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Algorithm for finite family of variational inequality with fixed point of two non-expansive mappings

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Abstract. In the recent work, a new hybrid technique for computing the common solution of fixed point of a finite family of two non-expansive mapping and variational inequality problem for inverse strongly monotone mapping in a real Hilbert space is provided. We also demonstrate the convergence of the hybrid approach using an example.

Keywords. Non-expansive mapping, fixed point theory, inverse strongly monotone mapping, variational inequality

1 Introduction

All over, let $\mathcal{H}$ be a real Hilbert space with norm and inner product expressed as $||.||$ and $\langle ., . \rangle$, jointly. Let $\mathcal{D}$ be a nonempty closed and convex subset of $\mathcal{H}$ and $P_{\mathcal{D}}$ represent metric projection of $\mathcal{H}$ onto $\mathcal{D}$. Let $\mathcal{S} : \mathcal{D} \to \mathcal{D}$ be a non-expansive mapping defined as

$$||\mathcal{S}a - \mathcal{S}b|| \leq ||a - b|| \ \forall \ a, b \in \mathcal{D}.$$

Here, $Fix(\mathcal{S})$ serve as a set of fixed point of $\mathcal{S}$.

Let $\mathcal{A} : \mathcal{D} \to \mathcal{H}$ be a nonlinear mapping. The problem of classical variational inequality is to find a point $a \in \mathcal{D}$ such that

$$\langle \mathcal{A}a, b - a \rangle \geq 0, \ \forall \ b \in \mathcal{D} \tag{1.1}$$

Here, $\mathcal{VI}(\mathcal{D}, \mathcal{A})$ serves as a solution set of the variational inequality problem.

Firslty, $\mathcal{VIP}$ was stated by Lions [7], after Lions [7] many researcher worked on $\mathcal{VIP}$ (1.1) to find its solution in finite and infinte dimensional spaces. There are various well-liked methods for solving variational inequalities. The first approaches, which were based on Lemke and Howson [1], were referred to as a fixed point approaches. Scarf [5] put out the first algorithm to approximate a continuous mapping's fixed point. These techniques have been used in numerous applications. Also, nonlinear optimization, which reformulate the variational inequality problem into similar optimization problems, is another technique for resolving variational inequalities. Recently many

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researchers work in this direction, for instance, see [10, 15] .

In 1976, Korpelevich [4] planned an iterative algorithm $\{x_n\}$ for $\mathcal{VIP}$ (1.1) defined as:

$$y_n = P_{\mathcal{D}}(x_n - \xi \mathcal{A} x_n)$$

$$x_{n+1} = P_{\mathcal{D}}(x_n - \xi \mathcal{A} y_n), n \geq 0$$

where $P_{\mathcal{D}}$ is the metric projection from $\mathcal{R}^n$ onto $\mathcal{D}, \mathcal{A} : \mathcal{D} \to \mathcal{H}$ is a monotone operator and ξ is a constant and proved the strong convergence of sequence $\{x_n\}$.

Takahashi and Toyoda [16], defined an algorithm to find the common element of $Fix(\mathcal{S}) \cap \mathcal{VI}(\mathcal{A}, \mathcal{D})$ by considering $\mathcal{S}$ and $\mathcal{A}$ must be non-expansive mapping and inverse strongly monotone mapping, respectively, as

$$x_1 = x \in \mathcal{D}$$

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\mathcal{S} P_{\mathcal{D}}(x_n - \eta_n \mathcal{A} x_n), n \geq 1$$

where $\{\beta_n\}$ is a sequence in (0,1) and $\{\eta_n\}$ is a sequence in $(0,\ 2\beta)$ where $\beta > 0$.

Marino and Xu [6] represents the consecutive iteration which depends on viscosity approximation method, as:

$$x_{n+1} = (I - \alpha_n \mathcal{A})\mathcal{S} x_n + \alpha_n \xi g(x_n), n \geq 0,$$

where $\mathcal{A}$ is a strongly positive bounded linear operator on $\mathcal{H}$. They proved the strong convergence, under some convinient conditions, to unique solution of the following variational inequality

$$\langle (\mathcal{A} - \xi g)x^*, x - x^* \rangle \geq 0, x \in \mathcal{D}. \tag{1.2}$$

Cho and Qin [18] proposed a continued composite iterative algorithm $\{x_n\}$ as follows:

$$\begin{aligned} x_1 &= x \in \mathcal{D}, \\ z_n &= \xi_n x_n + (1 - \xi_n)\mathcal{S} x_n, \\ y_n &= \beta_n x_n + (1 - \beta_n)\mathcal{S} z_n, \\ x_{n+1} &= \alpha_n \xi g(x_n) + \gamma_n x_n + [(1 - \gamma_n)I - \alpha_n \mathcal{A}]y_n,\ \forall\ n \geq 1, \end{aligned} \tag{1.3}$$

where $g, \mathcal{S}, \mathcal{A}$ are contraction, non-expansive and strongly positive linear bounded self adjoint operator respectively. Iteration (1.3) converges strongly to a fixed point of $\mathcal{S}$, which also solves (1.2).

Husain and Singh [14] proposed an iteration which is combination of Korpelevich's extra-gradient method, viscosity approximation method and the hybrid steepest method, as follows:

$$\begin{aligned} y_{n,1} &= \alpha_{n,1}\mathcal{S}_1 x_n + (1 - \beta_{n,1})x_n, \\ y_{n,l} &= \beta_{n,l}\mathcal{S}_l x_n + (1 - \beta_{n,l})x_{n,l-1}, \quad l = 1, 2, 3...., M \\ x_{n+1} &= \alpha_n \xi g(x_n) + \gamma_n x_n + [(1 - \gamma_n)I - \alpha_n \mathcal{A}]P_{\mathcal{D}}(y_{n,l} - \theta_l \mathcal{F}_l y_{n,l}), \forall\ n \geq 1. \end{aligned}$$

where $g, \mathcal{S}, \mathcal{A}$ are contraction, non-expansive and strongly positive linear bounded self adjoint operator, respectively. Iteration (**??**) converges strongly to common solution of finite family of non-expansive mappings and variational inequality, whose solution set is denoted by Ω, defined as:

$$\langle \xi g(x) - \mathcal{A} z, x - z \rangle \geq 0\ \forall\ z \in \Omega. \tag{1.4}$$

Furthermore, research is being carried out in the same direction [9, 11, 12]. In this paper, motivated from [14, 18], we proposed and evaluate an iteration and prove its strong convergence to fixed point of finite family of two non-expansive mappings and solution set of variational inequality.

2 Preliminaries

In this section, we collect some lemma's and definition's which are useful for our result.

Definition 1. $P_{\mathcal{D}}$ is metric projection from $\mathcal{H}$ onto $\mathcal{D}$, if for every point $s \in \mathcal{H}$ there exist a unique nearest point in $\mathcal{D}$, such that

$$||s - P_{\mathcal{D}}s|| \leq ||s - t|| \;\forall\; t \in \mathcal{D} \tag{2.1}$$

Also, $x \in \mathcal{VI}(\mathcal{A}, \mathcal{D}) \iff x = P_{\mathcal{D}}(x - \xi\mathcal{A}x), \xi > 0.$

Definition 2. A mapping $B : \mathcal{D} \to \mathcal{H}$ is said to be

1. monotone if,
$$\langle Bs - Bt, s - t\rangle \geq 0 \;\forall s, t \in \mathcal{D};$$
2. L-lipschitz, if there exists a constant $L > 0$ such that
$$||Bs - Bt|| \leq L||s - t|| \;\forall s, t \in \mathcal{D};$$
3. α-inverse strongly monotone, if there exists a positive real number α such that
$$\langle Bs - Bt, s - t\rangle \geq \alpha||Bs - Bt||^2 \;\forall s, t \in \mathcal{D}.$$

Lemma 2.1. *[2] Assume that A is strongly positive linear bounded self adjoint operator on a Hilbert space $\mathcal{H}$ with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq ||A||^{-1}$. Then $||I - \rho A|| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.2. *[6] Assume $\{a_n\}$ is a sequence of non-negative real numbers such that*
$$a_{n+1} \leq (1 - \nu_n)a_n + \delta_n,$$
where $\{\nu_n\}$ is a sequence in (0,1) and δ_n is a sequence such that
(i) $\sum_{n=1}^{\infty} \nu_n = \infty$
(ii) $\limsup_{n\to\infty}(\delta_n/\nu_n) \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. *[17] Let $\mathcal{H}$ be a real Hilbert space. Let $\mathcal{D}$ be a closed convex set and $T : \mathcal{D} \to \mathcal{D}$ be an L-Lipschitz quasi-pseudocontractive operator. Then we have*

$$\|(1 - \sigma)a + \sigma T((1 - \xi)a + \xi Ta) - b\|^2 \leq \|a - b\|^2 + \sigma(\sigma - \xi)\|T((1 - \xi)a + \xi Ta) - a\|^2,$$

for all $a \in \mathcal{D}$ and $b \in Fix(T)$ when $0 < \sigma < \xi < \frac{1}{\sqrt{1+L^2}+1}$.

Lemma 2.4. *[17] Let $\mathcal{H}$ be a real Hilbert space. Let $\mathcal{D}$ be a closed convex set. If the operator $T : \mathcal{D} \to \mathcal{D}$ is L-Lipschitz with $L \geq 1$, then*

$$Fix(((1 - \delta)I + \delta T)T) = Fix(T(1 - \delta)I - \delta T) = Fix(T),$$

where $\delta \in (0, \frac{1}{L})$.

