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On the Bari basis property for even-order differential operators with involution

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Abstract. By using the method of similar operators we study even-order differential operators with involution. The domain of these operators are defined by periodic and antiperiodic boundary conditions. We obtain estimates for spectral projections and we prove the Bari basis property for the system of eigenfunctions and associated functions.

Keywords. even-order differential operator, basis, spectral projection, system of eigenfunctions and associated functions

1 Introduction and the main result

We consider the operator $S_\theta : D(S_\theta) \subset L^2(-1,1) \to L^2(-1,1)$, $\theta \in \{0,1\}$, given by

$$(S_\theta y)(x) = (-1)^k y^{(2k)}(-x) - p(x)y(x) - q(x)y(-x), \quad k > 1,$$

where p and q are complex-valued functions and belong to $L^2(-1,1)$. The domain $D(S_\theta) = \{y \in H^{2k}(-1,1)\}$, $k > 1$, is defined by the following boundary conditions:
(a) periodic ($\theta = 0$): $y^{(j)}(-1) = y^{(j)}(1)$, $j = 0, 1, \dots, 2k-1$;
(b) antiperiodic ($\theta = 1$): $y^{(j)}(-1) = -y^{(j)}(1)$, $j = 0, 1, \dots, 2k-1$.

The differential operator S_θ contains the transformation of involution. Recall that the function φ such that $\varphi(\varphi(x)) = x$ on a set X is called an involution on X. In this paper we consider only the involution of the reflection $\varphi(x) = -x$.

Differential equations with an involution are a separate class of problems in the theory of functional differential equations. The algebraic and analytic aspects of such equations were studied by [1] and [2]. Spectral problems for differential operators with involution have been investigated in most detail for first order differential operators (see [3] – [5] and the references therein). The spectral properties of second-order differential operators were studied for particular operators

$$(Ly)(x) = -y''(-x), \qquad (\widetilde{L}y)(x) = -y''(x) + \alpha y''(-x), \qquad \alpha \in (-1,1),$$

acting in $L^2(-1,1)$, with various boundary conditions. If $\alpha \neq 0$, then the spectral problems require a special approach, since the term with involution is not subordinate to the term without

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involution. The basis property of the system of eigenfunctions and associated functions (SEAF) for the operator $\widetilde{L}$ was investigated in [6] – [10]. Inverse spectral problem for this operator was solved in [11]. The convergence of spectral decompositions and basis property of the SEAF for the operator L were established in [12] – [14].

Spectral properties of higher-order operators with involution are much less investigated. The classification of general ordinary differential operators with involution depending on the type of boundary conditions can be found in [15]. Theorems about unconditional basis property of the SEAF with parentheses in $L^2(-1,1)$ of general ordinary differential operators with involution and regular boundary conditions were proved in [16].

In the present work we determine the estimates of spectral projections. In order to obtain these estimates we apply the method of similar operators [17], [18], [19] to the operator S_θ. The main idea of this method is to construct a similarity transform of the operator S_θ to an operator of block-diagonal form. Using the estimates of spectral projections, we prove that the SEAF of the operator S_θ forms the Bari basis in $L^2(-1,1)$. Recall that the basis generated by projection systems quadratically close to complete and minimal systems of orthogonal projections is called the Bari basis. The Bari basis is a special class and the study of such bases has separate interest.

We have the following result.

Theorem 1.1. *The system of eigenfunctions and associated functions of the operator S_θ, $\theta \in \{0,1\}$, forms the Bari basis in the space $L^2(-1,1)$.*

Note that any basis quadratically close to an orthogonal basis is the Riesz basis by the Bari-Markus theorem (see [20, Ch. VI, Theorem 5.2]). Thus, the result on the Bari basis property is a stronger than the Riesz basis property.

2 Auxiliary results and research technique

2.1 The unperturbed case.

Let $\theta \in \{0,1\}$. If $p = q = 0$, then S_θ is an unperturbed operator. We denote this operator by A_θ. Direct verification shows that the operator A_θ with periodic (antiperiodic) boundary conditions is self-adjoint operator with discrete spectrum. The eigenvalues $\lambda_n = \lambda_{n,\theta}$, $n \in \mathbb{Z}$, of the operator A_θ are simple and have the form:

$$\lambda_n = -\Big(\frac{\pi}{2}\theta + \pi n\Big)^{2k}, \quad \text{for} \quad n < 0, \qquad\qquad \lambda_n = \Big(\frac{\pi}{2}\theta + \pi n\Big)^{2k}, \quad \text{for} \quad n \geq 0.$$

