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Biconservative Lorentz hypersurfaces with at least three principal curvatures

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Abstract. Biconservative submanifolds, with important role in mathematical physics and differential geometry, arise as the conservative stress-energy tensor associated to the variational problem of biharmonic submanifolds. Many examples of biconservative hypersurfaces have constant mean curvature. A famous conjecture of Bang-Yen Chen on Euclidean spaces says that every biharmonic submanifold has null mean curvature. Inspired by Chen conjecture, we study biconservative Lorentz submanifolds of the Minkowski spaces. Although the conjecture has not been generally confirmed, it has been proven in many cases, and this has led to its spread to various types of submenifolds. As an extension, we consider a advanced version of the conjecture (namely, L_1-conjecture) on Lorentz hypersurfaces of the pseudo-Euclidean space $\mathbb{M}^5 := \mathbb{E}^5_1$ (i.e. the Minkowski 5-space). We show every L_1-biconservative Lorentz hypersurface of $\mathbb{M}^5$ with constant mean curvature and at least three principal curvatures has constant second mean curvature.

Keywords. Lorentz hypersurface, L_1-biconservative, isoparametric, Newton transformation

1 Introduction

From mathematical point of view, the biharmonic surfaces appear as solutions of strongly elliptic semilinear differential equations of order four. Also, the biharmonic Bezier surfaces play important roles in computational geometry. From physical points of view, the biharmonic surfaces play central roles in elastics and fluid mechanics. A differential geometric motivation of the matter of biharmonic maps is a well-known conjecture of Bang-Yen Chen which states that each biharmonic submanifold of an Euclidean space in minimal. Later on, Dimitrić proved that any biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures is minimal ([7]). An equivalent statement say that every biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures is harmonic, which means that there is no proper biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures. Remember that, a biharmonic hypersurface which is not harmonic is called *proper biharmonic*.

Clearly, harmonic maps are biharmonic but not vis versa. In the homotopy class of Brower of degree ± 1, one cannot find a harmonic map as $\mathbb{T}^2 \to \mathbb{S}^2$, although, there exists a biharmonic

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one [8]. From a geometric point of view, the variational problem associated to the bienergy functional on the set of Riemannian metrics on a domain has given rise to the biharmonic stress-energy tensor. In 1995, Hasanis and Vlachos affirmed Chen's conjecture on hypersurfaces in Euclidean 4-spaces ([10]). In 2013, Akutagawa and Maeta ([1]) have studied Chen's conjecture on biharmonic submanifolds in Euclidean n-space. On the other hand, Chen himself had found a good relation between the finite type hypersurfaces and biharmonic ones. The theory of finite type hypersurfaces is a well-known subject interested by Chen (for instance, in [5, 6]) and also L.J. Alias, S.M.B. Kashani and others. In [11], Kashani has studied the notion of L_1-finite type Euclidean hypersurfaces as an extension of finite type ones. One can see main results in Chapter 11 of Chen's book ([5]).

The linearized operator L_1 is an extension of the Laplace operator $L_0 = \Delta$, which stands for the linearized map of the first variation of the 2th mean curvature of the hypersurface (see, for instance, [2, 12, 16, 17, 19]). This operator is defined by $L_1(f) = tr(P_1 \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where $P_1 = nHI - S$ denotes the first Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of f. It is interesting to generalize the definition of biharmonic hypersurface by replacing Δ by L_1. Recently, in [14], we have studied the L_1-biharmonic spacelike hypersurfaces in 4-dimentional Minkowski space $\mathbb{M}^4$. In this paper, we study the L_1-biharmonic Lorentzian hypersurfaces in $\mathbb{M}^5$. We show that, every L_1-biconservative Lorentzian hypersurface $x : M_1^4 \to \mathbb{M}^5$, with constant mean curvature and three distinct principal curvatures is L_1-harmonic.

Now we present the structure of paper. In section 2 we remember some notations and definitions which will be needed in paper. In section 3, we study the L_1-biconservative Lorentzian hypersurfaces with constant mean curvature, separately according to four possible types of shape operator of hypersurfaces in four subsections. In subsection 3.1, by three propositions, we show that if a hypersurface M_1^4 of type I has constant ordinary mean curvature and three distinct principal curvatures, then it has constant second mean curvature. In subsection 3.2, we study on L_1-biconservative Lorentzian hypersurfaces M_1^4 with non-diagonal shape operator and at least three distinct principal curvatures.

2 Preliminaries

In this section, we recall some preliminaries from [2, 12, 13] and [15]-[18]. The 5-dimensional Minkowski space, $\mathbb{M}^5$, is the Euclidean 5-space $\mathbb{E}^5$ equipped with a scalar product as

$$\langle x, y\rangle := -x_1y_1 + \Sigma_{i=2}^5 x_iy_i,$$

for every $x, y \in \mathbb{E}^5$.

Throughout the paper, we study the Lorentzian hypersurface of $\mathbb{M}^5$, defined by an isometric immersion $\mathbf{x} : M_1^4 \to \mathbb{M}^5$.

The symbols $\tilde{\nabla}$ and $\bar{\nabla}$ stand for the Levi-Civita connection on M_1^4 and $\mathbb{M}^5$, respectively. For every tangent vector fields X and Y on M, the Gauss formula is given by $\bar{\nabla}_X Y = \tilde{\nabla}_X Y + \langle AX, Y\rangle \mathbf{n}$, for every $X, Y \in \chi(M)$, where, $\mathbf{n}$ is a (locally) unit normal vector field on M and A is the shape operator of M relative to $\mathbf{n}$. For each non-zero vector $X \in \mathbb{M}^5$, the real value $\langle X, X\rangle$ may be a negative, zero or positive number and then, the vector X is said to be time-like, light-like or space-like, respectively.

Definition 1. For a 4-dimensional Lorentzian vector space V_1^4, a basis $\mathcal{B} := \{e_1, \cdots, e_4\}$ is said to be *orthonormal* if it satisfies $\langle e_i, e_j\rangle = \epsilon_i \delta_i^j$ for $i, j = 1, \cdots, 4$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for

$i = 2,3,4$. As usual, δ_i^j stands for the Kronecker function. $\mathcal{B}$ is called *pseudo-orthonormal* if it satisfies $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0$, $\langle e_1, e_2\rangle = -1$ and $\langle e_i, e_j\rangle = \delta_i^j$, for $i = 1,2,3,4$ and $j = 3,4$.

As well-known, the shape operator A of the Lorentzian hypersurface M_1^4 in $\mathbb{M}^5$, as a self-adjoint linear map on the tangent bundle of M_1^4, locally can be put into one of four possible canonical matrix forms, usually denoted by I, II, III and IV. Where, in cases I and IV, with respect to an orthonormal basis of the tangent space of M_1^4, the matrix representation of the induced metric on M_1^4 is

$$G_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the shape operator of M_1^4 can be put into matrix forms

$$B_1 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \qquad \text{and} \qquad B_4 = \begin{pmatrix} \kappa & \lambda & 0 & 0 \\ -\lambda & \kappa & 0 & 0 \\ 0 & 0 & \eta_1 & 0 \\ 0 & 0 & 0 & \eta_2 \end{pmatrix}, \quad (\lambda \neq 0)$$

respectively. For cases II and III, using a pseudo-orthonormal basis of the tangent space of M_1^4, the induced metric on which has matrix form

$$G_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the shape operator of M_1^4 can be put into matrix forms

$$B_2 = \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 1 & \kappa & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \qquad \text{and} \qquad B_3 = \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \kappa & 1 & 0 \\ -1 & 0 & \kappa & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

respectively. In case IV, the matrix B_4 has two conjugate complex eigenvalues $\kappa \pm i\lambda$, but in other cases the eigenvalues of the shape operator are real numbers.