Lemma 2.5. *(Demiclosed principle)[8] Let $\mathcal{D}$ be a non empty closed and convex subset of a real Hilbert space $\mathcal{H}$. Let $\mathcal{S}$ be a non-expansive self mapping on $\mathcal{H}$ with $Fix(\mathcal{S}) \neq \phi$. Then $I - \mathcal{S}$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in $\mathcal{H}$ weakly converging to some $x \in \mathcal{H}$ and the sequence $\{(I - \mathcal{S})x_n\}$ strongly converges to some t, it follows that $(I - \mathcal{S})x = t$. I is identity operator in $\mathcal{H}$.*

Lemma 2.6. *[3] Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{D}$ be a closed convex subset of $\mathcal{H}$. For any integer $N \geq 1$, assume that for each $1 \leq i \leq N, \delta_i : \mathcal{D} \to \mathcal{H}$ is a k_i-strictly pseudocontractive mapping for some $0 \leq k_i < 1$. Assume that $\{\varphi_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \varphi_i = 1$. Then $\sum_{i=1}^N \varphi_i \delta_i$ is a k-strictly pseudocontractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$.*

Lemma 2.7. *[3] Let $\{\delta_i\}_{i=1}^N$ and $\{\varphi_i\}_{i=1}^N$ be as in Lemma 2.6. Suppose that $\{\delta_i\}_{i=1}^N$ has a common fixed point in $\mathcal{D}$. Then $Fix(\sum_{i=1}^N \varphi_i \delta_i) = \cap_{i=1}^N Fix(\delta_i)$.*

Rockafellar[13] defined monotone mapping for set valued mapping and maximal monotone with the help of normal cone.

3 Main Result

Let $\mathcal{D}$ be closed and convex subset of real Hilbert space $\mathcal{H}$, let $\mathfrak{A}$ be strongly positive linear bounded self adjoint operator, $g : \mathcal{D} \to \mathcal{D}$ be a ξ-contraction with coefficient $\xi \in [0, 1]$, $F_l : \mathcal{D} \to \mathcal{H}$ be a ς_l- inverse strongly monotone mappings and $\mathcal{S}_l : \mathcal{D} \to \mathcal{D}$, $\mathcal{T}_l : \mathcal{D} \to \mathcal{D}$ are two finite family of non-expansive mappings for all $1 \leq l \leq M$, where M is some positive integer. Let $\{\sigma_{n,l}\}, \{\delta_{n,l}\}, \{\beta_{n,l}\}, \{\eta_{n,l}\}$ for all $1 \leq l \leq M$, $\{a_n\}, \{c_n\}$ be the sequences in [0,1]. For an arbitrary given $x_1 = x \in \mathcal{D}$, we recommend the accompanying hybrid iterative algorithm:

$$\begin{aligned}
y_{n,1} &= (1 - \sigma_{n,1})x_n + \sigma_{n,1}\mathcal{T}_1((1 - \delta_{n,1})x_n + \delta_{n,1}\mathcal{T}_1 x_n) \\
y_{n,l} &= (1 - \sigma_{n,l})y_{n,l-1} + \sigma_{n,l}\mathcal{T}_l((1 - \delta_{n,l})y_{n,l-1} + \delta_{n,l}\mathcal{T}_l y_{n,l-1}) \\
z_{n,1} &= (1 - \beta_{n,1})y_{n,1} + \beta_{n,1}\mathcal{S}_1((1 - \eta_{n,1})y_{n,1} + \eta_{n,1}\mathcal{S}_1 y_{n,1}) \\
z_{n,l} &= (1 - \beta_{n,l})z_{n,l-1} + \beta_{n,l}\mathcal{S}_l((1 - \eta_{n,l})z_{n,l-1} + \eta_{n,l}\mathcal{S}_l z_{n,l-1}) \\
x_{n+1} &= a_n \xi g(x_n) + c_n x_n + [(1 - c_n)I - a_n \mathfrak{A}]P_{\mathcal{D}}(z_{n,l} - \theta_l F_l z_{n,l}) + a_n P_{\mathcal{D}}(y_{n,l} - \theta_l F_l y_{n,l})
\end{aligned} \tag{3.1}$$

The sequence $\{x_n\}$ defined by iteration (3.1) converges strongly to a common point of the solution set of finite family of variational inequalities for an inverse strongly monotone mapping and solution set of common fixed point of a finite family of two non-expansive mapping.

Theorem 3.1. *Let $\mathcal{D}$ be closed and convex subset of real Hilbert space $\mathcal{H}$, let $\mathfrak{A}$ be strongly positive linear bounded self adjoint operator with the coefficient $\zeta > 0$, $g : \mathcal{D} \to \mathcal{D}$ be a ξ-contraction with coefficient $\xi \in [0, 1]$, $F_l : \mathcal{D} \to \mathcal{H}$ be a ς_l- inverse strongly monotone mappings for each $1 \leq l \leq M$, where M is some positive integer, and $\mathcal{S}_l : \mathcal{D} \to \mathcal{D}$, $\mathcal{T}_l : \mathcal{D} \to \mathcal{D}$ are two finite family of non-expansive mappings with $\Omega = \cap_{l=1}^M((Fix(\mathcal{S}_l) \cap Fix(\mathcal{T}_l)) \cap (\cap_{l=1}^M \mathcal{V}I(F_l, \mathcal{D})) \neq \phi$. Assume $0 < \xi \leq \frac{\zeta}{\varsigma}$ and let $\{\theta_l\}$ be a real numbers in $(0, 2\varsigma_l)$. Let $\{\sigma_{n,l}\}, \{\delta_{n,l}\}, \{\beta_{n,l}\}, \{\eta_{n,l}\}$, and $\{c_n\}$ be sequences in (0,1) fulfilling the following conditions:*

(i) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$

(ii) $\sum_{n=1}^{\infty} |a_n - a_{n-1}| < \infty$ *and* $\sum_{n=1}^{\infty} |c_n - c_{n-1}| < \infty$.

(iii) $\sum_{n=1}^{\infty} |\sigma_{n,l} - \sigma_{n-1,l}| < \infty, \sum_{n=1}^{\infty} |\beta_{n,l} - \beta_{n-1,l}| < \infty, \sum_{n=1}^{\infty} |\delta_{n,l} - \delta_{n-1,l}| < \infty$ *and* $\sum_{n=1}^{\infty} |\eta_{n,l} - \eta_{n-1,l}| < \infty$. *for each l=1,2.....M.*

(iv) $0 < \liminf_{n\to\infty} \sigma_{n,l} \leq \limsup_{n\to\infty} \sigma_{n,l} < 1$,
$0 < \liminf_{n\to\infty} \beta_{n,l} \leq \limsup_{n\to\infty} \beta_{n,l} < 1$,
$0 < \liminf_{n\to\infty} \delta_{n,l} \leq \limsup_{n\to\infty} \delta_{n,l} < 1$
and $0 < \liminf_{n\to\infty} \eta_{n,l} \leq \limsup_{n\to\infty} \eta_{n,l} < 1$.

Then, the $\{x_n\}$ *sequence characterized by iteration* (3.1) *converges strongly to* $x \in \Omega$, *where* $x = P_\Omega(\xi g + (I - \mathfrak{A}))x$ *is also unique solution of the consecutive variational inequality:*

$$\langle \xi g(x) - \mathfrak{A}z, z - x\rangle \leq 0 \ \forall\ z \in \Omega. \tag{3.2}$$

Proof. Consider any $x, y \in \mathcal{D}$ and $\theta_l = (0, 2\varsigma_l)$, we get

$$\begin{aligned}
&\|(I - \theta_l F_l)x - (I - \theta_l F_l)y\|^2 \\
&\quad = \|(x - y) - \theta_l(F_l x - F_l y)\|^2 \\
&\quad = \|x - y\|^2 - 2\theta_l\langle F_l x - F_l y, x - y\rangle + \theta_l^2\|F_l x - F_l y\|^2 \\
&\quad \leq \|x - y\|^2 - \theta_l(2\varsigma_l - \theta_l)\|F_l x - F_l y\|^2
\end{aligned}$$

This implies that, $(I - \theta_l F_l)$ is non-expansive. Since $\mathfrak{A}$ is linear bounded self adjoint operator , so

$$\|\mathfrak{A}\| = \sup\{|\langle \mathfrak{A}s, s\rangle| : s \in \mathbb{H}, \|\mathrm{s}\| = 1\}.$$

We assume that

$$\|\mathfrak{A}\|a_n < 1 - c_n \ \forall\ n \geq 0,$$

and since $\lim_{n\to\infty} a_n = 0$, so without loss of generality, we have

$$\begin{aligned}
\langle [(1 - c_n)I - a_n\mathfrak{A}]s, s\rangle &= 1 - c_n - a_n\langle \mathfrak{A}s, s\rangle \\
&\leq 1 - c_n - a_n\|\mathfrak{A}\| \leq 0,
\end{aligned}$$

so $(1 - c_n)I - a_n\mathfrak{A}$ is positive. So, it gives us

$$\begin{aligned}
\|(1 - c_n)I - a_n\mathfrak{A}\| &= \sup\{\langle [(1 - c_n)I - a_n\mathfrak{A}]s, s\rangle : s \in \mathbb{H}, \|\mathrm{s}\| = 1\} \\
&= \sup\{1 - c_n - a_n\langle \mathfrak{A}s, s\rangle : s \in \mathbb{H}.\|\mathrm{s}\| = 1\} \\
&\leq 1 - c_n - a_n\zeta.
\end{aligned}$$

Next, we prove that the sequence $\{x_n\}$ is bounded. Taking $x^* \in \Omega$, for $l = 1$

$$\|y_{n,1} - x^*\|^2 = \|(1 - \sigma_{n,1})x_n + \sigma_{n,1}\mathcal{T}_1((1 - \delta_{n,1})x_n + \delta_{n,1}\mathcal{T}_1 x_n) - x^*\|^2$$

By Lemma 2.3, we get

$$\begin{aligned}
&\leq \|x_n - x^*\|^2 + \sigma_{n,1}(\sigma_{n,1} - \delta_{n,1})\|\mathcal{T}_1((1 - \delta_{n,1})x_n + \delta_{n,1}\mathcal{T}_1 x_n) - x_n\|^2 \\
&\leq \|x_n - x^*\|^2
\end{aligned}$$