We index these eigenvalues by the following way

$$\cdots < \lambda_{-n-1} < \lambda_{-n} < \cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} < \ldots .$$

The eigenvalues λ_n, $n < 0$, correspond to the eigenfunctions $e_n(x) = \sin\big(\pi\theta/2 + \pi n\big)x$, $x \in L^2(-1,1)$, $n < 0$, and the eigenvalues λ_n, $n > 0$, correspond to the eigenfunctions $e_n(x) = \cos\big(\pi\theta/2 + \pi n\big)x$, $x \in L^2(-1,1)$, $n > 0$. These eigenfunctions form orthonormal basis in $L^2(-1,1)$. The Riesz projections P_n, $n \in \mathbb{Z}$, are defined by

$$P_n x = (x, e_n)e_n, \qquad n \in \mathbb{Z}, \tag{2.1}$$

for any vector $x \in L^2(-1,1)$.

We represent the operator S_θ in the form $S_\theta = A_\theta - B$, where A_θ is the unperturbed self-adjoint operator and B is the operator of multiplication by p and q. In order to study the operator S_θ we use the method of similar operators (see [17], [18], [19]). We describe the scheme of this method in the next subsection.

2.2 The method of similar operators

Now we present the scheme of the method of similar operators. The theoretical facts of this method can be found in [18, Sec. 2], [19, Sec. 3.1]. Note that our presentation will be adapted to the operator A_θ.

We denote the space $L_2(-1,1)$ by $\mathcal{H}$ with respect to norm $\|\cdot\|$. Moreover, we denote the space of all linear bounded operators acting in $\mathcal{H}$ and the ideal of the Hilbert-Schmidt operators by $\mathcal{B}(\mathcal{H})$ and $\mathfrak{S}_2(\mathcal{H})$ with respect to norm $\|\cdot\|_{\mathcal{B}(\mathcal{H})}$ and $\|\cdot\|_2$, respectively. We introduce the space $\mathfrak{L}_{A_\theta}(\mathcal{H})$ of all A_θ-bounded linear operators with the domain equal $D(A_\theta)$. This space is a Banach space with respect to the norm $\|\cdot\|_A$. Recall that a linear operator $\widetilde{B} : D(\widetilde{B}) \subset \mathcal{H} \to \mathcal{H}$ is A_θ-bounded if $D(\widetilde{B}) \supseteq D(A_\theta)$ and $\|\widetilde{B}\|_A = \inf\{C > 0 : \|\widetilde{B}x\| \le C(\|x\| + \|A_\theta x\|), x \in D(A_\theta)\} < \infty$. Here and below we denote different positive constants by the general symbol $C > 0$.

The spectral properties S_θ are well-known for the operator A_θ, but the operator S_θ is not similar to the operator A_θ. We select a Banach space $\mathfrak{U}$ from $\mathfrak{L}_{A_\theta}(\mathcal{H})$ such that it is possible to transform the operator S_θ to an operator $A_\theta - \widetilde{B}$, where $\widetilde{B} \in \mathfrak{U}$. Furthermore, the operator $A_\theta - \widetilde{B}$ has the block-diagonal structure and its spectral properties are easy to study. Since the operators S_θ and $A_\theta - \widetilde{B}$ are similar, the operator S_θ has the same properties as $A_\theta - \widetilde{B}$. In order to realize this idea, we need some technical tools.

Further in this section, we investigate the operator $A_\theta - \widetilde{B}$, where $\widetilde{B} \in \mathfrak{U}$.

Definition 1 ([19]). Let $\mathfrak{U}$ be a linear subspace of $\mathfrak{L}_{A_\theta}(\mathcal{H})$, and let $J : \mathfrak{U} \to \mathfrak{U}$, $\Gamma : \mathfrak{U} \to \mathcal{B}(\mathcal{H})$ be linear operators. A triple $(\mathfrak{U}, J, \Gamma)$ is called an admissible for the operator A_θ and $\mathfrak{U}$ is the space of admissible perturbations, if the following properties hold:

1) $\mathfrak{U}$ is a Banach space with norm $\|\cdot\|_*$ such that there is a constant $C > 0$ that gives $\|X\|_A \le C\|X\|_*$ for any $X \in \mathfrak{U}$;

2) J and Γ are bounded linear operators, J is a projection;