Remark 1. In two cases II and III, one can substitute the pseudo-orthonormal basis $\mathcal{B} := \{e_1, e_2, e_3, e_4\}$ by a new orthonormal basis $\tilde{\mathcal{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3, e_4\}$ where $\tilde{e_1} := \frac{1}{2}(e_1 + e_2)$ and $\tilde{e_2} := \frac{1}{2}(e_1 - e_2)$. Therefore, we obtain new matrix representations $\tilde{B_2}$ and $\tilde{B_3}$ (instead of B_2 and B_3, respectively) as

$$\tilde{B_2} = \begin{pmatrix} \kappa + \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \kappa - \frac{1}{2} & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \qquad \text{and} \qquad \tilde{B_3} = \begin{pmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \kappa & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

After this changes, to unify the notations we denote the orthonormal basis by $\mathcal{B}$ in all cases.

Notation: According to four possible matrix representations of the shape operator of M_1^4, we define its principal curvatures, denoted by unified notations κ_i for $i = 1, \cdots, 4$, as follow. In case I, we put $\kappa_i := \lambda_i$, for $i = 1, \cdots, 4$, where λ_i's are the eigenvalues of B_1.

In cases II, where the matrix representation of A is $\tilde{B_2}$, we take $\kappa_i := \kappa$ for $i = 1,2$, and $\kappa_i := \lambda_{i-2}$, for $i = 3,4$.

In case III, where the shape operator has matrix representation $\tilde{B_3}$, we take $\kappa_i := \kappa$ for $i = 1,2,3$, and $\kappa_4 := \lambda$.

Finally, in the case IV, where the shape operator has matrix representation $\tilde{B}_4$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$, and $\kappa_i := \eta_{i-2}$, for $i = 3, 4$.

The characteristic polynomial of A on M_1^4 is of the form $Q(t) = \prod_{i=1}^{4}(t-\kappa_i) = \sum_{j=0}^{4}(-1)^j s_j t^{4-j}$, where, $s_0 := 1$, $s_i := \sum_{1\leq j_1<\cdots<j_i\leq 4} \kappa_{j_1}\cdots\kappa_{j_i}$ for $i = 1, 2, 3, 4$.

For $j = 1, \cdots, 4$, the *jth mean curvature* H_j of M_1^4 is defined by $H_j = \frac{1}{\binom{4}{j}} s_j$. When H_j is identically null, M_1^4 is said to be $(j-1)$-*minimal*.

Definition 2. (i) A timelike hypersurface $x : M_1^4 \to \mathbb{M}^5$, with diagonalizable shape operator, is said to be *isoparametric* if all of it's principal curvatures are constant.
(ii) A timelike hypersurface $x : M_1^4 \to \mathbb{M}^5$, with non-diagonalizable shape operator, is said to be *isoparametric* if the minimal polynomial of it's shape operator is constant.

Remark 2. Here we remember Theorem 4.10 from [13], which assures us that there is no isoparametric timelike hypersurface of $\mathbb{M}^5$ with complex principal curvatures.

The well-known Newton transformations $P_j : \chi(M) \to \chi(M)$ on M_1^4, is defined by

$$P_0 = I, \quad P_j = s_j I - A \circ P_{j-1}, \qquad (j = 1, 2, 3, 4), \tag{2.1}$$

where, I is the identity map. Using its explicit formula, $P_j = \sum_{i=0}^{j}(-1)^i s_{j-i} A^i$ (where $A^0 = I$), which gives, by the Cayley-Hamilton theorem (stating that any operator is annihilated by its characteristic polynomial), that $P_4 = 0$. It can be seen that, P_j is self-adjoint and commutative with A (see [2, 16]).

Now, we define a notation as

$$\mu_{i;k} = \sum_{1\leq j_1<\cdots<j_k\leq 4;\, j_l\neq i} \kappa_{j_1}\cdots\kappa_{j_k}, \qquad (i = 1, 2, 3, 4; \quad 1 \leq k \leq 3). \tag{2.2}$$

Corresponding to four possible forms $\tilde{B}_i$ (for $1 \leq i \leq 4$) of A, the Newton transformation P_j has different representations. In the case I, where $A = \tilde{B}_1$, we have $P_j = diag[\mu_{1;j}, \cdots, \mu_{4;j}]$, for $j = 1, 2, 3$.

When $A = B_2$ (in the case II), we have

$$P_1 = \begin{pmatrix} \lambda_1+\lambda_2+\kappa-\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \lambda_1+\lambda_2+\kappa+\frac{1}{2} & 0 & 0 \\ 0 & 0 & 2\kappa+\lambda_2 & 0 \\ 0 & 0 & 0 & 2\kappa+\lambda_1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} \lambda_1\lambda_2+(\kappa-\frac{1}{2})(\lambda_1+\lambda_2) & -\frac{1}{2}(\lambda_1+\lambda_2) & 0 & 0 \\ \frac{1}{2}(\lambda_1+\lambda_2) & \lambda_1\lambda_2+(\kappa+\frac{1}{2})(\lambda_1+\lambda_2) & 0 & 0 \\ 0 & 0 & \kappa(\kappa+2\lambda_2) & 0 \\ 0 & 0 & 0 & \kappa(\kappa+2\lambda_1) \end{pmatrix}.$$

In the case III, we have $A = B_3$, and

$$P_1 = \begin{pmatrix} 2\kappa+\lambda & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 2\kappa+\lambda & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\kappa+\lambda & 0 \\ 0 & 0 & 0 & 3\kappa \end{pmatrix}, \quad P_2 = \begin{pmatrix} 2\kappa\lambda+\kappa^2-\frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2}(\kappa+\lambda) & 0 \\ \frac{1}{2} & 2\kappa\lambda+\kappa^2+\frac{1}{2} & \frac{\sqrt{2}}{2}(\kappa+\lambda) & 0 \\ \frac{\sqrt{2}}{2}(\kappa+\lambda) & \frac{\sqrt{2}}{2}(\kappa+\lambda) & 2\kappa\lambda+\kappa^2 & 0 \\ 0 & 0 & 0 & 3\kappa^2 \end{pmatrix}.$$

In the case IV, $A = B_4$,

$$P_1 = \begin{pmatrix} \kappa+\eta_1+\eta_2 & -\lambda & 0 & 0 \\ \lambda & \kappa+\eta_1+\eta_2 & 0 & 0 \\ 0 & 0 & 2\kappa+\eta_2 & 0 \\ 0 & 0 & 0 & 2\kappa+\eta_1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} \kappa(\eta_1+\eta_2)+\eta_1\eta_2 & -\lambda(\eta_1+\eta_2) & 0 & 0 \\ \lambda(\eta_1+\eta_2) & \kappa(\eta_1+\eta_2)+\eta_1\eta_2 & 0 & 0 \\ 0 & 0 & \kappa^2+\lambda^2+2\kappa\eta_2 & 0 \\ 0 & 0 & 0 & \kappa^2+\lambda^2+2\kappa\eta_1 \end{pmatrix}.$$

Fortunately, in all cases we have the following important identities, similar to those in [2, 16].

$$\mu_{i,1} = 4H_1 - \lambda_i, \quad \mu_{i,2} = 6H_2 - \lambda_i\mu_{i,1} = 6H_2 - 4\lambda_i H_1 + \lambda_i^2, \ (1 \le i \le 4), \tag{2.3}$$

$$tr(P_1) = 12H_1, \quad tr(P_2) = 12H_2, \quad tr(P_1 \circ A) = 12H_2, \quad tr(P_2 \circ A) = 12H_3, \tag{2.4}$$

$$trA^2 = 4(4H_1^2 - 3H_2), \quad tr(P_1 \circ A^2) = 12(2H_1H_2 - H_3), \quad tr(P_2 \circ A^2) = 4(4H_1H_3 - H_4). \tag{2.5}$$

The *linearized operator* of the $(j+1)$th mean curvature of M, $L_j : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is defined by the formula $L_j(f) := tr(P_j \circ \nabla^2 f)$, where, $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X \nabla f, Y\rangle$ for every $X, Y \in \chi(M)$.