By induction over $l = 2$ to $l = M$, we obtain

$$\begin{aligned}
\|y_{n,l} - x^*\|^2 &= \|(1-\sigma_{n,l})y_{n,l-1} + \sigma_{n,l}\mathcal{T}_l((1-\delta_{n,l})y_{n,l-1} + \delta_{n,l}\mathcal{T}_l y_{n,l-1}) - x^*\|^2 \\
&\leq \|y_{n,l-1} - x^*\|^2 + \sigma_{n,l}(\sigma_{n,l} - \delta_{n,l})\|\mathcal{T}_l((1-\delta_{n,l})y_{n,l-1} \\
&\quad + \delta_{n,l}\mathcal{T}_l y_{n,l-1} - y_{n,l-1}||^2 \\
&\leq \|y_{n,l-1} - x^*\|^2 \\
&\leq \cdots \\
&\leq \|y_{n,1} - x^*\|^2
\end{aligned}$$

Then for every $l = 1, 2, ..., M$, we get

$$\|y_{n,l} - x^*\|^2 \leq \|x_n - x^*\|^2$$

Similarly, for the $z_{n,1}$ for $l = 2$ to $l = M$, we have

$$\begin{aligned}
\|z_{n,1} - x^*\|^2 &\leq \|y_{n,1} - x^*\|^2 \leq \|x_n - x^*\|^2 \\
\|z_{n,l} - x^*\|^2 &\leq \|z_{n,l-1} - x^*\|^2 \\
&\leq \cdots \leq \|z_{n,1} - x^*\|^2 \leq \|x_n - x^*\|^2.
\end{aligned}$$

From iteration (3.1), we calculate

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|a_n\xi g(x_n) + c_n x_n + [(1-c_n)I - a_n\mathfrak{A}]P_{\mathcal{D}}(z_{n,l} - \theta_l F_l z_{n,l}) \\
&\quad + a_n P_{\mathcal{D}}(y_{n,l} - \theta_l F_l y_{n,l}) - x^*\| \\
&= \|a_n(\xi g(x_n) - \mathfrak{A}x^* + x^*) + c_n(x_n - x^*) \\
&\quad + (1 - c_n - a_n\zeta)(P_{\mathcal{D}}(z_{n,l} - \theta_l F_l z_{n,l}) - x^*) + a_n(P_{\mathcal{D}}(y_{n,l} - \theta_l F_l y_{n,l}) - x^*)\| \\
&\leq \|a_n(\xi g(x_n) - \xi g x^*) + a_n(\xi g(x^*) - \mathfrak{A}x^* + x^*) + c_n(x_n - x^*) \\
&\quad + ((1-c_n)I - a_n\mathfrak{A})(P_{\mathcal{D}}(z_{n,l} - \theta_l F_l z_{n,l}) - x^*) + a_n(P_{\mathcal{D}}(y_{n,l} - \theta_l F_l y_{n,l}) - x^*)\| \\
&\leq a_n\|\xi g(x_n) - \xi g x^*|| + a_n||\xi g(x^*) - \mathfrak{A}x^* + x^*\| + c_n\|x_n - x^*\| \\
&\quad + (1 - c_n - a_n\zeta)\|P_{\mathcal{D}}(z_{n,l} - \theta_l F_l z_{n,l}) - x^*\| + a_n\|P_{\mathcal{D}}(y_{n,l} - \theta_l F_l y_{n,l}) - x^*)\| \\
&\leq a_n\xi\tau||x_n - x^*|| + a_n||\xi g(x^*) - \mathfrak{A}x^* + x^*\| + c_n\|x_n - x^*\| \\
&\quad + (1 - c_n - a_n\zeta)\|x_n - x^*\| + a_n\|x_n - x^*\| \\
&= (1 - a_n(\zeta - \xi\tau - 1))\|x_n - x^*\| + a_n\frac{\zeta - \xi\tau - 1}{\zeta - \xi\tau - 1}\|\xi g(x^*) - \mathfrak{A}x^* + x^*\| \\
&\leq \max\{\|x_n - x^*\|, \frac{1}{\zeta - \xi\tau - 1}\|\xi g(x^*) - \mathfrak{A}x^* + x^*\|\}
\end{aligned}$$

By induction on n, we have

$$\leq \max\{\|x_0 - x^*\|, \frac{1}{\zeta - \xi\tau - 1}\|\xi g(x^*) - \mathfrak{A}x^* + x^*\|\}$$

Thus $\{x_n\}$ is bounded and so is $\{z_{n,l}\}$ and $\{y_{n,l}\}$ for $l = 1, 2, ..., M$.
Next, we prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$
For $l = 1$

$$\begin{aligned}
\|y_{n,1} - y_{n-1,1}\| &= \|(1-\sigma_{n,1})x_n + \sigma_{n,1}\mathcal{T}_1((1-\delta_{n,1})x_n + \delta_{n,1}\mathcal{T}_1 x_n) - (1-\sigma_{n-1,1})x_{n-1} \\
&\quad - \sigma_{n-1,1}\mathcal{T}_1((1-\delta_{n-1,1})x_{n-1} + \delta_{n-1,1}\mathcal{T}_1 x_{n-1})\| \\
&= \|(x_n - x_{n-1}) - \sigma_{n,1}(x_n - x_{n-1}) + x_{n-1}(\sigma_{n-1,1} - \sigma_{n,1}) \\
&\quad + \sigma_{n,1}(\mathcal{T}_1((1-\delta_{n,1})x_n + \delta_{n,1}\mathcal{T}_1 x_n) - \mathcal{T}_1((1-\delta_{n-1,1})x_{n-1} + \delta_{n-1,1}\mathcal{T}_1 x_{n-1}) \\
&\quad + (\sigma_{n,1} - \sigma_{n-1,1})\mathcal{T}_1((1-\delta_{n-1,1})x_{n-1} + \delta_{n-1,1}\mathcal{T}_1 x_{n-1})\|
\end{aligned}$$

Replace

$$\mathcal{T}_1((1-\delta_{n-1,1})x_{n-1} + \delta_{n-1,1}\mathcal{T}_1 x_{n-1}) \text{ with } \mathcal{T}_1(n-1,1)x_{n-1}$$

and

$$\mathcal{T}_1((1-\delta_{n,1})x_n+\delta_{n,1}\mathcal{T}_1x_n) \text{ with } \mathcal{T}_1(n,1)x_n$$

and similiarly for the $\mathcal{S}_l$, we have

$$\begin{aligned}\|y_{n,1}-y_{n-1,1}\| &\le \|x_n-x_{n-1}\|+|\sigma_{n,1}-\sigma_{n-1,1}|.\|\mathcal{T}_1(n-1,1)x_{n-1}-x_{n-1}\|\\ &\quad+|\sigma_{n,1}|\cdot\|(\mathcal{T}_1(n,1)x_n-x_n)-(\mathcal{T}_1(n-1,1)x_{n-1}-x_{n-1})\|\\ \|y_{n,l}-y_{n-1,l}\| &= \|(1-\sigma_{n,l})y_{n,l-1}+\sigma_{n,l}\mathcal{T}_l(n,l-1)y_{n,l-1}-(1-\sigma_{n-1,l})y_{n-1,l-1}\\ &\quad-\sigma_{n-1,l}\mathcal{T}_l(n-1,l-1)y_{n-1,l-1}\|\\ &\le \|y_{n,l-1}-y_{n-1,l-1}\|+|\sigma_{n,l}|\cdot\|(\mathcal{T}_l(n,l-1)y_{n,l-1}-y_{n,l-1})\\ &\quad-(\mathcal{T}_l(n-1,l-1)y_{n-1,l-1}-y_{n-1,l-1})\|\\ &\quad+|\sigma_{n,l}-\sigma_{n-1,l}|\cdot\|\mathcal{T}_l(n-1,l)y_{n-1,l}-y_{n-1,l}\|\end{aligned}$$

$$\begin{aligned}\|y_{n,l}-y_{n-1,l}\| \le \|x_n-x_{n-1}\|+\sum_{i=1}^{M}\{&|\sigma_{n,i}|\{(\mathcal{T}_i(n,i-1)y_{n,i-1}-y_{n,i-1})\\ &-(\mathcal{T}_i(n-1,i-1)y_{n-1,i-1}-y_{n-1,i-1})\}\\ &+|\sigma_{n,i}-\sigma_{n-1,i}|\cdot\|\mathcal{T}_i(n-1,i-1)y_{n-1,i-1}-y_{n-1,i-1}\|\}\end{aligned} \tag{3.3}$$

In a similar manner, we solve $\{z_{n,l}\}$ for $l=1$,

$$\begin{aligned}\|z_{n,1}-z_{n-1,1}\| \le \|y_{n,1}-y_{n-1,1}\|&+|\beta_{n,1}|.\|(\mathcal{S}_1(n,1)y_{n,1}-y_{n,1})-(\mathcal{S}_1(n-1,1)y_{n-1,1}\\ &-y_{n-1,1})\|+|\beta_{n,1}-\beta_{n-1,1}|.\|\mathcal{S}_1(n-1,1)y_{n-1,1}-y_{n-1,1}\|\end{aligned}$$

and for $l=2$ to M,

$$\begin{aligned}\|z_{n,l}-z_{n-1,l}\| \le \|z_{n,l-1}-z_{n-1,l-1}\|&+|\beta_{n,l}|.\|(\mathcal{S}_l(n,l)z_{n,l-1}-z_{n,l-1})\\ &-(\mathcal{S}_l(n-1,l)z_{n-1,l}-z_{n-1,l})\|+|\beta_{n,l}-\beta_{n-1,l})|.\\ &||\mathcal{S}_l(n-1,l)z_{n-1,l}-z_{n-1,l}\|\end{aligned}$$

Thus, for overall $l=1,...,M$, we obtain

$$\begin{aligned}\|z_{n,l}-z_{n-1,l}\| \le \|y_{n,1}-y_{n-1,l}\|&+\sum_{i=1}^{M}\{|\beta_{n,i}|.||\mathcal{S}_i(n,i)z_{n,i}-z_{n,i-1}\\ &-(\mathcal{S}_i(n-1,i)z_{n-1,i}-z_{n-1,i})\|\}+|\beta_{n,i}-\beta_{n-1,i}|.\|\mathcal{S}_i(n-1,i)z_{n-1,i}-z_{n-1,i}\|\end{aligned}$$