3) $(\Gamma X)D(A_\theta) \subset D(A_\theta)$ and

$$A_\theta(\Gamma X)x - (\Gamma X)A_\theta x = (X - JX)x, \quad x \in D(A_\theta), \quad X \in \mathfrak{U}, \tag{2.2}$$

where $\Gamma X \in \mathcal{B}(\mathcal{H})$ is a unique solution of the equation $A_\theta Y - Y A_\theta = X - JX$ that satisfies $JY = 0$;

4) $X\Gamma Y$, $(\Gamma X)Y \in \mathfrak{U}$ for all $X, Y \in \mathfrak{U}$ and there is a constant $\gamma > 0$ such that

$$\|\Gamma\| \le \gamma, \qquad \max\{\|X\Gamma Y\|_*, \|(\Gamma X)Y\|_*\} \le \gamma\|X\|_*\|Y\|_*;$$

5) for every $X \in \mathfrak{U}$ and $\varepsilon > 0$ there exists a number $\lambda_\varepsilon \in \rho(A_\theta)$ such that

$$\|X(A_\theta - \lambda_\varepsilon I)^{-1}\| < \varepsilon,$$

where $\rho(A_\theta)$ is the resolvent set of the operator A_θ.

To get an idea about this definition, one should think of the operators involved in terms of their matrices. The operator A_θ is represented by a diagonal matrix. The operator $\widetilde{B}$ has a matrix with some kind of off-diagonal decay. The operator J acts as projection that picks out the main (block) diagonal of an infinite matrix of the operator $\widetilde{B}$. The operator Γ annihilates the main (block) diagonal thereby enhancing the off-diagonal decay. Therefore, using the operators J and Γ, we construct a sequence of transformations that yields to a stronger off-diagonal decrease in the matrix of the operator $\widetilde{B}$.

Now we formulate the main theorem of the method of similar operators for operator $A_\theta - \widetilde{B}$. Introduce the function $\Phi : \mathfrak{U} \to \mathfrak{U}$ by

$$\Phi(X) = B\Gamma X - (\Gamma X)(JB) + B. \tag{2.3}$$

Theorem 2.1. *Let $(\mathfrak{U}, J, \Gamma)$ be an admissible triple for the operator A_θ and $\widetilde{B} \in \mathfrak{U}$. Assume that*

$$\gamma \|J\| \|\widetilde{B}\|_* < \frac{1}{4},$$

where γ comes from Definition 1. Then the operator $A_\theta - \widetilde{B}$ is similar to the operator $A_\theta - JX_*$, where X_* is a unique fixed point of the map $\Phi : \mathfrak{U} \to \mathfrak{U}$ given by (2.3) in the ball

$$\mathcal{B} = \{X \in \mathfrak{U} : \|X - \widetilde{B}\|_* \le 3\|\widetilde{B}\|_*\}.$$

The similarity transform of $A_\theta - \widetilde{B}$ into $A - JX_*$ is given by $I + \Gamma X_*$.

The proof of this theorem can be found in [19, Theorem 3.1].

3 Preliminary similarity transformation

3.1 Construction of admissible triple

In this Section we apply the scheme of the method of similar operators to study the basis properties of the operator $S_\theta = A_\theta - B$. Recall that the operator A_θ is the unperturbed operator and B is the operator of the multiplication on the function p and q. Note that the operator B is A_θ-bounded. Therefore, $B \in \mathfrak{L}_{A_\theta}(\mathcal{H})$.

Now we construct an admissible triple for the operator A_θ. We introduce a space $\mathfrak{U}$ and two operators J and Γ. Note that the admissible triple may be not unique. We choose the ideal $\mathfrak{S}_2(\mathcal{H})$ of the Hilbert-Schmidt operators as the space of admissible perturbations $\mathfrak{U}$. We define the operators J and $\Gamma : \mathfrak{U} \to \mathfrak{U}$ by

$$JX = \sum_{j \in \mathbb{Z}} P_j X P_j, \qquad \Gamma X = \sum_{\substack{s,j \in \mathbb{Z} \\ s \ne j}} \frac{P_s X P_j}{\lambda_s - \lambda_j}, \qquad X \in \mathfrak{U}, \tag{3.1}$$

where P_j, $j \in \mathbb{Z}$ are defined by (2.1). These operators are well-defined, bounded, and the series in (3.1) unconditionally converge in the uniform operator topology (see [19, Sec. 3.2]).