Associated to the orthonormal frame $\{e_1, \cdots, e_4\}$ of tangent space on a local coordinate system in the hypersurface $x : M_1^4 \to \mathbb{M}^5$, $L_1(f)$ has an explicit expression as

$$L_1(f) = \sum_{i=1}^{4} \epsilon_i \mu_{i,1}(e_i e_i f - \nabla_{e_i} e_i f). \tag{2.6}$$

For a Lorentzian hypersurface $x : M_1^4 \to \mathbb{M}^5$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a} \in \mathbb{M}^5$ we use the decomposition $\mathbf{a} = \mathbf{a}^T + \mathbf{a}^N$ where $\mathbf{a}^T \in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^N \perp TM$, and we have the following formulae from [2, 16].

$$\nabla\langle x, \mathbf{a}\rangle = \mathbf{a}^T, \quad \nabla\langle \mathbf{n}, \mathbf{a}\rangle = -S\mathbf{a}^T. \tag{2.7}$$

$$L_1 x = 12H_2\mathbf{n}, \quad L_1\mathbf{n} = -6\nabla(H_2) - 12[2H_1H_2 - H_3]\mathbf{n}, \tag{2.8}$$

$$L_1^2 x = 12L_1(H_2\mathbf{n}) = 24[P_2\nabla H_2 - 9H_2\nabla H_2] + 12[L_1H_2 - 12H_2(2H_1H_2 - H_3)]\mathbf{n}. \tag{2.9}$$

Assume that a hypersurface $x : M_1^4 \to \mathbb{M}^5$ satisfies the condition $L_1^2 x = 0$, then it is said to be L_1-*biharmonic*. By equalities (2.8) and (2.9), from the condition $L_1(H_2\mathbf{n}) = 0$ (which is equivalent to L_1-biharmonicity) we obtain simpler conditions on M_1^4 to be a L_1-biharmonic hypersurface in $\mathbb{M}^5$, as:

$$(i)\ L_1H_2 = 12H_2(2H_1H_2 - H_3) = H_2 tr(P_1 \circ A^2), \qquad (ii)\ P_2\nabla H_2 = 9H_2\nabla H_2. \tag{2.10}$$

The hypersurface $x : M_1^4 \to \mathbb{M}^5$ is said to be L_1-biconservative if it satisfies condition (2.10)(ii).

The well-known structure equations on $\mathbb{M}^5$ are given by $d\omega_i = \sum_{j=1}^{5} \omega_{ij} \wedge \omega_j$, $\omega_{ij} + \omega_{ji} = 0$ and $d\omega_{ij} = \sum_{l=1}^{5} \omega_{il} \wedge \omega_{lj}$. Restricted on M, we have $\omega_5 = 0$ and then, $0 = d\omega_5 = \sum_{i=1}^{4} \omega_{5,i} \wedge \omega_i$. So, by Cartan's lemma, there exist functions h_{ij} such that $\omega_{5,i} = \sum_{j=1}^{4} h_{ij}\omega_j$ and $h_{ij} = h_{ji}$ Which give the second fundamental form of M, as $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_5$. The mean curvature H is given by $H = \frac{1}{4}\sum_{i=1}^{4} h_{ii}$. Therefore, we obtain the structure equations on M as follow.

$$d\omega_i = \sum_{j=1}^{4} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.11}$$

$$d\omega_{ij} = \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{4} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.12}$$

for $i, j = 1, 2, 3$, and the Gauss equations $R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$, where R_{ijkl} denotes the components of the Riemannian curvature tensor of M. Denoting the covariant derivative of h_{ij} by h_{ijk}, we have

$$dh_{ij} = \sum_{k=1}^{4} h_{ijk}\omega_k + \sum_{k=1}^{4} h_{kj}\omega_{ik} + \sum_{k=1}^{4} h_{ik}\omega_{jk}, \tag{2.13}$$

and by the Codazzi equation we get $h_{ijk} = h_{ikj}$.

3 Biconservative hypersurfaces in Minkowski 5-space with different types of shape operators

This section is followed in several subsections by considering different possible cases for the shape operator of hypersurfaces.

3.1 Diagonal shape operator

The next lemma can be proved by the same manner of similar one in [19].

Lemma 3.1. *Let M_1^4 be a Lorentz hypersurface in $\mathbb{M}^5$ with diagonal shape operator and real principal curvatures of constant multiplicities. Then distribution generated by principal directions is completely integrable. In addition, if a principal curvature be of multiplicity greater than one, then it will be constant on each integral submanifold of its corresponding distribution.*

Proposition 3.1. *Let M_1^4 be a L_1-biconservative orientable Lorentz hypersurface in $\mathbb{M}^5$ having diagonal shape operator A, constant 1st mean curvature and non-constant 2nd mean curvature. Then, M_1^4 has a non-constant principal curvature of multiplicity one.*

Proof. By assumption, there exists an open connected subset $\mathcal{U}$ of M, on which we have $\nabla H_2 \neq 0$. By the biconservativity condition (2.10)(ii), $e_1 := \frac{\nabla H_2}{||\nabla H_2||}$ is an eigenvector of P_2 with the corresponding eigenvalue $9H_2$, on $\mathcal{U}$. Without loss of generality, we can take a suitable orthonormal local basis $\{e_1, e_2, e_3, e_4\}$ for the tangent bundle of M, consisting of the eigenvectors of the shape operator A such that $Ae_i = \lambda_i e_i$ and $P_2 e_i = \mu_{i,2} e_i$, (for $i = 1, 2, 3, 4$) and then

$$\mu_{1,2} = 9H_2. \tag{3.1}$$

By the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} e_i(H_2)e_i$, we get

$$e_1(H_2) \neq 0, \quad e_2(H_2) = e_3(H_2) = e_4(H_2) = 0. \tag{3.2}$$

By (2.3) and (3.1) we have

$$H_2 = \frac{1}{3}\lambda_1(\lambda_1 - 4H). \tag{3.3}$$

Then, having assumed H to be constant, from (3.2) we get

$$e_1(\lambda_1) \neq 0, \quad e_2(\lambda_1) = e_3(\lambda_1) = e_4(\lambda_1) = 0, \tag{3.4}$$

which gives that λ_1 is non-constant. Now, putting $\nabla_{e_i} e_j = \sum_{k=1}^4 \omega_{ij}^k e_k$ (for $i, j = 1, 2, 3, 4$), the identity $e_k < e_i, e_j >= 0$ gives $\epsilon_j \omega_{ki}^j = -\epsilon_i \omega_{kj}^i$ (for $i, j, k = 1, 2, 3, 4$). Furthermore, for distinct $i, j, k = 1, 2, 3, 4$, the Codazzi equation implies

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \quad (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j. \tag{3.5}$$

Since by (3.4) we have $e_1(\lambda_1) \neq 0$, we claim $\lambda_j \neq \lambda_1$ for $j = 2, 3, 4$. Because, assuming $\lambda_j = \lambda_1$ for some integer $j \neq 1$, we have $e_1(\lambda_j) = e_1(\lambda_1) \neq 0$. On the other hand, from (3.5) we obtain $0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1)$. So, we get a contradiction. □

One can find the similar ordinary version of Proposition 3.1 in [9] and [20].