Next,

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|(a_n\xi g(x_n)+c_nx_n+[(1-c_n)I-a_n\mathfrak{A}]P_{\mathcal{D}}(z_{n,l}-\theta_lF_lz_{n,l})+a_nP_{\mathcal{D}}(y_{n,l}\\ &\quad-\theta_lF_ly_{n,l}))-(a_{n-1}\xi g(x_{n-1})+c_{n-1}x_{n-1}\\ &\quad+[(1-c_{n-1})I-a_{n-1}\mathfrak{A}]P_{\mathcal{D}}(z_{n-1,l}-\theta_lF_lz_{n-1,l})+a_{n-1}P_{\mathcal{D}}(y_{n-1,l}-\theta_lF_ly_{n-1,l}))\|\\ &= \|a_n\xi(g(x_n)-g(x_{n+1})+\xi g(x_{n-1})(a_n-a_{n-1})+c_n(x_n-x_{n-1})+x_{n-1}(c_n-c_{n-1})\\ &\quad+[(1-c_n)I-a_n\mathfrak{A}][P_{\mathcal{D}}(z_{n,l}-\theta_lF_lz_{n,l})-P_{\mathcal{D}}(z_{n-1,l}-\theta_lF_lz_{n-1,l})]\\ &\quad+P_{\mathcal{D}}(z_{n-1,l}-\theta_lF-lz_{n-1,l})[(1-c_n)-a_nG-((1-c_{n-1})I-a_{n-1}G)]\\ &\quad+a_n[P_{\mathcal{D}}(y_{n,l}-\theta_lF_ly_{n,l})-P_{\mathcal{D}}(y_{n-1,l}-\theta_lF_ly_{n-1,l})]\\ &\quad+P_{\mathcal{D}}(y_{n-1,l}-\theta_lF_ly_{n-1,l})(a_n-a_{n-1})\|\\ &= |a_n\xi\tau|.\|x_n-x_{n-1}\|+c_n\|x_n-x_{n-1}\|+\|\xi g(x_{n-1})\|.|a_n-a_{n-1}|\\ &\quad+|c_n-c_{n-1}|.\|x_{n-1}\|+|a_n-a_{n-1}|.\|P_{\mathcal{D}}(y_{n-1,l}-\theta_lF_ly_{n-1,l}\|\\ &\quad+(1-c_n-a_n\zeta).\|z_{n,l}-z_{n-1,l}\|+|c_{n-1}-c_n|.\|P_{\mathcal{D}}(z_{n-1,l}-\theta_lF_lz_{n-1,l})\|\\ &\quad+|a_{n-1}-a_n|.\|\mathfrak{A}.P_{\mathcal{D}}(z_{n-1,l}-\theta_lF_lz_{n-1,l})\|+a_n\|y_{n,l}-y_{n-1,l}\|\\ &=\|a_n\xi\tau+c_n|.\|x_n-x_{n-1}\|+|a_n-a_{n-1}|\{\|\xi g(x_{n-1})\|+\|P_{\mathcal{D}}(y_{n-1,l}-\theta_lF_lz_{n-1,l})\|\\ &\quad+\|\mathfrak{A}.P_{\mathcal{D}}(z_{n-1,l}-\theta_lF_lz_{n-1,l})\|\}+|c_n-c_{n-1}|\{\|x_{n-1}\|+\|P_{\mathcal{D}}(z_{n-1,l}-\theta_lF_lz_{n-1,l})\|\}\end{aligned}$$

$$
\begin{aligned}
&+|1-c_n-a_n\zeta|.\|z_{n,l}-z_{n-1,l}\|+|a_n|.\|y_{n,l}-y_{n-1,l}\|\\
=&\|1-a_n(\zeta-\xi\tau-1)|.\|x_n-x_{n-1}\|+|a_n-a_{n-1}|.\{\|\xi g(x_{n-1}\|\\
&+\|P_{\mathcal{D}}(y_{n-1,l}-\theta_l F_l y_{n-1,l}\|+\|G.P_{\mathcal{D}}(z_{n-1,l}-\theta_l F_l z_{n-1,l})\|\}\\
&+|c_n-c_{n-1}|.\{\|x_{n-1}\|+\|P_{\mathcal{D}}(z_{n-1,l}-\theta_l F_l z_{n-1,l})\|\\
&+a_n\{\sum_{i=2}^{M}|\sigma_{n,i}|.\|(\mathcal{T}_i(n,i-1)y_{n,i-1}-y_{n,i-1})-(\mathcal{T}_i(n-1,i-1)y_{n-1,i-1}-y_{n-1,i-1})\|\\
&+|\sigma_{n,i}-\sigma_{n-1,i}|.\|\mathcal{T}_i(n-1,i-1)y_{n-1,i-1}-y_{n-1,i-1}\|\}\\
&+|1-c_n-a_n\zeta|\{|\sigma_{n,1}-\sigma_{n-1,1}|.\|\mathcal{T}_1(n-1,1)x_{n-1}-x_{n-1}\|\\
&+|\sigma_{n,1}|.\|(\mathcal{T}_1(n,1)x_n-x_n)-(\mathcal{T}_1(n-1,1)x_{n-1}+x_{n-1})\|\\
&+\sum_{i=2}^{M}(\beta_{n,i}(\|(\mathcal{S}_i(n,i)z_{n,i}-z_{n,i})-(\mathcal{S}_i(n-1,i)z_{n-1,i}-z_{n-1,i})\|\\
&+|\beta_{n,i}-\beta_{n-1,i}|.\|\mathcal{S}_i(n-1,i)z_{n-1,i}-z_{n-1,i}\|)\}\\
\le&|1-a_n(\zeta-\xi\tau-1)|.\|x_n-x_{n-1}\|\\
&+M_1\big[|a_n-a_{n-1}|+|c_n-c_{n-1}|+a_n\sum_{i=1}^{M}|\sigma_{n,i}-\sigma_{n-1,i}|+|1-c_n-a_n\zeta|.\\
&\sum_{i=1}^{M}|\beta_{n,i}-\beta_{n-1,i}|+\sum_{i=1}^{M}|\sigma_{n,i}|+\sum_{i=1}^{M}|\beta_{n,i}|\big]
\end{aligned}
$$

where
$M_1=\max\{\sup_{n\ge1}(\|\xi g(x_{n-1})\|+\|P_{\mathcal{D}}(z_{n-1,l}-\theta_l F_l z_{n-1,l})\|+\|\mathfrak{A}.P_{\mathcal{D}}(z_{n-1,l}-\theta_l F_l z_{n-1,l})\|),\sup_{n\ge1}(\|x_{n-1}\|+\|P_{\mathcal{D}}(y_{n-1,i}-\theta_l F_l y_{n-1,l}\|),\ \sup_{n\ge1}(\sum_{i=1}^{M}\|(\mathcal{T}_i(n,i-1)-y_{n,i-1})-(\mathcal{T}_i(n-1,i-1)-y_{n-1,i-1})\|,$ $\sup_{n\ge1}\|(\mathcal{S}_i(n,i)z_{n,i-1}-z_{n,i-1})-(\mathcal{S}_i(n-1,i)z_{n-1,i}-z_{n-1,i})\|,\sup_{n\ge1}\|(\mathcal{T}_i(n,i-1)y_{n,i-1}-y_{n,i-1})-(\mathcal{T}_i(n-1,i-1)y_{n-1,i-1}-y_{n-1,i-1})\|,\ \sup_{n\ge1}\|(\mathcal{S}_i(n,i)z_{n,i}-z_{n,i})-(\mathcal{S}_i(n-1,i)z_{n-1,i}-z_{n-1,i})\|\}$
From imposed conditions, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.4}$$

After setting $u_{n,l}=P_{\mathcal{D}}(z_{n,l}-\theta_l F_l z_{n,l})$ and $v_{n,l}=P_{\mathcal{D}}(y_{n,l}-\theta_l F_l y_{n,l})$, we have

$$x_{n+1}=a_n\xi g(x_n)+c_n x_n+[(1-c_n)I-a_n\mathfrak{A}]u_{n,l}+a_n v_{n,l}$$

Next for $x^*\in\Omega$, we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2&=\|a_n\xi g(x_n)+c_n x_n+[(1-c_n)I-a_n\mathfrak{A}]u_{n,l}+a_n v_{n,l}-x^*\|^2\\
&=\|a_n(\xi g(x_n)-\mathfrak{A}x^*+x^*)+c_n(x_n-x^*)+[(1-c_n)I-a_n\mathfrak{A}](u_{n,l}-x^*)\\
&\quad+a_n(v_{n,l}-x^*)\|^2\\
&=\|a_n(\xi g(x_n)-\mathfrak{A}x^*+v_{n,l})+c_n(x_n-x^*)+[(1-c_n)I-a_n\mathfrak{A}](u_{n,l}-x^*)\|^2\\
&=\|a_n(\xi g(x_n)-\mathfrak{A}x^*+v_{n,l})\|^2+\|c_n(x_n-x^*)+[(1-c_n)I-a_n\mathfrak{A}]\\
&\quad(u_{n,l}-x^*)\|^2+2a_n\langle((1-c_n)I-a_n\mathfrak{A})(u_{n,l}-x^*)+c_n(x_n-x^*),\xi g(x_n)-\mathfrak{A}x^*+v_{n,l}\rangle\\
&=a_n^2\|(\xi g(x_n)-\mathfrak{A}x^*+v_{n,l})\|^2+\big[|c_n|.\|(x_n-x^*)\|+|1-c_n-a_n\zeta|.\|(u_{n,l}-x^*)\|\big]^2\\
&\quad+2a_n\langle((1-c_n)I-a_n\mathfrak{A})(u_{n,l}-x^*),\xi g(x_n)-\mathfrak{A}x^*+v_{n,l}\rangle\\
&\quad+2a_n c_n\langle x_n-x^*,\xi g(x_n)-\mathfrak{A}x^*+v_{n,l}\rangle\\
&=(1-c_n-a_n\zeta)^2.\|u_{n,l}-x^*\|^2+c_n^2\|x_n-x^*\|^2\\
&\quad+2c_n(1-c_n-a_n\zeta)\|u_{n,l}-x^*\|.\|x_n-x^*\|+d_n
\end{aligned}\tag{3.5}
$$

where

$$d_n = a_n^2\|(\xi g(x_n) - \mathfrak{A}x^* + v_{n,l})\|^2 + 2a_n\langle((1-c_n)I - a_n\mathfrak{A})(u_{n,l}-x^*), \xi g(x_n) - \mathfrak{A}x^* + v_{n,l}\rangle + 2a_nc_n\langle x_n - x^*, \xi g(x_n) - \mathfrak{A}x^* + v_{n,l}\rangle \tag{3.6}$$