The extensions of the operators J and Γ to the space $\mathfrak{L}_{A_\theta}(\mathcal{H})$ (which will be denoted by the same symbols) are defined by

$$\begin{aligned} JX &= J(X(A_\theta - \psi I)^{-1})(A_\theta - \psi I), \qquad X \in \mathfrak{L}_{A_\theta}(\mathcal{H}), \\ \Gamma X &= \Gamma(X(A_\theta - \psi I)^{-1})(A_\theta - \psi I), \qquad X \in \mathfrak{L}_{A_\theta}(\mathcal{H}), \end{aligned} \tag{3.2}$$

where $\psi \in \rho(A_\theta)$. These definitions are well-defined and independent on the choice of ψ.

We need to consider resolutions of the identity given by

$$I = \sum_{|j|\geq m+1} P_j + P_{(m)}, \qquad P_{(m)} = \sum_{|j|\leq m} P_j, \qquad m \in \mathbb{Z}_+.$$

We define two sequences of operators J_m and Γ_m, $m \in \mathbb{Z}_+$, as follows

$$\begin{aligned} J_m X &= JX - P_{(m)}(JX)P_{(m)} + P_{(m)}XP_{(m)} = P_{(m)}XP_{(m)} + \sum_{|j|\geq m+1} P_j X P_j, \\ \Gamma_m X &= \Gamma X - P_{(m)}(\Gamma X)P_{(m)} = \sum_{\substack{\max\{|s|,|j|\}\geq m+1 \\ s\neq j}} \frac{P_s X P_j}{\lambda_s - \lambda_j}. \end{aligned} \tag{3.3}$$

Obviously, the operators J_0 and Γ_0 coincide with the operators J and Γ, respectively, and $J_m X$ and $\Gamma_m X$, $m \in \mathbb{Z}_+$, are finite-rank perturbations of JX and ΓX, $X \in \mathfrak{U}$. Furthermore,

$$\lim_{m\to\infty} \Gamma_m X = 0$$

in the topology $\mathfrak{U}$. Therefore, $\|\Gamma_m X\|_* < 1$ for $m \in \mathbb{Z}_+$ large enough. Moreover, the formulas (3.2) imply that the operators $J_m X$ and $\Gamma_m X$ are well-defined for the operator $X \in \mathfrak{L}_{A_\theta}(\mathcal{H})$. Therefore, $J_m B$ and $\Gamma_m B$ also are well-defined for the perturbation B.

Thus, we construct the triple $(\mathfrak{U}, J_m, \Gamma_m)$. It remains to prove that this triple is admissible.

Lemma 3.1. *Assume that $\mathfrak{U} = \mathfrak{S}_2(\mathcal{H})$ and the operators J_m and Γ_m are defined by (3.3). Then $(\mathfrak{U}, J_m, \Gamma_m)$ is admissible triple for the operator A_θ.*

Proof. We prove that all properties of Definition 1 hold. Property 1) immediately follows from the definition of the space $\mathfrak{U}$. Property 2) holds by the definitions of J_m, Γ_m, and [19, Lemma 3.4].

We prove Property 3). Let $x \in D(A_\theta)$ and $\psi \in \rho(A_\theta)$. There exists $y \in \mathcal{H}$ such that

$$x = (A_\theta - \psi I)^{-1} y = \sum_{j\in\mathbb{Z}} \frac{1}{\lambda_j - \psi} P_j y,$$

where P_j, $j \in \mathbb{Z}$, are defined by (2.1). Then

$$\begin{aligned} (\Gamma X)x &= (\Gamma X)(A_\theta - \psi I)^{-1} y = \sum_{\substack{s,j\in\mathbb{Z} \\ s\neq j}} \frac{P_s X P_j y}{(\lambda_s - \lambda_j)(\lambda_j - \psi)} \\ &= \sum_{\substack{s,j\in\mathbb{Z} \\ s\neq j}} \frac{P_s X P_j y}{(\lambda_s - \lambda_j)(\lambda_s - \psi)} + \sum_{\substack{s,j\in\mathbb{Z} \\ s\neq j}} \frac{P_s X P_j y}{(\lambda_s - \psi)(\lambda_j - \psi)} \\ &= (A_\theta - \psi I)^{-1}(\Gamma X)y + (A_\theta - \psi I)^{-1}(\Gamma X)(A_\theta - \psi I)^{-1} y \\ &= (A_\theta - \psi I)^{-1}(\Gamma X)(x + y) \in D(A_\theta). \end{aligned}$$

