n+2}$.

Proof. If (M, g, N) is one of the null hypersurfaces mentioned then, equality (7.1) show that (M, g, N) satisfies $L_r x = \mathcal{U} x + b$. Conversely, let us assume that $x : (M, g, N) \to \mathbb{R}^{n+2}_1$ satisfies the condition $L_r x = \mathcal{U} x + b$, for some field of screen constant matrix $\mathcal{U} \in \mathbb{R}^{(n+2) \times (n+2)}$ and field of screen constant vector $b \in \mathbb{R}^{n+2}$. By Lemma 7.1 we know that $\overset{\star}{H}_{r+1}$ is screen constant. Let

us assume that $\overset{\star}{H}_{r+1}$ is a non-zero constant (otherwise, there is nothing to prove). Then, from (7.7) and (7.8), one obtains

$$UX = (-1)^{r+1}(r+1)\, \overset{\star}{S}_{r+1} \overset{\star}{A}_\xi(X), \tag{7.17}$$

$$\mathcal{U}N + \mathcal{U}\xi = -tr\left(\overset{\star}{A}_\xi^2 \circ \overset{\star}{T}_r\right)\xi = \alpha\xi. \tag{7.18}$$

Covariant derivative of (7.18) gives us $\overline{\nabla}_X(\mathcal{U}N + \mathcal{U}\xi) = \langle \overset{\star}{\nabla}\, \alpha, X\rangle \xi - \alpha\, \overset{\star}{A}_\xi(X)$, and by using (7.17), $\overline{\nabla}_X(\mathcal{U}N + \mathcal{U}\xi) = -2\mathcal{U}\, \overset{\star}{A}_\xi(X) = 2(-1)^r(r+1)\, \overset{\star}{S}_{r+1}\overset{\star}{A}_\xi^2(X)$. Hence, α is screen constant and

$$\overset{\star}{A}_\xi\left(\overset{\star}{A}_\xi - \lambda I\right) = 0,$$

where $\lambda = (-1)^{r+1}\dfrac{\alpha}{2(r+1)\overset{\star}{S}_{r+1}} = \dfrac{tr\left(\overset{\star}{A}_\xi^2 \circ \overset{\star}{T}_r\right)}{2tr\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r\right)}$ is screen constant. Then (M, g, N) has at most one

non-zero principal curvature which is screen constant. Thus $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ is an almost isoparametric normalized null hypersurface of $\mathbb{R}_1^{n+2}$. $\qquad\square$

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