As from equation(3.5), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= (1-c_n-a_n\zeta)^2.\|u_{n,l}-x^*\|^2 + c_n^2.\|x_n-x^*\|^2 \\
&\quad + (1-c_n-a_n\zeta)c_n\left[\|u_{n,l}-x^*\|^2 + \|x_n-x^*\|^2\right] + d_n \\
&= ((1-a_n\zeta)^2 - 2(1-a_n\zeta)c_n + c_n^2).\|u_{n,l}-x^*\|^2 + c_n^2\|x_n-x^*\|^2 \\
&\quad + \left[(1-a_n\zeta)c_n - c_n^2\right](\|u_{n,l}-x^*\|^2 + \|x_n-x^*\|^2) + d_n \\
&= (1-a_n\zeta)^2.\|u_{n,l}-x^*\|^2 - (1-a_n\zeta)c_n\|u_{n,l}-x^*\|^2 \\
&\quad + (1-a_n\zeta)c_n\|x_n-x^*\|^2 + d_n \qquad (3.7)\\
&= (1-a_n\zeta)(1-a_n\zeta-c_n)\|u_{n,l}-x^*\|^2 + (1-a_n\zeta)c_n\|x_n-x^*\|^2 + d_n \\
&\leq (1-a_n\zeta)(1-a_n\zeta-c_n)\left[\|(z_{n,l}-\theta_lF_lz_{n,l}) - (x^*-\theta_lF_lx^*\|\right]^2 \\
&\quad + (1-a_n\zeta)c_n\|x_n-x^*\|^2 + d_n \\
&\leq (1-a_n\zeta)(1-a_n\zeta-c_n)\left[\|z_{n,l}-x^*\|^2 - \theta_l(\theta_l - 2\varsigma_l)\right. \\
&\quad \left.\|F_lz_{n,l}-F_lx^*\|^2\right] + (1-a_n\zeta)c_n\|x_n-x^*\|^2 + d_n \\
&\leq \|x_n-x^*\|^2 + \theta_l(\theta_l-2.\varsigma_l)\|F_lz_{n,l}-F_lx^*\|^2 + d_n
\end{aligned}$$

From equation (3.5), we get

$$\begin{aligned}
-\theta_l(\theta_l-2.\varsigma_l)\|F_lz_{n,l}-F_lx^*\|^2 &\leq \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + d_n \\
&\leq (\|x_n-x^*\| + \|x_{n+1}-x^*\|)\|x_n-x_{n+1}\| + d_n
\end{aligned} \tag{3.8}$$

From condition (i),

$$\lim_{n\to\infty} d_n = 0. \tag{3.9}$$

Using equations (3.4) and (3.9) in (3.8), we get

$$\lim_{n\to\infty} ||F_lz_{n,l} - F_lx^*|| = 0 \quad \forall \quad l = 1,2...M. \tag{3.10}$$

Now

$$\begin{aligned}
\|x_n-u_{n,l}\| &= \|(x_n-x_{n+1}) + (x_{n+1}-u_{n,l})\| \\
&= \|x_n - x_{n+1} + a_n\xi g(x_n) + c_nx_n + ((1-c_n)I - a_n\mathfrak{A})u_{n,l} + a_nv_{n,l} - u_{n,l}\| \\
&= \|x_n-x_{n+1}\| + a_n\|\xi g(x_n) - \mathfrak{A}u_{n,l} + v_{n,l}\| + c_n\|x_n-u_{n,l}\| \\
(1-c_n)\|x_n-u_{n,l}\| &= \|x_n-x_{n+1}\| + a_n\|\xi g(x_n) - \mathfrak{A}u_{n,l} + v_{n,l}\|
\end{aligned}$$

Thus,

$$\|x_n-u_{n,l}\| \to 0 \text{ as } n\to\infty \tag{3.11}$$

Similarly, we find

$$\|x_n-v_{n,l}\| \to 0 \text{ as } n\to\infty \tag{3.12}$$

Now, from equation (2.1), we have

$$\begin{aligned}
\|u_{n,l}-x^*\|^2 &= \|P_{\mathfrak{D}}(z_{n,l}-\theta_lF_lz_{n,l}) - P_{\mathfrak{D}}(x^*-\theta_lF_lx^*)\|^2 \\
&\leq \langle(z_{n,l}-\theta_lF_lz_{n,l}) - (x^*-\theta_lF_lx^*), u_{n,l}-x^*\rangle \\
&= \frac{1}{2}\{\|(z_{n,l}-\theta_lF_lz_{n,l}) - (x^*-\theta_lF_lx^*)\|^2 + \|u_{n,l}-x^*\|^2 - \|(z_{n,l}-\theta_lF_lz_{n,l}) \\
&\quad - (x^*-\theta_lF_lx^*) - (u_{n,l}-x^*)\|^2\} \\
&\leq \frac{1}{2}\{\|z_{n,l}-x^*\|^2 + \|u_{n,l}-x^*\|^2 - \|(z_{n,l}-u_{n,l}) - \theta_l(F_lz_{n,l}-F_lx^*\|^2\}
\end{aligned}$$

$$\le \frac{1}{2}\{\|z_{n,l}-x^*\|^2+\|u_{n,l}-x^*\|^2-\|z_{n,l}-u_{n,l}\|^2-\theta_l^2\|F_lz_{n,l}-F_lx^*\|^2 + 2\theta_l\langle z_{n,l}-u_{n,l},F_lz_{n,l}-F_lx^*\rangle$$
$$\le \|z_{n,l}-x^*\|^2-\|z_{n,l}-u_{n,l}\|^2-\theta_l^2\|F_lz_{n,l}-F_lx^*\|^2+2\theta_l\langle z_{n,l}-u_{n,l},F_lz_{n,l}-F_lx^*\rangle$$

Further, from equation (3.7)

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-a_n\zeta)(1-c_n-a_n\zeta)\|u_{n,l}-x^*\|^2+(1-a_n\zeta)c_n\|x_n-x^*\|^2+d_n\\ &\le (1-a_n\zeta)(1-c_n-a_n\zeta)\{\|z_{n,l}-x^*\|^2-\|z_{n,l}-u_{n,l}\|^2-\theta_l^2\|F_lz_{n,l}-F_lx^*\|^2\\ &\quad + 2\theta_l\langle z_{n,l}-u_{n,l},F_lz_{n,l}-F_lx^*\rangle\}+(1-a_n\zeta)c_n\|x_n-x^*\|^2+d_n\\ &\le \|x_n-x^*\|^2-(1-a_n\zeta)(1-c_n-a_n\zeta)\\ &\quad \left[\|z_{n,l}-u_{n,l}\|^2+2\theta_l\|z_{n,l}-u_{n,l}\|.\|F_lz_{n,l}-F_lx^*\|-\theta_l^2\|F_lz_{n,l}-F_lx^*\|^2\right]+d_n,\end{aligned}$$

implies

$$\begin{aligned}(1-a_n\zeta)(1-c_n-a_n\zeta)\|z_{n,l}-u_{n,l}\|^2 \le\ & \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ & + 2\theta_l(1-a_n\zeta)(1-c_n-a_n\zeta).\|z_{n,l}-u_{n,l}\|.\|F_lz_{n,l}-F_lx^*\|\\ & -\theta_l^2(1-a_n\zeta)(1-c_n-a_n\zeta)\|F_lz_{n,l}-F_lx^*\|^2+d_n\\ \le\ & \|x_n-x_{n+1}\|\left[\|x_n-x^*\|+\|x_{n+1}-x^*\|\right]+2\theta_l(1-a_n\zeta)(1-c_n-a_n\zeta)\\ & \|z_{n,l}-u_{n,l}\|.\|F_lz_{n,l}-F_lx^*\|-\theta_l^2(1-a_n\zeta)(1-c_n-a_n\zeta)\|F_lz_{n,l}-F_lx^*\|^2\\ & + d_n\end{aligned}$$

After using equations (3.4), (3.9) and (3.10), we get

$$\lim_{n\to\infty}\|z_{n,l}-u_{n,l}\| = 0 \tag{3.13}$$

From equations (3.11) and (3.13), we get

$$\|x_n-z_{n,l}\| \le \|x_n-u_{n,l}\|+\|u_{n,l}-z_{n,l}\|\to 0. \tag{3.14}$$

Similarly, with the help of $v_{n,l}$ we prove that $\lim_{n\to\infty}\|y_{n,l}-v_{n,l}\|=0$ and from equation (3.11) we obtain

$$\|x_n-y_{n,l}\|\to 0. \tag{3.15}$$

At last from equation (3.14) and (3.15) we have

$$\|z_{n,l}-y_{n,l}\|\to 0. \tag{3.16}$$

In view of equation (3.3), we have

$$\begin{aligned}\sigma_{n,l}(\sigma_{n,l}-\delta_{n,l}).&\|\mathcal{T}_l((1-\delta_{n,l})y_{n,l-1}+\delta_{n,l}\mathcal{T}_ly_{n,l-1})-y_{n,l-1}\|^2+\xi_{n,l}(\eta_{n,l}-\xi_{n,l})\|\mathcal{S}_l((1-\eta_{n,l})\\ & z_{n,l-1}+\eta_{n,l}\mathcal{S}_lz_{n,l-1})-z_{n,l-1}\|^2\\ &\le \|x_n-x^*\|^2-\|z_{n,l}-x^*\|^2\\ &\le \|x_n-z_{n,l}\|(\|x_n-x^*\|+\|z_{n,l}-x^*\|)\end{aligned}$$

$$\Rightarrow \lim_{n\to\infty}.\|\mathcal{T}_l((1-\delta_{n,l})y_{n,l-1}+\delta_{n,l}\mathcal{T}_ly_{n,l-1})-y_{n,l-1}\|=0. \tag{3.17}$$