Therefore, $(\Gamma X)D(A_\theta) \subset D(A_\theta)$. Moreover, for $X \in \mathfrak{S}_2(\mathcal{H})$ and $x \in D(A_\theta)$ we get

$$A_\theta(\Gamma X)x - (\Gamma X)A_\theta x = \sum_{\substack{s,j\in\mathbb{Z} \\ s\neq j}} \frac{\lambda_s P_s X P_j x}{\lambda_s - \lambda_j} - \sum_{\substack{s,j\in\mathbb{Z} \\ s\neq j}} \frac{\lambda_j P_s X P_j x}{\lambda_s - \lambda_j} = \sum_{\substack{s,j\in\mathbb{Z} \\ s\neq j}} P_s X P_j x = (X - JX)x.$$

Thus, the identity (2.2) holds. It follows from (3.1) that $J(\Gamma X) = 0$ for all $X \in \mathfrak{S}_2(\mathcal{H})$. Obviously, this proof holds for the operators J_m and Γ_m. Property 3) is proved.

We establish Property 4). Let X, $Y \in \mathfrak{S}_2(\mathcal{H})$. We prove that $X\Gamma Y$ belongs to $\mathfrak{S}_2(\mathcal{H})$. Recall that the operator Γ is bounded. Then

$$\|X\Gamma Y\|_2^2 = \sum_{\substack{s,j\in\mathbb{Z}\\ s\neq j}} \frac{\|XP_sYP_j\|_2^2}{|\lambda_s-\lambda_j|^2} \leq \gamma^2\|X\|_2^2 \sum_{\substack{s,j\in\mathbb{Z}\\ s\neq j}} \|P_sYP_j\|_2^2 \leq \gamma^2\|X\|_2^2\|Y\|_2^2,$$

where γ has the form

$$\gamma = \Big(\max_{s\in\mathbb{Z}} \sum_{j\in\mathbb{Z}\setminus\{s\}} |\lambda_s-\lambda_j|^{-2}\Big)^{1/2}.$$

These estimates give $\|X\Gamma Y\|_2 \leq \gamma\|X\|_2\|Y\|_2$. The same inequality for the operator $(\Gamma X)Y$ is established by a similar way. Obviously, this proof holds for the operators J_m and Γ_m. Property 4) is proved.

Property 5) is easily established by direct computation. We put $\theta = 0$ for simplicity. Assume that $X \in \mathfrak{S}_2(\mathcal{H})$, $\varepsilon > 0$, and $\lambda_\varepsilon = \pi^{2k}n$. Then

$$\|X(A_\theta-\lambda_\varepsilon I)^{-1}\| \leq \|X\|_2 \max_{s\in\mathbb{Z}} \frac{1}{|\lambda_s-\pi^{2k}n|} \leq C\|X\|_2\Big(\max_{s\geq 0}\frac{1}{|(\pi s)^{2k}-\pi^{2k}n|}\Big) \leq \frac{C}{n}.$$

Therefore, by these estimates and an appropriate choice of n, we get the inequality $\|X(A_\theta - \lambda_\varepsilon I)^{-1}\| < \varepsilon$. In the case $\theta = 1$ this estimate is established in a similar way. Property 5) is proved. Therefore, $(\mathfrak{U}, J_m, \Gamma_m)$ is admissible triple for the operator A_θ. □

3.2 Preliminary similarity transformation for the operator S_θ

We constructed the admissible triple for the operator A_θ in the previous subsection. The operator A_θ satisfies the assumptions of Theorem 2.1, but the perturbation B does not belong to the space $\mathfrak{U}$. Therefore, we can't directly use Theorem 2.1. We need to transform the operator $S_\theta = A_\theta - B$ to the operator $A_\theta - \widetilde{B}$, where $\widetilde{B} \in \mathfrak{U}$. In the method of similar operators this procedure is preliminary similarity transformation. Then we can apply the results of Theorem 2.1.

Now we study the properties of the operators B, J_mB, and Γ_mB. Introduce the Fourier coefficients of a function $f \in L^2(-1,1)$ by

$$\widehat{f}_0 = \int_{-1}^{1} f(x)\,dx, \quad \widehat{f}_{cn,n} = \int_{-1}^{1} f(x)\cos\pi nx\,dx, \quad \widehat{f}_{sn,n} = \int_{-1}^{1} f(x)\sin\pi nx\,dx, \quad n\in\mathbb{Z}.$$

Without loss of generality we assume that $\widehat{p}_0 = \widehat{q}_0 = 0$ for the functions p and q. Since the eigenvalues of the operator A_θ are simple, the corresponding matrix has size 1×1. Therefore, the matrix $(B_{sj})_{s,j\in\mathbb{Z}}$ of the perturbation B has the same size and $B_{sj} = (Be_j, e_s)$. We consider all possible situations.