Proposition 3.2. *Let M_1^4 be a L_1-biconservative Lorentz hypersurface in $\mathbb{M}^5$ with diagonal shape operator, which has exactly three distinct principal curvatures, constant 1st mean curvature and non-constant 2nd mean curvature. Then, there exists a locally moving orthonormal tangent frame $\{e_1, e_2, e_3, e_4\}$ of principal vectors of M_1^4 with associated principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4$, which satisfy the following equalities:*

$$\begin{aligned}
&(i)\nabla_{e_1} e_1 = 0, \ \nabla_{e_2} e_1 = \alpha e_2, \ \nabla_{e_3} e_1 = \alpha e_3, \ \nabla_{e_4} e_1 = -\beta e_4, \\
&(ii)\nabla_{e_2} e_2 = -\alpha e_1 + \omega_{22}^3 e_3 + \gamma e_4, \ \nabla_{e_i} e_2 = \omega_{i2}^3 e_3 \quad for\ i = 1, 3, 4\ ; \\
&(iii)\nabla_{e_3} e_3 = -\alpha e_1 - \omega_{32}^3 e_3 + \gamma e_4, \quad \nabla_{e_i} e_3 = -\omega_{i2}^3 e_2 \quad for\ i = 1, 2, 4\ , \\
&(iv)\nabla_{e_1} e_4 = 0, \ \nabla_{e_2} e_4 = -\gamma e_2, \ \nabla_{e_3} e_4 = -\gamma e_3, \ \nabla_{e_4} e_4 = \beta e_1,
\end{aligned} \tag{3.6}$$

where $\alpha := \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}$, $\beta := \frac{e_1(\lambda_1 + 2\lambda_2)}{\lambda_1 - \lambda_4}$, $\gamma := \frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4}$.

Proof. Similar to the proof of Proposition 3.1, taking a suitable local basis $\{e_1, e_2, e_3, e_4\}$ for TM, one can see that the equalities (3.1) – (3.5) occur and λ_1 is of multiplicity one. Also, direct calculations give $[e_2, e_3](\lambda_1) = [e_3, e_4](\lambda_1) = [e_2, e_4](\lambda_1) = 0$, which yields

$$\omega_{23}^1 = \omega_{32}^1, \quad \omega_{34}^1 = \omega_{43}^1, \quad \omega_{24}^1 = \omega_{42}^1. \tag{3.7}$$

Now, having assumed M_1^4 to has three distinct principal curvatures, (without loss of generality) we can take $\lambda_2 = \lambda_3$, and then $\lambda_4 = 4H_1 - \lambda_1 - 2\lambda_2$. Hence, applying equalities (3.5) for distinct positive integers i, j and k less than 5, we get $e_2(\lambda_2) = e_3(\lambda_2) = 0$ and then,

$$\begin{aligned}
&(i)\ \omega_{11}^1 = \omega_{12}^1 = \omega_{13}^1 = \omega_{14}^1 = \omega_{31}^2 = \omega_{21}^3 = \omega_{34}^2 = \omega_{24}^3 = \omega_{42}^4 = \omega_{43}^4 = 0, \\
&(ii)\ \omega_{21}^2 = \omega_{31}^3 = \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}, \ \omega_{41}^4 = \frac{-e_1(\lambda_1 + 2\lambda_2)}{\lambda_1 - \lambda_4}, \ \omega_{24}^2 = \omega_{34}^3 = \frac{-e_4(\lambda_2)}{\lambda_2 - \lambda_4}, \\
&(iii)\ (\lambda_1 - \lambda_4)\omega_{24}^1 = (\lambda_1 - \lambda_2)\omega_{42}^1, \ (\lambda_1 - \lambda_4)\omega_{34}^1 = (\lambda_1 - \lambda_2)\omega_{43}^1.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we get $\omega_{24}^1 = \omega_{42}^1, = \omega_{34}^1 = \omega_{43}^1 = \omega_{12}^4 = \omega_{13}^4 = 0$. Therefore, all items of the proposition obtain from the above results. □

Proposition 3.3. *Let M_1^4 be a L_1-biconservative orientable Lorentz hypersurface in the Minkowski 5-space $\mathbb{M}^5$ with diagonal shape operator, which has three distinct principal curvatures, constant 1st mean curvature and non-constant 2nd mean curvature. Then, there exists an orthonormal (local) tangent frame $\{e_1, e_2, e_3, e_4\}$ of principal vectors of M_1^4 with associated principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4$, satisfying $e_4(\lambda_2) = 0$ and*

$$e_1(\lambda_2)e_1(\lambda_1 + 2\lambda_2) = \frac{1}{2}\lambda_2(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_1)(2\lambda_1 + 4\lambda_2 + \lambda_4). \tag{3.9}$$

Proof. From Gauss curvature tensor $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, by substituting X, Y and Z by different choices from e_1, e_2, e_3 and e_4, using the results of Proposition 3.2, we get the following equalities:

$$\begin{aligned}
&(i)\ e_1(\alpha) + \alpha^2 = -\lambda_1\lambda_2, \quad \beta^2 - e_1(\beta) = -\lambda_1\lambda_4;\\
&(ii)\ e_1\left(\frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4}\right) + \alpha\frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4} = 0;\\
&(iii)\ e_4(\alpha) - (\alpha + \beta)\frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4} = 0;\\
&(iv)\ e_4\left(\frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4}\right) + \alpha\beta - \left(\frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4}\right)^2 = \lambda_2\lambda_4.
\end{aligned} \tag{3.10}$$

Now, from (2.6) and (2.10), applying Proposition (3.2) we obtain

$$\begin{aligned}
&(\lambda_1 - 4H_1)e_1e_1(H_2) - (2(\lambda_2 - 4H_1)\alpha + (\lambda_1 + 2\lambda_2)\beta)e_1(H_2)\\
&= 12H_2(2H_1H_2 - H_3),
\end{aligned} \tag{3.11}$$

where $\alpha := \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}$ and $\beta := \frac{e_1(\lambda_1 + 2\lambda_2)}{\lambda_1 - \lambda_4}$.

On the other hand, from (3.2) and (3.6), we obtain

$$e_ie_1(H_{k+1}) = 0, \tag{3.12}$$

for $i = 2, 3, 4$. Also, by differentiating α and β along e_4, we get

$$(\lambda_1 - \lambda_2)e_4(\alpha) - \alpha e_4(\lambda_2) = e_4e_1(\lambda_2) = \frac{1}{2}(\lambda_1 - \lambda_4)e_4(\beta) + \beta e_4(\lambda_2),$$

then

$$\frac{1}{2}(\lambda_1 - \lambda_4)e_4(\beta) = (\lambda_1 - \lambda_2)e_4(\alpha) - (\alpha + \beta)e_4(\lambda_2),$$

which, by substituting the value of $e_4(\alpha)$ from (3.10), gives

$$e_4(\beta) = \frac{-8e_4(\lambda_2)(\alpha + \beta)(\lambda_2 - H_1)}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)}.$$

Again, differentiating (3.11)along e_4 and using (3.12), (3.10) and the last value of $e_4(\beta)$, we get $e_4(\lambda_2) = 0$ or

$$\frac{4(\alpha + \beta)[-H_1(8\lambda_1 + 12\lambda_2) + {\lambda_1}^2 + 3\lambda_1\lambda_2 + 16H_1^2]e_1(H_2)}{\lambda_4 - \lambda_1} + 6H_2(\lambda_2 - \lambda_4)^2 = 0. \tag{3.13}$$

Finally, we claim that $e_4(\lambda_2) = 0$.