Similarly,

$$\Rightarrow \lim_{n\to\infty}.\|\mathcal{S}_l((1-\eta_{n,l})z_{n,l-1}+\eta_{n,l}\mathcal{S}_lz_{n,l-1})-z_{n,l-1}\|=0. \tag{3.18}$$

Next, we prove that $\limsup_{n\to\infty}\langle\xi g(x)-\mathfrak{A}x,u_{n,l}-x\rangle\le 0$.
Let $\{u_{n_k,l}\}$ be a subsequence of $\{u_{n,l}\}$ such that

$$\lim\sup_{n\to\infty}\langle\xi g(x)-\mathfrak{A}x,u_{n,l}-x\rangle=\lim_{k\to\infty}\langle\xi gx-\mathfrak{A}x,u_{n_k,l}-x\rangle$$

Since $\{u_{n,l}\}$ is bounded so there exist a subsequence $\{u_{n_{k_m},l}\}$ of $\{u_{n_k,l}\}$ which converges weakly to $z\in\mathcal{D}$. Now, W.L.O.G., we assume $\{u_{n_{k_m},l}\}\rightharpoonup z$. Therefore, we have

$$\lim\sup_{n\to\infty}\langle\xi g(x)-\mathfrak{A}x,u_{n,l}-x\rangle$$

$$= \lim_{k\to\infty} \langle \xi gx - \mathfrak{A}x, u_{n_k,l} - x\rangle$$
$$= \langle \xi g(x) - \mathfrak{A}(x), z - x\rangle \tag{3.19}$$

From equation (3.12) and (3.16), we get $y_{n,l} \rightharpoonup z$ and $z_{n,l} \rightharpoonup z$
Apply Lemma 2.3 to equation (3.17) and (3.18) to deduce $z \in Fix(\mathcal{T}_l)$ and $z \in Fix(\mathcal{S}_l)$ respectively for each $l = 1, 2, 3..., M$
$\Rightarrow z \in \cap_{l=1}^{M}(Fix\mathcal{T}_l \cap Fix\mathcal{S}_l)$
Next we prove $z \in VI(F_l, \mathcal{D})$.
Let

$$\mathfrak{B}v = \begin{cases} F_l v + N_w v, & v \in \mathcal{D} \\ \phi, & v \notin \mathcal{D} \end{cases} \tag{3.20}$$

where $\mathfrak{B}$ is maximal monotone mapping. For any given $(a, b) \in G(\mathfrak{B})$, we have $(b - F_l a) \in N_W a$. For $u_{n,l} \in \mathcal{D}$, we have
$\langle a - u_{n,l}, b - F_l a\rangle \geq 0$ Since $u_{n,l} = P_{\mathcal{D}}(z_{n,l} - \theta_l F_l z_{n,l})$, we have (By definition of Projection mapping)

$$\langle a - u_{n,l}, u_{n,l} - (z_{n,l} - \theta_l F_l z_{n,l})\rangle \geq 0.$$
$$\langle a - u_{n,l}, \frac{u_{n,l} - z_{n,l}}{\theta_l} + F_l z_{n,l}\rangle \geq 0.$$

So, we have

$$\begin{aligned}
\langle a - u_{n_k,l}, b\rangle &\geq \langle a - u_{n_k,l}, F_l a\rangle \\
&\geq \langle a - u_{n_k,l}, F_l a\rangle - \langle a - u_{n_k,l}, \frac{u_{n_k,l} - z_{n_k,l}}{\theta_{l_k}} + F_l z_{n_k,l}\rangle \\
&= \langle a - u_{n_k,l}, F_l a - F_l z_{n_k,l} - \frac{u_{n_k,l} - z_{n_k,l}}{\theta_{l_k}}\rangle \\
&= \langle a - u_{n_k,l}, F_l a - F_l u_{n_k,l}\rangle + \langle a - u_{n_k,l}, F_l u_{n_k,l} - F_l z_{n_k,l}\rangle \\
&\quad - \langle a - u_{n_k,l}, \frac{u_{n_k,l} - z_{n_k,l}}{\theta_{l_k}}\rangle \\
&\geq \langle a - u_{n_k,l}, F_l u_{n_k,l} - F_l z_{n_k,l}\rangle - \langle a - u_{n_k,l}, \frac{u_{n_k,l} - z_{n_k,l}}{\theta_{l_k}}\rangle
\end{aligned}$$

Since, $\|u_{n_k,l} - z_{n_k,l}\| \to 0$ as $n \to \infty$ and F_l is Lipschitz continuous, therefore, we get
$\langle a - z, b\rangle \geq 0$
As $\mathfrak{A}$ is maximal monotone, $z \in \mathfrak{B}^{-1}0$ and we have $z \in \mathcal{V}I(F_l, \mathcal{D})$.
Therefore,
$z \in \Omega = \cap_{l=1}^{M}((Fix(\mathcal{S}_l) \cap Fix(\mathcal{T}_l)) \cap (\cap_{l=1}^{M}\mathcal{V}I(F_l, \mathcal{D}))$
From equation (3.19), we obtain,

$$\lim_{n\to\infty} \sup\langle \xi g(x) - \mathfrak{A}x, x_n - x\rangle$$
$$= \lim_{n\to\infty} \langle \xi g(x) - \mathfrak{A}(x), z - x\rangle \leq 0 \tag{3.21}$$

Since $x = P_{\mathcal{W}}(\xi g + (I - \mathfrak{A}))x$, $\|x_n - u_{n,l}\| \to 0$ and from (3.21), we get

$$\begin{aligned}
\lim_{n\to\infty} \sup\langle \xi g(x) - \mathfrak{A}x, x_n - x\rangle &= \lim_{n\to\infty} \sup\langle \xi g(x) - \mathfrak{A}x, (x_n - u_{n,l}) + (u_{n,l} - x)\rangle \\
&\leq \lim_{n\to\infty} \sup\langle \xi g(x) - \mathfrak{A}x, u_{n,l} - x\rangle \\
&\leq 0
\end{aligned}$$

Now, we show that $x_n \to x$

$$\begin{aligned}
\|x_{n+1} - x\|^2 =& \|a_n \xi g(x_n) + c_n x_n + [(1 - c_n)I - a_n\mathfrak{A}]u_{n,l} + a_n.v_{n,l} - x||^2 \\
=& \|a_n(\xi g(x_n) - \mathfrak{A}x + x) + c_n(x_n - x) + ((1 - c_n)I - a_n\mathfrak{A})(u_{n,l} - x) + a_n(v_{n,l} - x)\|^2
\end{aligned}$$

$$
\begin{aligned}
=&\|a_n(\xi g(x_n)-\mathfrak{A}x+v_{n,l})+c_n(x_n-x)+((1-c_n)I-a_n\mathfrak{A})(u_{n,l}-x)\|^2\\
=&a_n^2\|\xi g(x_n)-\mathfrak{A}x+v_{n,l}\|^2+\|c_n(x_n-x)+((1-c_n)I-a_n\mathfrak{A})(u_{n,l}-x)\|^2\\
&+2c_na_n\langle x_n-x,\xi g(x_n)-\mathfrak{A}x+v_{n,l}\rangle\\
&+2a_n\langle((1-c_n)I-a_n\mathfrak{A})(u_{n,l}-x),\xi g(x_n)-\mathfrak{A}x+v_{n,l}\rangle\\
=&\big[(1-c_n-a_n\zeta)\|u_{n,l}-x\|+c_n\|x_n-x\|\big]^2+a_n^2\|\xi g(x_n)-\mathfrak{A}x+v_{n,l}\|^2\\
&+2c_na_n\xi\langle x_n-x,g(x_n)-g(x)\rangle+2a_nc_n\langle x_n-x,\xi g(x)-\mathfrak{A}x+v_{n,l}\rangle\\
&+2(1-c_n)\xi a_n\langle u_{n,l}-x,g(x_n)-g(x)\rangle+2(1-c_n)a_n\langle u_{n,l}-x,\xi gx-\mathfrak{A}x+v_{n,l}\rangle\\
&-2a_n^2\xi\langle\mathfrak{A}(u_{n,l}-x),g(x_n)-gx\rangle-2a_n^2\langle\mathfrak{A}(u_{n,l}-x),\xi gx-\mathfrak{A}x+v_{n,l}\rangle\\
\le&\big[(1-c_n-a_n\zeta)\|x_n-x\|+c_n\|x_n-x\|\big]^2+a_n^2\|\xi g(x_n)-\mathfrak{A}x+v_{n,l}\|^2\\
&+2a_n\xi\tau\|x_n-x\|^2+2c_na_n\langle x_n-x,\xi gx-\mathfrak{A}x+v_{n,l}\rangle-2a_n^2\xi\tau\zeta\|x_n-x\|^2\\
&+2(1-c_n)a_n\langle u_{n,l}-x,\xi gx-\mathfrak{A}x+v_{n,l}\rangle-2a_n^2\langle\mathfrak{A}(u_{n,l}-x),\xi gx-\mathfrak{A}x+v_{n,l}\rangle\\
=&(1-a_n(\zeta-2\xi\tau))\|x_n-x\|^2-2a_n^2\zeta\xi\tau\|x_n-x\|^2\\
&+a_n^2\|\xi g(x_n)-\mathfrak{A}x+v_{n,l}\|^2+2c_na_n\langle x_n-x,\xi gx-\mathfrak{A}x+v_{n,l}\rangle\\
&+2(1-c_n)a_n\langle u_{n,l}-x,\xi gx-\mathfrak{A}x+v_{n,l}\rangle+2a_n^2\|\mathfrak{A}(u_{n,l}-x)\|.\|\xi gx-\mathfrak{A}x+v_{n,l}\|\\
=&(1-a_n(\zeta-2\xi\tau))\|x_n-x\|^2+a_n\{a_n(2\zeta\xi\tau\|x_n-x\|^2\\
&+\|\xi g(x_n)-\mathfrak{A}x+v_{n,l}\|+2\|\mathfrak{A}(u_{n,l}-x)\|.\|\xi gx-\mathfrak{A}x+v_{n,l}\|)\\
&+2c_n\langle x_n-x,\xi gx-\mathfrak{A}x+v_{n,l}\rangle+2(1-c_n)\langle u_{n,l}-x,\xi gx-\mathfrak{A}x+v_{n,l}\rangle\}
\end{aligned}
$$