1) Let s, $j \geq 0$. Then

$$\begin{aligned} B^1_{sj} = (Be_j,e_s) &= \int_{-1}^{1}\Big(p(x)\cos\big(\frac{\pi}{2}\theta+\pi j\big)x + q(x)\cos\big(\frac{\pi}{2}\theta+\pi j\big)x\Big)\cos\big(\frac{\pi}{2}\theta+\pi s\big)x\,dx \\ &= \frac{1}{2}\int_{-1}^{1}\Big(p(x)+q(x)\Big)\Big(\cos\pi(j-s)x+\cos\pi(j+s+\theta)x\Big)\,dx \qquad (3.4) \\ &= \frac{1}{2}\big(\widehat{p}_{cn,j-s}+\widehat{q}_{cn,j-s}+\widehat{p}_{cn,j+s+\theta}+\widehat{q}_{cn,j+s+\theta}\big). \end{aligned}$$

2) Let $s,\ j<0$. Then

$$B^2_{sj}=(Be_j,e_s)=\int_{-1}^{1}\Big(p(x)\sin\big(\frac{\pi}{2}\theta+\pi j\big)x-q(x)\sin\big(\frac{\pi}{2}\theta+\pi j\big)x\Big)\sin\big(\frac{\pi}{2}\theta+\pi s\big)x\,dx$$
$$=\frac{1}{2}\int_{-1}^{1}\Big(p(x)-q(x)\Big)\Big(\cos\pi(j-s)x-\cos\pi(j+s+\theta)x\Big)\,dx \tag{3.5}$$
$$=\frac{1}{2}\big(\widehat{p}_{cn,j-s}-\widehat{q}_{cn,j-s}-\widehat{p}_{cn,j+s+\theta}+\widehat{q}_{cn,j+s+\theta}\big).$$

3) Let $s\geq 0,\ j<0$. Then

$$B^3_{sj}=(Be_j,e_s)=\int_{-1}^{1}\Big(p(x)\sin\big(\frac{\pi}{2}\theta+\pi j\big)x-q(x)\sin\big(\frac{\pi}{2}\theta+\pi j\big)x\Big)\cos\big(\frac{\pi}{2}\theta+\pi s\big)x\,dx$$
$$=\frac{1}{2}\int_{-1}^{1}\Big(p(x)-q(x)\Big)\Big(\sin\pi(j-s)x+\sin\pi(j+s+\theta)x\Big)\,dx \tag{3.6}$$
$$=\frac{1}{2}\big(\widehat{p}_{sn,j-s}-\widehat{q}_{sn,j-s}+\widehat{p}_{sn,j+s+\theta}-\widehat{q}_{sn,j+s+\theta}\big).$$

4) Let $s<0,\ j\geq 0$. Then

$$B^4_{sj}=(Be_j,e_s)=\int_{-1}^{1}\Big(p(x)\cos\big(\frac{\pi}{2}\theta+\pi j\big)x+q(x)\cos\big(\frac{\pi}{2}\theta+\pi j\big)x\Big)\sin\big(\frac{\pi}{2}\theta+\pi s\big)x\,dx$$
$$=\frac{1}{2}\int_{-1}^{1}\Big(p(x)+q(x)\Big)\Big(\sin\pi(s-j)x+\sin\pi(s+j+\theta)x\Big)\,dx \tag{3.7}$$
$$=\frac{1}{2}\big(\widehat{p}_{sn,s-j}+\widehat{q}_{sn,s-j}+\widehat{p}_{sn,s+j+\theta}+\widehat{q}_{sn,s+j+\theta}\big).$$

Since the matrix of the operator B has different values depending on the sign s and j, we use the general symbol B_{sj} to denote the matrix of the operator B. Moreover, the identities (3.1) imply that the elements of matrices of operators JB and ΓB have the form

$$(JB)_{sj}=\delta_{s-j}B_{sj},\qquad (\Gamma B)_{sj}=\frac{B_{sj}}{\lambda_s-\lambda_j},\qquad s,j\in\mathbb{Z}. \tag{3.8}$$

Now we proceed to preliminary similarity transformation of the operator S_θ. The next auxiliary lemma is basis of this procedure.