Indeed, if the claim be false, then we have

$$\frac{4(\alpha+\beta)\gamma e_1(H_2)}{\lambda_1-\lambda_4} = 6H_2(\lambda_2-\lambda_4)^2, \tag{3.14}$$

where $\gamma = -8H_1\lambda_1 + {\lambda_1}^2 + 3\lambda_1\lambda_2 - 12H_1\lambda_2 + 16H_1^2$. Differentiating (3.14) along e_4, we get

$$\begin{aligned}&\frac{2(\alpha+\beta)\left[6\gamma(\lambda_2-H_1)+(3\lambda_1-12H_1)(\lambda_1+\lambda_2-2H_1)(\lambda_1+3\lambda_2-4H_1)\right]e_1(H_2)}{(\lambda_1+\lambda_2-2H_1)^2}\\ &= 36H_2(4H_1+\lambda_1+3\lambda_2)^2.\end{aligned} \tag{3.15}$$

Eliminating $e_1(H_2)$ from (3.14) and (3.15), we obtain

$$\gamma(2\lambda_1-2H_1) = (\lambda_1-4H_1)(\lambda_1+\lambda_2-2H_1)(-4H_1+\lambda_1+3\lambda_2). \tag{3.16}$$

By differentiating (3.16) along e_4, we get $4H_1 = \lambda_1$, which is not possible since λ_1 is not constant. Consequently, $e_4(\lambda_2) = 0$. Therefore, the latest equality in (3.10) gives the main result. ☐

Theorem 3.1. *Let $x : M_1^4 \to \mathbb{M}^5$ be a L_1-biconservative Lorentz hypersurface with diagonal shape operator and constant 1st mean curvature which has three distinct principal curvatures. Then it has constant 2nd mean curvature.*

Proof. We assume H_2 is non-constant on M and try to get a contradiction.

By differentiating (3.3) in direction of e_1 and using the definition of β, we get

$$e_1(H_2) = \frac{4}{3}(2H_1-\lambda_1)e_1(\lambda_2) + \frac{4}{3}(\lambda_1+\lambda_2-2H_1)(\lambda_1-2H_1)\beta. \tag{3.17}$$

By Proposition 3.3 and equalities (3.10), from (3.17) we obtain

$$\begin{aligned}e_1e_1(H_2) = {}&\frac{4}{3}\lambda_1\lambda_2(\lambda_1-\lambda_2)(\lambda_1+2H_1)\\ &+\frac{4}{3}(4H_1-\lambda_1-2\lambda_2)(\lambda_1-2H_1)(4\lambda_1\lambda_2+{\lambda_1}^2-4H_1\lambda_2-2H_1\lambda_1)\\ &+\left[3\beta-4\alpha+2\frac{(\lambda_1+\lambda_2-2H_1)\beta-(\lambda_1-\lambda_2)\alpha}{\lambda_1-2H_1}\right]e_1(H_2).\end{aligned} \tag{3.18}$$

Combining (3.11) and (3.18), we get

$$(P_{1,2}\alpha + P_{2,2}\beta)e_1(H_2) = P_{3,6}, \tag{3.19}$$

where $P_{1,2}$, $P_{2,2}$ and $P_{3,6}$ are polynomials in terms of λ_1 and λ_2 of degrees 2, 2 and 6, respectively.

Differentiating (3.19) along e_1 and using equalities (3.9), (3.10)-(i) and (3.19), we get the following equality

$$P_{4,8}\alpha + P_{5,8}\beta = P_{6,5}e_1(H_2), \tag{3.20}$$

where $P_{4,8}$, $P_{5,8}$ and $P_{6,5}$ are polynomials in terms of λ_1 and λ_2 of degrees 8, 8 and 5, respectively.

Combining (3.17) and (3.20), we obtain

$$\begin{aligned}&\left(P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\right)\alpha\\ &+\left(P_{5,8}-\frac{4}{3}P_{6,5}(\lambda_1+\lambda_2-2H_1)(\lambda_1-2H_1)\right)\beta = 0.\end{aligned} \tag{3.21}$$

On the other hand, combining (3.17) with (3.19) and using Proposition 3.3, we get

$$P_{2,2}(\lambda_1+\lambda_2-2H_1)(\lambda_1-2H_1)\beta^2-P_{1,2}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\alpha^2=\zeta, \tag{3.22}$$

where ζ is given by

$$\zeta=\lambda_2(4H_1-\lambda_1-2\lambda_2)(\lambda_1-2H_1)\Big(P_{2,2}(\lambda_1-\lambda_2)-P_{1,2}(\lambda_1+\lambda_2-2H_1)\Big)+\frac{3}{4}P_{3,6}.$$

Using Proposition 3.3 and equality (3.21), we get

$$\begin{aligned}\alpha^2&=\frac{\frac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)+P_{5,8}}{P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)}\lambda_2\lambda_4,\\ \beta^2&=\frac{\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)-P_{4,8}}{P_{5,8}-\frac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)}\lambda_2\lambda_4.\end{aligned} \tag{3.23}$$

Eliminating α^2 and β^2 from (3.22), we obtain

$$\begin{aligned}&-\lambda_2\lambda_4(\lambda_1+2H_1)(\lambda_2-\lambda_1)P_{1,2}\Big(P_{5,8}-\frac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)\Big)^2\\ &-\frac{1}{2}\lambda_2\lambda_4(\lambda_1+2H_1)(\lambda_1-\lambda_4)P_{2,2}\Big(P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\Big)^2\\ &=\zeta\left(P_{5,8}-\frac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)\right)\left(P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\right),\end{aligned} \tag{3.24}$$

which is a polynomial equation of degree 22 in terms of λ_2 and λ_1.

Now consider an integral curve of e_1 passing through $p=\gamma(t_0)$ as $\gamma(t)$, $t\in I$. Since $e_i(\lambda_1)=e_i(\lambda_2)=0$ for $i=2,3,4$ and $e_1(\lambda_1)$, $e_1(\lambda_2)\neq 0$, we can assume $\lambda_2=\lambda_2(t)$ and $\lambda_1=\lambda_1(\lambda_2)$ in some neighborhood of $\lambda_0=\lambda_2(t_0)$. Using (3.21), we have

$$\begin{aligned}\frac{d\lambda_1}{d\lambda_2}&=\frac{d\lambda_1}{dt}\frac{dt}{d\lambda_2}=\frac{e_1(\lambda_1)}{e_1(\lambda_2)}\\ &=2\frac{(\lambda_1+\lambda_2-2H_1)\beta-(\lambda_1-\lambda_2)\alpha}{(\lambda_1-\lambda_2)\alpha}\\ &=\frac{2\left(P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\right)(\lambda_1+\lambda_2-2H_1)}{\left(\frac{4}{3}P_{6,5}(\lambda_1+\lambda_2-2H_1)(\lambda_1-2H_1)-P_{5,8}\right)(\lambda_1-\lambda_2)}-2\end{aligned} \tag{3.25}$$

Differentiating (3.24) with respect to λ_2 and substituting $\frac{d\lambda_1}{d\lambda_2}$ from (3.25), we get

$$f(\lambda_1,\lambda_2)=0, \tag{3.26}$$

another algebraic equation of degree 30 in terms of λ_1 and λ_2.