Since $\{x_n\}, \{g(x_n)\}$ and $\{u_{n,l}\}$ are bounded, we have

$$\zeta\xi\tau\|x_n-x\|^2+\|\xi g(x_n)-\mathfrak{A}x+v_{n,l}\|+2\|\mathfrak{A}(u_{n,l}-x)\|.\|\xi gx-\mathfrak{A}x+v_{n,l}\|\le M_2$$

where

$M_2 > 0$

Further we have

$$\|x_{n+1}-x\|^2 \le (1-(\zeta-2\xi\tau)a_n)\|x_n-x\|^2+a_n\sigma_n, \tag{3.22}$$

where

$$\sigma_n = 2c_n\langle x_n-x,\xi gx-\mathfrak{A}x+v_{n,l}\rangle+2(1-c_n)\langle u_{n,l}-x,\xi gx-\mathfrak{A}x+v_{n,l}\rangle$$

Using assumption (i) and Lemma 2.2 in (3.22), we have

$x_n \to x$.

Hence the result obtained. □

4 Numerical Example

In this part, we present an example with efficiency and convergence of $\{x_n\}$.

Let $\mathbb{H} = \mathbb{R}$ and $\mathcal{D} = [0, 1]$. The mapping F_l, G and g be defined by

$$
\begin{aligned}
F_l(x) &= 2x\\
\mathfrak{A}(x) &= \frac{x}{5}\\
g(x) &= \frac{x}{7}
\end{aligned}
\tag{4.1}
$$

for all $x \in \mathcal{D}$. Let $\mathcal{S}_l(x) = \frac{x}{2l}$ and $\mathcal{T}_l(x) = \frac{x}{3l}$for all $x \in C$.

Define $\{a_n\}, \{\sigma_{n,l}\}, \{\beta_{n,l}\}, \{\delta_{n,l}\}, \{\eta_{n,l}\}$ and $\{c_n\}$ in (0,1) by $a_n = \frac{1}{2n}$, $\{\sigma_{n,l}\} = \{\beta_{n,l}\} = \frac{l}{n}$,

$\{\delta_{n,l}\} = \{\eta_{n,l}\} = \frac{l}{n^4}$ and $c_n = \frac{1}{2n}$. As $\mathcal{S}_l$ and $\mathcal{T}_l$ are non-expansive mappings and F_l is 2-inverse strongly monotone mapping. g is a $(\frac{1}{7})$-contraction mapping.

Now we check that the sequences $\{a_n\}, \{\sigma_{n,l}\}, \{\beta_{n,l}\}, \{\delta_{n,l}\}, \{\eta_{n,l}\}$ and $\{c_n\}$ satisfies the given condition.

$$\lim_{n\to\infty} a_n = \frac{1}{2}\lim_{n\to\infty}\frac{1}{n} = 0$$

and

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n} = \infty.$$

The sequence $\{a_n\}$ satisfies the condition (i) of theorem 3.1
Next we calculate

$$\begin{aligned} a_n - a_{n-1} = \frac{1}{2n} - \frac{1}{2(n-1)} &= \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n-1}\right) \\ &= \frac{-1}{2n(n-1)}. \end{aligned}$$

So,

$$\sum_{n=1}^{\infty} |a_n - a_{n-1}| < \infty.$$

Similarly, we have

$$\sum_{n=1}^{\infty} |c_n - c_{n-1}| < \infty.$$

Also, the sequences $\{a_n\}, \{\sigma_{n,l}\}, \{\beta_{n,l}\}, \{\delta_{n,l}\}, \{\eta_{n,l}\}$ and $\{c_n\}$ satisfies the conditions (ii)-(iv) of Theorem 3.1. Clearly,

$\Omega = \cap_{l=1}^{2}((Fix(\mathcal{S}_l) \cap Fix(\mathcal{T}_l)) \cap (\cap_{l=1}^{2}\mathcal{V}I(F_l, \mathcal{D})) = 0$
The iterative algorithm (3.1) is written as:
when $l = 1$

$$\begin{aligned} y_{n,1} &= x_n - \frac{2x_n}{3n} - \frac{2x_n}{9n^3} \\ z_{n,1} &= y_{n,1} - \frac{y_{n,1}}{2n} - \frac{y_{n,1}}{4n^3} \\ x_{n+1} &= \frac{25x_n}{49n} + \frac{z_{n,1}}{3} - \frac{z_{n,1}}{5n} + \frac{y_{n,1}}{6n}, \end{aligned}$$

when $l = 2$

$$\begin{aligned} y_{n,2} &= y_{n,1} - \frac{5y_{n,1}}{3n} - \frac{5y_{n,1}}{9n^3} \\ z_{n,1} &= z_{n,1} - \frac{3z_{n,1}}{2n} - \frac{3z_{n,1}}{4n^3} \\ x_{n+1} &= \frac{25x_n}{49n} + \frac{z_{n,2}}{3} - \frac{z_{n,2}}{5n} + \frac{y_{n,2}}{6n}. \end{aligned}$$

Remark Table (1) shows the sequences $\{y_{n,l}\}, \{z_{n,l}\}$ and $\{x_n\}$ converges to 0. Also, $\{0\} \in \Omega$

	$l=1$ x_1=1			$l=2$ x_1=1		
n	x_n	$y_{n,l}$	$z_{n,l}$	x_n	$y_{n,l}$	$z_{n,l}$
1	1	0.111111111	0.027777778	1	-0.135802469	-0.034722222
2	0.270842782	0.173038444	0.124371381	0.235683317	0.016823182	0.019433028
3	0.088840646	0.068367082	0.056339539	0.046198959	0.02897864	0.026604783
4	0.030143213	0.02501468	0.021790132	0.014638189	0.014374755	0.013363479
5	0.010301433	0.008909595	0.008000816	0.005892805	0.005900132	0.005552566
6	0.003523706	0.003128558	0.002864223	0.002330751	0.002251467	0.002138222
7	0.001204226	0.001088757	0.001010195	0.000875135	0.000827766	0.000791516
8	0.000410959	0.000376534	0.000352817	0.000317108	0.000297681	0.000286147
9	0.000140035	0.00012962	0.000122374	0.000112513	0.000105517	0.000101852
10	4.76488E-05	4.44617E-05	4.22275E-05	3.94128E-05	3.70267E-05	3.58617E-05
25	4.06891E-12	3.96035E-12	3.88107E-12	3.81308E-12	3.69618E-12	3.64802E-12
26	1.36907E-12	1.33395E-12	1.30828E-12	1.28646E-12	1.2484E-12	1.23274E-12
44	4.01967E-21	3.95876E-21	3.91376E-21	3.87911E-21	3.80878E-21	3.7803E-21
45	1.34743E-21	1.32746E-21	1.31271E-21	1.30139E-21	1.27829E-21	1.26894E-21
50	5.69328E-24	5.61736E-24	5.56118E-24	5.51865E-24	5.43009E-24	5.39431E-24
51	1.90723E-24	1.8823E-24	1.86384E-24	1.8499E-24	1.82078E-24	1.80901E-24
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 1: The values of $y_{n,l}, z_{n,l}$ and x_n with initial values $x_1 = 1$ when $l = 1, 2$.

5 Application

In ths part, by implimenting Theorem 3.1 we find the following results.

Theorem 5.1. *Let $\mathcal{D}$ be closed and convex subset of real Hilbert space $\mathbb{H}$, let $g : \mathcal{D} \to \mathcal{D}$ be a ξ-contraction mapping with coefficient $\xi \in [0,1]$, $\mathfrak{A}$ be strongly positive linear bounded self adjoint operator with the coefficient $\zeta > 0$, $F_l : \mathcal{D} \to \mathbb{H}$ be a ς_l- inverse strongly monotone mapping s for each $1 \le l \le M$, where M is some positive integer, and $\mathcal{S}_l : \mathcal{D} \to \mathcal{D}$, $\mathcal{T}_l : \mathcal{D} \to \mathcal{D}$ are two finite family of non-expansive mappings with $\Omega = \cap_{l=1}^{M}((Fix(\mathcal{S}_l) \cap Fix(\mathcal{T}_l)) \cap (\cap_{l=1}^{M} F_l^{-1}(0)) \neq \phi$. Assume $0 < \xi \le \frac{\zeta}{\varsigma}$ and let $\{\theta_l\}$ be a real numbers in (0,2ς_l). Let $\{y_{n,l}\}, \{z_{n,l}\}$ and $\{x_n\}$ for all $l = 1, 2, ...M$ defined by*

$$\begin{aligned} y_{n,1} &= (1-\sigma_{n,1})x_n + \sigma_{n,1}\mathcal{T}_1((1-\delta_{n,1})x_n + \delta_{n,1}\mathcal{T}_1 x_n) \\ y_{n,l} &= (1-\sigma_{n,l})y_{n,l-1} + \sigma_{n,l}\mathcal{T}_l((1-\delta_{n,l})y_{n,l-1} + \delta_{n,l}\mathcal{T}_l y_{n,l-1}) \\ z_{n,1} &= (1-\beta_{n,1})y_{n,1} + \beta_{n,1}\mathcal{S}_1((1-\eta_{n,1})y_{n,1} + \eta_{n,1}\mathcal{S}_1 y_{n,1}) \\ z_{n,l} &= (1-\beta_{n,l})z_{n,l-1} + \beta_{n,l}\mathcal{S}_l((1-\eta_{n,l})z_{n,l-1} + \eta_{n,l}\mathcal{S}_l z_{n,l-1}) \\ x_{n+1} &= a_n\xi g(x_n) + c_n x_n + [(1-c_n)I - a_n\mathfrak{A}]P_{\mathcal{D}}(z_{n,l} - \theta_l F_l z_{n,l}) + a_n P_{\mathcal{D}}(y_{n,l} - \theta_l F_l y_{n,l}) \quad (5.1) \end{aligned}$$

where $\{\sigma_{n,l}\}, \{\delta_{n,l}\}, \{\beta_{n,l}\}, \{\eta_{n,l}\}$, and $\{c_n\}$ are the sequences in (0,1) satisfying the same as-

sumptions as in Theorem 3.1. Then the sequence $\{x_n\}$ characterized by (5.1) converges strongly to $x \in \Omega$, where $x = P_\Omega(\xi g + (I - \mathfrak{A}))x$ is also unique solution of the following variational inequality:

$$\langle \xi g(x) - \mathfrak{A}z, z - x\rangle \leq 0, \forall z \in \Omega. \tag{5.2}$$