Lemma 3.2. *Let $m\in\mathbb{Z}_+$ be large enough. The operators B, J_mB, and Γ_mB satisfy the following properties:*
(a) $\Gamma_mB\in\mathfrak{S}_2(\mathcal{H})$ and $\|\Gamma_mB\|_2<1$. Moreover,

$$\|P_n(\Gamma_mB)\|_2\leq\frac{C}{n^{2k-1}},\qquad \|(\Gamma_mB)P_n\|_2\leq\frac{C}{n^{2k-1}}, \tag{3.9}$$

where $C>0$ is some constant;
(b) $(\Gamma_mB)D(A_\theta)\subset D(A_\theta)$;
(c) $B\Gamma_mB$, $(\Gamma_mB)J_mB\in\mathfrak{U}$;
(d) $A_\theta(\Gamma_mB)x-(\Gamma_mB)A_\theta x=(B-J_mB)x$, $x\in D(A_\theta)$;
(e) for any $\varepsilon>0$ there is $\lambda_\varepsilon\in\rho(A_\theta)$ such that $\|B(A_\theta-\lambda_\varepsilon I)^{-1}\|<\varepsilon$.

Proof. We prove Property (a). Recall that (see [20, Sec. 9]) a operator X is the Hilbert-Schmidt operator if

$$\sum_{s,j\in\mathbb{Z}} |(Xe_j, e_s)|^2 < \infty$$

for some orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$. Now we establish this condition for the operator ΓB. Using the second identity from (3.8), we have

$$\begin{aligned}
&\sum_{\substack{s,j\in\mathbb{Z}\\ s\neq j}} |(\Gamma Be_j, e_s)|^2 \\
&= \frac{2^{4k}}{\pi^{4k}}\Bigg(\sum_{\substack{s,j\geq 0\\ s\neq j}} \frac{|B_{sj}^1|^2}{|(2s+\theta)^{2k}-(2j+\theta)^{2k}|^2} + \sum_{\substack{s,j<0\\ s\neq j}} \frac{|B_{sj}^2|^2}{|(2s+\theta)^{2k}-(2j+\theta)^{2k}|^2} \\
&\quad + \sum_{\substack{s\geq 0, j<0\\ s\neq j}} \frac{|B_{sj}^3|^2}{|(2s+\theta)^{2k}+(2j+\theta)^{2k}|^2} + \sum_{\substack{s<0, j\geq 0\\ s\neq j}} \frac{|B_{sj}^4|^2}{|(2s+\theta)^{2k}+(2j+\theta)^{2k}|^2}\Bigg),
\end{aligned} \tag{3.10}$$

where B_{sj}^i, $i = 1, 2, 3, 4$, are defined by the definitions (3.4) – (3.7). Now we estimate the first sum in (3.10). We get

$$\begin{aligned}
&\sum_{\substack{s,j\geq 0\\ s\neq j}} \frac{|B_{sj}^1|^2}{|(2s+\theta)^{2k}-(2j+\theta)^{2k}|^2} = \sum_{\substack{s,j\geq 0\\ s\neq j}} \frac{|B_{sj}^1|^2}{\big((2s+\theta)^k-(2j+\theta)^k\big)^2\big((2s+\theta)^k+(2j+\theta)^k\big)^2} \\
&\leq \frac{1}{4}\sum_{s>0}\frac{1}{(2s+\theta)^{2k}}\sum_{\substack{j\geq 0\\ j\neq s}} \frac{|B_{sj}^1|^2}{(s-j)^2((2s+\theta)^{k-1}+(2s+\theta)^{k-2}(2j+\theta)+\cdots+(2j+\theta)^{k-1})^2} \\
&\leq \frac{\|p\|^2+\|q\|^2}{4}\sum_{s>0}\frac{1}{(2s+\theta)^{4k-2}} < \infty.
\end{aligned}$$

We estimate the other terms in the right-hand side of (3.10) by a similar way. Therefore, ΓB is the Hilbert-Schmidt operator. The formula (3.3) shows that the operator $\Gamma_m B$ differs from ΓB by a finite-rank operator. Therefore, $\Gamma_m B \in \mathfrak{S}_2(\mathcal{H})$ and $\|\Gamma_m B\|_2 < 1$ for $m \in \mathbb{Z}_+$ large enough (see also (3.3)).