We rewrite (3.24) and (3.26) respectively in the following forms

$$\sum_{i=0}^{22}f_i(\lambda_1)\lambda_2^i=0,\qquad \sum_{i=0}^{30}g_i(\lambda_1)\lambda_2^i=0, \tag{3.27}$$

where $f_i(\lambda_1)$ and $g_j(\lambda_1)$ are polynomial functions of λ_1. We eliminate λ_2^{30} between these two polynomials of (3.27) by multiplying $g_{30}\lambda_2^8$ and f_{22} respectively on the first and second equations of (3.27), we obtain a new polynomial equation in λ_2 of degree 29. Combining this equation with the first equation of (3.27), we successively obtain a polynomial equation in λ_2 of degree 28. In a similar way, by using the first equation of (3.27) and its consequences we are able to gradually eliminate λ_2. At last, we obtain a non-trivial algebraic polynomial equation in λ_1 with constant coefficients. Therefore, we get that the real function λ_1 is constant and then by (3.3), H_2 is constant, which contradicts with the first assumption. Hence, H_2 is constant on M_1^4. □

3.2 Non-diagonal shape operator

Proposition 3.4. *Let $x : M_1^4 \to \mathbb{M}^5$ be a L_1-biconservative connected orientable Lorentz hypersurface with non-diagonal shape operator of form II, which has three distinct principal curvatures and constant 1st mean curvature. Then, its 2nd mean curvature has to be constant.*

Proof. Assuming H_2 to be non-constant, we try to get a contradiction. We show that the open subset $\mathcal{U} = \{p \in M : \nabla H_{k+1}^2(p) \neq 0\}$ is empty. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on M, the shape operator A has the matrix form $\tilde{B}_2$, such that $Ae_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Ae_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Ae_3 = \lambda_1 e_3$ and $Ae_4 = \lambda_2 e_4$, and then, for $j = 1, 2, 3$ we have $P_j e_1 = [\mu_{1,2;j} + (\kappa - \frac{1}{2})\mu_{1,2;j-1}]e_1 + \frac{1}{2}\mu_{1,2;j-1}e_2$, $P_2 e_2 = -\frac{1}{2}\mu_{1,2;j-1}e_1 + [\mu_{1,2;j} + (\kappa - \frac{1}{2})\mu_{1,2;j-1}]e_2$, and $P_2 e_3 = \mu_{3;j}e_3$ and $P_2 e_4 = \mu_{4;j}e_4$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2)e_i$, from conditions (2.10)(ii), we get

$$\begin{aligned}
&(i)\ [\lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_1 e_1(H_2) = \frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_2 e_2(H_2),\\
&(ii)\ [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_2 e_2(H_2) = -\frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_1 e_1(H_2),\\
&(iii)\ (\kappa^2 + 2\kappa\lambda_2 - 9H_2)\epsilon_3 e_3(H_2) = 0,\\
&(iv)\ (\kappa^2 + 2\kappa\lambda_1 - 9H_2)\epsilon_3 e_4(H_2) = 0.
\end{aligned} \tag{3.28}$$

Now, we prove the following claim.

Claim: $e_i(H_2) = 0$ for $i = 1, 2, 3, 4$.

If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (3.28(i, ii)) by $\epsilon_1 e_1(H_2)$ we get

$$\begin{aligned}
&(i)\ \lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2 = \frac{1}{2}(\lambda_1 + \lambda_2)u,\\
&(ii)\ [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]u = -\frac{1}{2}(\lambda_1 + \lambda_2),
\end{aligned} \tag{3.29}$$

where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. By substituting (i) in (ii), we obtain $(\lambda_1 + \lambda_2)(1 + u)^2 = 0$, then $\lambda_1 + \lambda_2 = 0$ or $u = -1$.

If $\lambda_1 + \lambda_2 = 0$, then, from (3.29)-$(i)$ we obtain $9H_2 = -\lambda_1^2$, which gives $3\kappa^2 = -\lambda_1^2$. Since H_1 is assumed to be constant on M, then $\kappa = 2H_1$ is constant on M. Hence, λ_1 and λ_2 are also constant on M. Therefore, M_1^4 is an isoparametric Lorentz hypersurface of real principal curvatures in E_1^5, which by Corollary 2.7 in [13], cannot has more than one nonzero principal curvature contradicting with the assumptions. So, $\lambda_1 + \lambda_2 \neq 0$ and then $u = -1$.

From $u = -1$, we get $\lambda_1\lambda_2 + \kappa(\lambda_1 + \lambda_2) = 9H_2$, then

$$3\kappa^2 + 4\kappa(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = 0.$$

Since $4H_1 = 2\kappa + \lambda_1 + \lambda_2$ is assumed to be constant on M, by substituting which in the last equality, we get $\lambda^2 - H_1\lambda - 3H_1^2 = 0$, which means λ, κ and the kth mean curvatures (for

$k = 2,3,4$) are constant on M. So, we got a contradiction and therefore, the first part of the claim is proved.

By a similar manner, each of assumptions $e_i(H_2) \neq 0$ for $i = 2,3,4$, gives the equality $\lambda^2 + 2\kappa\lambda = 9H_2$, which implies the contradiction that H_2 is constant on M. So, the claim is affirmed. □

Proposition 3.5. *Let $x : M_1^4 \to \mathbb{M}^5$ be a L_1-biconservative connected orientable lorentz hypersurface with shape operator of type II which has one constant principal curvature. Then its 2nd mean curvature is constant. Furthermore, M_1^4 is isoparametric and its principal curvatures are constant.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on M, the shape operator A has the matrix form $\tilde{B}_2$, such that $Ae_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Ae_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Ae_3 = \lambda_1 e_3$ and $Ae_4 = \lambda_2 e_4$, and then, we have $P_2 e_1 = [\lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2)]e_1 + \frac{1}{2}(\lambda_1 + \lambda_2)e_2$, $P_2 e_2 = -\frac{1}{2}(\lambda_1 + \lambda_2)e_1 + [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2)]e_2$, and $P_2 e_3 = (\kappa^2 + 2\kappa\lambda_2)e_3$ and $P_2 e_4 = (\kappa^2 + 2\kappa\lambda_1)e_4$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2) e_i$, from condition (2.10(ii)) we get

$$\begin{aligned}
&(i) \quad [\lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_1 e_1(H_2) = \frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_2 e_2(H_2),\\
&(ii) \quad [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_2 e_2(H_2) = \frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_1 e_1(H_2),\\
&(iii) \quad (\kappa^2 + 2\kappa\lambda_2 - 9H_2)\epsilon_3 e_3(H_2) = 0,\\
&(iv) \quad (\kappa^2 + 2\kappa\lambda_1 - 9H_2)\epsilon_3 e_4(H_2) = 0.
\end{aligned} \tag{3.30}$$

Now, we prove some simple claims.