Now, we apply result of Theorem 3.1, to solve the problem of finding the common point of the solution set of fixed point of finite family of k-strictly pseudocontractive mappings and the set of fixed point of finite familty of two non-expansive mappings.

Theorem 5.2. *Let $\mathcal{D}$ be closed and convex subset of real Hilbert space $\mathcal{H}$, let $\mathfrak{A}$ be strongly positive linear bounded self adjoint operator with the coefficient $\zeta > 0$, $g : \mathcal{D} \to \mathcal{D}$ be a ξ-contraction with coefficient $\xi \in [0, 1]$, $\mathfrak{F}_l : \mathcal{D} \to \mathbb{H}$ is a k_l-strictly pseudocontractive mapping for some $0 \leq k_l < 1$, $F_l : \mathcal{D} \to \mathcal{H}$ be a ς_l- inverse strongly monotone mapping s, and $\mathcal{S}_l : \mathcal{D} \to \mathcal{D}$, $\mathcal{T}_l : \mathcal{D} \to \mathcal{D}$ are two finite family of non-expansive mappings with $\Omega = \cap_{l=1}^{M}((Fix(\mathcal{S}_l) \cap Fix(\mathcal{T}_l)) \cap (\cap_{l=1}^{M} F(\mathfrak{F}_l)) \neq \phi$. Assume $0 < \xi \leq \frac{\zeta}{\varsigma}$ and let $\{\theta_l\}$ be a real numbers in (0,2ς_l). Let $\{y_{n,l}\}, \{z_{n,l}\}$ and $\{x_n\}$ for all $l = 1, 2, ...M$, where M is some positive integer, defined by*

$$\begin{aligned}
y_{n,1} &=(1 - \sigma_{n,1})x_n + \sigma_{n,1}\mathcal{T}_1((1 - \delta_{n,1})x_n + \delta_{n,1}\mathcal{T}_1 x_n)\\
y_{n,l} &=(1 - \sigma_{n,l})y_{n,l-1} + \sigma_{n,l}\mathcal{T}_l((1 - \delta_{n,l})y_{n,l-1} + \delta_{n,l}\mathcal{T}_l y_{n,l-1})\\
z_{n,1} &=(1 - \beta_{n,1})y_{n,1} + \beta_{n,1}\mathcal{S}_1((1 - \eta_{n,1})y_{n,1} + \eta_{n,1}\mathcal{S}_1 y_{n,1})\\
z_{n,l} &=(1 - \beta_{n,l})z_{n,l-1} + \beta_{n,l}\mathcal{S}_l((1 - \eta_{n,l})z_{n,l-1} + \eta_{n,l}\mathcal{S}_l z_{n,l-1})
\end{aligned}$$

$$x_{n+1} = a_n\xi g(x_n) + c_n x_n + [(1 - c_n)I - a_n\mathfrak{A}]P_{\mathcal{D}}((1 - \lambda_n)z_{n,l} - \lambda_n \sum_{i=1}^{M} \varphi_l z_{n,l}) \tag{5.3}$$

$$+ a_n P_{\mathcal{D}}((1 - \lambda_n)y_{n,l} - \lambda_n \sum_{i=1}^{M} \varphi_l y_{n,l})$$

where $\{\sigma_{n,l}\}, \{\delta_{n,l}\}, \{\beta_{n,l}\}, \{\eta_{n,l}\}$, and $\{c_n\}$ are the sequences in (0,1) with the same assumptions as in Theorem 3.1. Then $\{x_n\}$ defined by (5.3) converge strongly to $x \in \Omega$, where $x = P_\Omega(\xi g + (I - \mathfrak{A}))x$ is also unique solution of the successive variational inequality:

$$\langle \xi g(x) - \mathfrak{A}z, z - x\rangle \leq 0, \forall z \in \Omega. \tag{5.4}$$

Proof Let us examine with $F_l = I - \sum_{i=1}^{M} \varphi_l \mathfrak{F}_l$ where $F_l : \mathcal{D} \to \mathbb{H}$, as F_l is ς_l- inverse strongly monotone with $\varsigma_l = (1 - k_l)/2$. Hence, F_l is monotone and B- Lipschitz continuous mapping with $B = 2/(1 - k_l)$. From Lemma 2.6, we have, $\sum_{l=1} \varphi_l \mathfrak{F}_l$ is a k-strictly peseudocontractive mapping with $k = \max\{k_l : 1 \leq l \leq M\}$ and then $F(\sum_{l=1}^{M} \varphi_l \mathfrak{F}_l) = VI(F_l, \mathcal{D})$. So, we obtain

$$P_{\mathcal{D}}(z_{n,l} - \theta_l F_l z_{n,l}) = P_{\mathcal{D}}((1 - \lambda_n)z_{n,l} - \lambda_n \sum_{i=1}^{M} \varphi_l z_{n,l}) \tag{5.5}$$

Similarly

$$P_{\mathcal{D}}(y_{n,l} - \theta_l F_l y_{n,l}) = P_{\mathcal{D}}((1 - \lambda_n)y_{n,l} - \lambda_n \sum_{i=1}^{M} \varphi_l y_{n,l}) \tag{5.6}$$

After using Theorem 3.1, we crave the final result.

Conclusion

We proved the common convergence of fixed point of a finite family of two non-expansive mapping and variational inequality problem for inverse strongly monotone mapping in a real Hilbert space, by the new hybrid technique. After that, we solved an example by using this hybrid approach and find the common solution for non-expansive mappings and variational inequlity problem.

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References

[1] C. E. Lemke and J. T. Howson, Equilibrium points of bimatrix games, J. Soc. Indust. Appl. Math., **12**(1964), 413–423.

[2] G. Marino and H. K. Xu, A general iterative method for non-expansive mappings in Hilbert spaces, Journal of Mathematical Analysis and Applications, **318**(1)(2006), 43-52.

[3] G. L. Acedo and H. K. Xu, Iterative methods for strict pseudocontractions in Hilbert spaces, Nonlinear Analysis: Theory, Methods & Applications, **67**(7)(2007), 2258-2271.

[4] G. M. Korpelevich, An extragradient method for finding saddle points and for other problem, Ekonomika I Matematicheskie Metody, **12**(4)(1976), 747-756.

[5] H. E. Scarf, The approximation of fixed points of a continuous mapping, SIAM J. Appl. Math., **15**(1967), 1328–1342.

[6] H. K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. , **66**(2002), 240-256.

[7] J. L. Lions, Quelques Methodes de Resolution des Problemes aux limites Non Lineaires, Dunod, Paris, France. **1969**

[8] K. Goebel and W. A. Kirk, Topics in Metric Fixed Point Theory, **28**(1990), Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, UK.

[9] L. Liu, S. Y. Cho and J.C. Yao, Convergence analysis of an inertial Tseng's extragradient algorithm for solving pseudomonotone variational inequalities and applications, J. Nonlinear Var. Anal., **5**(2021), 627-644.

[10] M. Khan, M. Zeeshan and S. Iqbal, Neutrosophic variational inequalities with applications in decision-making, Soft Comput., **26**(2022), 4641–4652.

[11] O. K. Oyewole and S. Reich, A totally relaxed self-adaptive algorithm for solving a variational inequality and fixed point problems in Banach spaces, Appl. Set-Valued Anal. Optim., **4**(2022), 349-366.

[12] Q. Cheng, Hybrid viscosity approximation methods with generalized contractions for zeros of monotone operators and fixed point problems, J. Nonlinear Funct. Anal., **2022**(2022), 23.

[13] R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Transaction of the American Mathematical Soceity, **149**(1970), 75-88.

[14] S. Husain and N. Singh, An iterative method for finding common solution of the fixed point problem of a finite family of nonexapnsive mappings and a finite family of variational inequality problems in Hilbert Space, Journal of Applied Mathematics, **2019**(2019), 11 pages.

[15] S. Treanţă, Well posedness of new optimization problems with variational inequality constraints, Fractal and Fractional, **5**(3)(2021), 123.

[16] W. Takahashi and M. Toyoda, Strong convergence theorems for nonexapnsive mappings and monotone mappings, Journal of Optimization Theory and Applications, **118**(2)(2003), 417-428.

[17] Y. Yao, Y. C. Liou and J. C. Yao, Split common fixed point problem for two quasi-pseudocontractive operators and its algorithm construction, Fixed Point Theory Appl., **2015**(127)(2015), 19 pages.

[18] Y. J. Cho and X. Qin, Convergence of a general iterative method for nonexapnsive mappings in Hilbert spaces, Journal of Computational and Applied Mathematics, **228**(1)(2009), 458-465.

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