Claim: $e_1(H_2) = e_2(H_2) = e_3(H_2) = e_4(H_2) = 0$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (3.30(i, ii)) by $\epsilon_1 e_1(H_2)$ we get

$$\begin{aligned}
&(i) \quad \lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2 = \frac{1}{2}(\lambda_1 + \lambda_2)u,\\
&(ii) \quad [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]u = -\frac{1}{2}(\lambda_1 + \lambda_2),
\end{aligned} \tag{3.31}$$

where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. By substituting (i) in (ii), we obtain $\frac{1}{2}(\lambda_1+\lambda_2)(1+u)^2 = 0$, Then $\lambda_1+\lambda_2 = 0$ or $u = -1$. If $\lambda_1 + \lambda_2 = 0$, then, by assumption we get that $\kappa = 2H_1$ is constant, and also, from (3.29(i)) we obtain $H_2 = \frac{-1}{9}\lambda_1^2$ which gives $\frac{1}{6}(\kappa^2 - \lambda_1^2) = \frac{-1}{9}\lambda_1^2$ and then $\lambda_1^2 = 3\kappa^2$. Hence, we get $H_2 = \frac{-1}{3}\kappa^2$, which means H_2 is constant.

Also, by assumption $\lambda_1+\lambda_2 \neq 0$ we get $u = -1$, which, using (3.31(i)) and $4H_1 = 2\kappa+\lambda_1+\lambda_2$, gives $5\kappa^2 - 16\kappa H_1 - \lambda_1(4H_1 - 2\kappa - \lambda_1) = 0$. Without loss of generality, we assume that λ_1 is constant on M. So, from the last equation we get that κ, λ_2 and H_2 are constant on $\mathcal{U}$, which is a contradiction. Therefore, the first claim is proved. The second claim (i.e. $e_2(H_2) = 0$) can be proven by a similar manner.

Now, if $e_3(H_2) \neq 0$, then using (3.30(iii)) and $4H_1 = 2\kappa + \lambda_1 + \lambda_2$ and by assuming λ_1 to be constant on M, we get

$$\kappa^2 - (\frac{16}{3}H_1 - \frac{2}{3}\lambda_1)\kappa - 4\lambda_1 H_1 + \lambda_1^2 = 0,$$

which gives that κ, λ_2 and H_2 are constant on $\mathcal{U}$, which is a contradiction. Therefore, the third claim is proved.

The forth claim (i.e. $e_4(H_2) = 0$) can be proven by a manner exactly similar to third one. □

Theorem 3.2. *Let $x : M_1^4 \to E_1^5$ be a L_1-biconservative Lorentz hypersurface with non-diagonal shape operator of form II, which has one constant principal curvature and constant ordinary mean curvature. Then, it is 1-minimal.*

Proof. By Proposition 3.5, all of principal curvatures of M_1^3 are constant and M_1^3 is isoparametric. We claim that H_2 is null. Since, by Corollary 2.7 in [13], an isoparametric Lorentz hypersurface of real principal curvatures in E_1^5 has at most one nonzero principal curvature, we get $H_2 = 0$. □

Proposition 3.6. *Let $x : M_1^4 \to \mathbb{M}^5$ be a L_1-biconservative connected orientable Lorentz hypersurface with non-diagonal shape operator of form III, then it has constant 2nd mean curvature.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on M, the shape operator A has the matrix form $\tilde{B}_3$, such that $Ae_1 = \kappa e_1 - \frac{\sqrt{2}}{2}e_3$, $Ae_2 = \kappa e_2 - \frac{\sqrt{2}}{2}e_3$, $Ae_3 = \frac{\sqrt{2}}{2}e_1 - \frac{\sqrt{2}}{2}e_2 + \kappa e_3$ and $Ae_4 = \lambda e_4$ and then, we have $P_2e_1 = (\kappa^2+2\kappa\lambda-\frac{1}{2})e_1+\frac{1}{2}e_2+\frac{\sqrt{2}}{2}(\kappa+\lambda)e_3$, $P_2e_2 = \frac{-1}{2}e_1+(\kappa^2+2\kappa\lambda+\frac{1}{2})e_2+\frac{\sqrt{2}}{2}(\kappa+\lambda)e_3$, $P_2e_3 = \frac{-\sqrt{2}}{2}(\kappa+\lambda)e_1 + \frac{\sqrt{2}}{2}(\kappa+\lambda)e_2 + (\kappa^2+2\kappa\lambda)e_3$ and $P_2e_4 = 3\kappa^2 e_4$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2)e_i$, from condition (2.10(ii)) we get

$$\begin{aligned}
&(i) \quad (\kappa^2 + 2\kappa\lambda - \frac{1}{2} - 9H_2)\epsilon_1 e_1(H_2) - \frac{1}{2}\epsilon_2 e_2(H_2) - \frac{\sqrt{2}}{2}(\kappa+\lambda)\epsilon_3 e_3(H_2) = 0,\\
&(ii) \quad \frac{1}{2}\epsilon_1 e_1(H_2) + (\kappa^2 + 2\kappa\lambda + \frac{1}{2} - 9H_2)\epsilon_2 e_2(H_2) + \frac{\sqrt{2}}{2}(\kappa+\lambda)\epsilon_3 e_3(H_2) = 0,\\
&(iii) \quad \frac{\sqrt{2}}{2}(\kappa+\lambda)(\epsilon_1 e_1(H_2) + \epsilon_2 e_2(H_2)) + (\kappa^2 + 2\kappa\lambda - 9H_2)\epsilon_3 e_3(H_2) = 0,\\
&(iv) \quad (3\kappa^2 - 9H_2)\epsilon_4 e_4(H_2) = 0.
\end{aligned} \tag{3.32}$$

Now, we prove some simple claims.

Claim: $e_1(H_2) = e_2(H_2) = e_3(H_2) = e_4(H_2) = 0$.

If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (3.32(i, ii, iii)) by $\epsilon_1 e_1(H_2)$, and using the identity $2H_2 = \kappa^2 + \kappa\lambda$ in Case III, putting $u_1 := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$ and $u_2 := \frac{\epsilon_3 e_3(H_2)}{\epsilon_1 e_1(H_2)}$, we get

$$\begin{aligned}
&(i) \quad -\frac{1}{2} - \frac{7}{2}\kappa^2 - \frac{5}{2}\kappa\lambda - \frac{1}{2}u_1 - \frac{\sqrt{2}}{2}(\kappa+\lambda)u_2 = 0,\\
&(ii) \quad \frac{1}{2} + (\frac{1}{2} - \frac{7}{2}\kappa^2 - \frac{5}{2}\kappa\lambda)u_1 + \frac{\sqrt{2}}{2}(\kappa+\lambda)u_2 = 0,\\
&(iii) \quad \frac{-\sqrt{2}}{2}(\kappa+\lambda)(1+u_1) - (\frac{7}{2}\kappa^2 + \frac{5}{2}\kappa\lambda)u_2 = 0,
\end{aligned} \tag{3.33}$$

which, by comparing (i) and (ii), gives $\frac{-1}{2}\kappa(7\kappa+5\lambda)(1+u_1) = 0$. If $\kappa = 0$, then $H_2 = 0$. Assuming $\kappa \neq 0$, we get $u_1 = -1$ or $\lambda = -\frac{7}{5}\kappa$. If $u_1 \neq -1$ then $\lambda = -\frac{7}{5}\kappa$, then by (3.33($iii$)) we obtain $u_1 = -1$, which is a contradiction. Hence we have $u_1 = -1$, which by (3.33(i, iii)) implies $u_2 = 0$.

Now we discuss on two cases $\lambda=-\frac{7}{5}\kappa$ or $\lambda\neq-\frac{7}{5}\kappa$. If $\lambda=-\frac{7}{5}\kappa$, then, $\kappa=\frac{5}{2}H_1$, $H_2=\frac{-1}{5}\kappa^2$, $H_3=\frac{-4}{5}\kappa^3$ and $H_4=\frac{-7}{5}\kappa^4$ are all constants on $\mathcal{U}$. Also, the case $\lambda\neq-\frac{7}{5}\kappa$ is in contradiction with (3.33(ii)).

Hence, the first claim $e_1(H_2)\equiv 0$ is affirmed. Similarly, the second claim (i.e. $e_2(H_2)=0$) can be proved.

Now, applying the results $e_1(H_2)=e_2(H_2)=0$, from (3.33(ii,iii)) we get $e_3(H_2)=0$.

The final claim (i.e. $e_2(H_2)=0$), can be proved using (3.33(iv)), in a straightforward manner. □

Proposition 3.7. *Let $x: M_1^4\to\mathbb{M}^5$ be an L_1-biconservative connected orientable Lorentz hypersurface with non-diagonal shape operator of form IV. If M_1^4 has constant mean curvature and a constant real principal curvature. Then its 2nd mean curvature has to be constant.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U}=\{p\in M:\nabla H_2^2(p)\neq 0\}$, we try to show $\mathcal{U}=\emptyset$. By assumption, the shape operator A of M_1^4 is of type IV, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1,\cdots,e_4\}$ on M, the shape operator A has the matrix form B_4, such that $Ae_1=\kappa e_1-\lambda e_2$, $Ae_2=\lambda e_1+\kappa e_2$, $Ae_3=\eta_1 e_3$, $Ae_4=\eta_2 e_4$ and then, we have $P_2e_1=[\kappa(\eta_1+\eta_2)+\eta_1\eta_2]e_1+\lambda(\eta_1+\eta_2)e_2$, $P_2e_2=-\lambda(\eta_1+\eta_2)e_1+[\kappa(\eta_1+\eta_2)+\eta_1\eta_2]e_2$, $P_2e_3=(\kappa^2+\lambda^2+2\kappa\eta_2)e_3$ and $P_2e_4=(\kappa^2+\lambda^2+2\kappa\eta_1)e_4$.

Using the polar decomposition $\nabla H_2=\sum_{i=1}^{4}\epsilon_i e_i(H_2)e_i$, from condition (2.10(ii)) we get

$$\begin{aligned}
&(i)\quad (\kappa\eta_1+\kappa\eta_2+\eta_1\eta_2-9H_2)\epsilon_1e_1(H_2)-\lambda(\eta_1+\eta_2)\epsilon_2e_2(H_2)=0,\\
&(ii)\quad \lambda(\eta_1+\eta_2)\epsilon_1e_1(H_2)+(\kappa\eta_1+\kappa\eta_2+\eta_1\eta_2-9H_2)\epsilon_2e_2(H_2)=0,\\
&(iii)\quad (\kappa^2+\lambda^2+2\kappa\eta_2-9H_2)\epsilon_3e_3(H_2)=0,\\
&(iv)\quad (\kappa^2+\lambda^2+2\kappa\eta_1-9H_2)\epsilon_4e_4(H_2)=0,
\end{aligned}\tag{3.34}$$

Now, assuming H_1 and η_1 to be constant on M, the we prove four simple claims.

Claim: $e_1(H_2)=e_2(H_2)=e_3(H_2)=e_4(H_2)=0$.

If $e_1(H_2)\neq 0$, then by dividing both sides of equalities (3.34(i,ii)) by $\epsilon_1e_1(H_2)$ and putting $u:=\frac{\epsilon_2e_2(H_2)}{\epsilon_1e_1(H_2)}$ we get

$$\begin{aligned}
&(i)\quad \kappa(\eta_1+\eta_2)+\eta_1\eta_2-9H_2=\lambda(\eta_1+\eta_2)u,\\
&(ii)\quad (\kappa(\eta_1+\eta_2)+\eta_1\eta_2-9H_2)u=-\lambda(\eta_1+\eta_2),
\end{aligned}\tag{3.35}$$

which, by substituting (i) in (ii), gives $\lambda(\eta_1+\eta_2)(1+u^2)=0$, then $\lambda(\eta_1+\eta_2)=0$. Since by assumption $\lambda\neq 0$, we get $\eta_1+\eta_2=0$. So, by (3.35(i)), we obtain $\kappa^2+\lambda^2=\frac{1}{3}\eta_1^2$. Since one of real principal curvatures η_1 and η_2 is assumed to be constant, we get that $9H_2=-\eta_1^2=-\eta_1^2$ is constant. Also, since $H_1=\frac{1}{2}\kappa$ is assumed to be constant, we get that $H_3=\frac{-1}{2}\kappa\eta_1^2$ and $H_4=\frac{-1}{3}\eta_1^4$ are constant. These results are in contradiction with the assumption $e_1(H_2)\neq 0$. Hence, the first claim is proved.

Similarly, if $e_2(H_2)\neq 0$, then by dividing both sides of equalities (3.34(i,ii)) by $\epsilon_2e_2(H_2)$ and taking $v:=\frac{\epsilon_1e_1(H_2)}{\epsilon_2e_2(H_2)}$, we get $\lambda(\eta_1+\eta_2)(1+v^2)=0$, which by a similar way gives the same results in contradiction with the assumption $e_2(H_2)\neq 0$. Hence, the second claim is satisfied.

Now, in order to prove the third claim, we assume that $e_3(H_2)\neq 0$. From equality (3.34(iii)) we have $\kappa^2+\lambda^2+2\kappa\eta_2=9H_2$, and by a straightforward computation we get

$$-3\kappa^2+2(4H_1-\eta_1)\kappa+3\eta_1(4H_1-\eta_1)=-\lambda^2<0,$$

then,

$$-2[2\kappa^2 + (\eta_1 - 4H_1)\kappa + 2\eta_1(\eta_1 - 3H_1)] = -(\lambda^2 + \kappa^2 + \eta_1^2) < 0.$$

Remember that the last inequality occurs if and only if we have $\delta < 0$ where

$$\delta = (\eta_1 - 4H_1)^2 - 16\eta_1(\eta_1 - 3H_1) = -15\eta_1^2 + 40\eta_1 H_1 + 16H_1^2.$$

The condition $\delta < 0$ is equivalent to a new inequality $\bar{\delta} < 0$ where

$$\bar{\delta} = (40H_1)^2 + (4 \times 15 \times 16)H_1^2 = 2560H_1^2,$$

which is a contradiction. So, the third claim is proved.

To prove the final part of the claim, we assume that $e_4(H_2) \neq 0$. From equality (3.34(iv)) we have $\kappa^2 + \lambda^2 + 2\kappa\eta_1 = 9H_2$, and by a straightforward computation we get

$$-11\kappa^2 + (24H_1 - 10\eta_1)\kappa + 12\eta_1 H_1 - 3\eta_1^2 = -\lambda^2 < 0,$$

then,

$$-2[6\kappa^2 + (5\eta_1 - 12H_1)\kappa + 2\eta_1(\eta_1 - 3H_1)] = -(\lambda^2 + \kappa^2 + \eta_1^2) < 0.$$

Remember that the last inequality occurs if and only if we have $\delta < 0$ where

$$\delta = (5\eta_1 - 12H_1)^2 - 48\eta_1(\eta_1 - 3H_1) = -23\eta_1^2 + 24\eta_1 H_1 + 144H_1^2.$$

The condition $\delta < 0$ is equivalent to a new inequality $\bar{\delta} < 0$ where

$$\bar{\delta} = (24H_1)^2 + (4 \times 23 \times 144)H_1^2 = 13824H_1^2,$$

which is a contradiction. So, the 4th claim is proved. The final result is that H_2 is constant on M. □

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