Now we prove the inequalities (3.9). Fix a number $n \geq 0$. The case $n < 0$ is considered by a similar way. Then

$$\begin{aligned}
\|P_n(\Gamma_m B)\|_2^2 &\leq \|P_n(\Gamma B)\|_2^2 = \sum_{\substack{s,j\in\mathbb{Z}\\ s\neq j}} |(P_n(\Gamma B)e_j, e_s)|^2 \\
&= \frac{2^{4k}}{\pi^{4k}}\Bigg(\sum_{\substack{j\geq 0\\ j\neq n}} \frac{|B_{nj}^1|^2}{|(2n+\theta)^{2k}-(2j+\theta)^{2k}|^2} + \sum_{\substack{j<0\\ j\neq n}} \frac{|B_{nj}^3|^2}{|(2n+\theta)^{2k}+(2j+\theta)^{2k}|^2}\Bigg),
\end{aligned} \tag{3.11}$$

where B_{sj}^i, $i = 1, 3$, are defined by the definitions (3.4) and (3.6). Now we estimate the first sum in (3.11). We have

$$\sum_{\substack{j\geq 0\\ j\neq n}} \frac{|B_{nj}^1|^2}{|(2n+\theta)^{2k}-(2j+\theta)^{2k}|^2} = \sum_{\substack{j\geq 0\\ j\neq n}} \frac{|B_{nj}^1|^2}{\big((2n+\theta)^k-(2j+\theta)^k\big)^2\big((2n+\theta)^k+(2j+\theta)^k\big)^2}$$

$$\leq \frac{1}{4(2n+\theta)^{2k}} \sum_{\substack{j\geq 0\\ j\neq n}} \frac{|B^1_{nj}|^2}{(n-j)^2((2n+\theta)^{k-1}+(2n+\theta)^{k-2}(2j+\theta)+\cdots+(2j+\theta)^{k-1})^2}$$
$$\leq \frac{\|p\|^2+\|q\|^2}{4(2n+\theta)^{4k-2}}.$$

Similar arguments yield the same estimate for the second sum in (3.11). Therefore,

$$\sum_{\substack{j\geq 0\\ j\neq n}} \frac{|B^1_{nj}|^2}{|(2n+\theta)^{2k}-(2j+\theta)^{2k}|^2} \leq \frac{\|p\|^2+\|q\|^2}{4(2n+\theta)^{4k-2}},$$
$$\sum_{\substack{j<0\\ j\neq n}} \frac{|B^3_{nj}|^2}{|(2n+\theta)^{2k}+(2j+\theta)^{2k}|^2} \leq \frac{\|p\|^2+\|q\|^2}{4(2n+\theta)^{4k-2}}.$$

Substituting these inequalities into (3.11), we obtain the first estimate in (3.9). Using similar arguments for the operator $(\Gamma_m B)P_n$, we get the second estimate in (3.9). Property (a) is proved.

The proof of Property (b) completely repeats the proof of Lemma 3.1 (Condition 3).

We prove Property (c). Show that the operator $J_m B$ is the Hilbert-Schmidt operator. Using the first definition from (3.8), and the identities (3.4), (3.5), and $\widehat{p}_0 = \widehat{q}_0 = 0$, we have

$$\sum_{j\in\mathbb{Z}} |B_{jj}|^2 = \sum_{j\geq 0} |B^1_{jj}|^2 + \sum_{j<0} |B^2_{jj}|^2$$
$$\leq \frac{1}{4}\sum_{j\geq 0} \left|\widehat{p}_{cn,2j+\theta} + \widehat{q}_{cn,2j+\theta}\right|^2 + \frac{1}{4}\sum_{j<0} \left|-\widehat{p}_{cn,2j+\theta} + \widehat{q}_{cn,2j+\theta}\right|^2 \leq \|p\|^2 + \|q\|^2.$$

Therefore, $JB \in \mathfrak{S}_2(\mathcal{H})$. It follows from (3.3) that the operator $J_m B$ differs from JB by a finite-rank operator. Thus, $J_m B \in \mathfrak{S}_2(\mathcal{H})$.

Now we establish that the operator $B\Gamma B$ is the Hilbert-Schmidt operator. Using the second definition from (3.8), we get

$$(B\Gamma B)_{sj} = \sum_{\substack{l\in\mathbb{Z}\\ l\neq j}} \frac{B_{sl}B_{lj}}{\lambda_l-\lambda_j}, \qquad s,j\in\mathbb{Z}.$$

Recall that B_{sj} is the matrix of operator B. We prove that $\sum_{s,j\in\mathbb{Z}} |(B\Gamma B)_{sj}|^2 < \infty$. It follows from [17, Lemma 7] that

$$\sum_{s,j\geq 0} \left| \sum_{\substack{l\geq 0\\ l\neq s}} \frac{B^1_{sl}B^1_{lj}}{l^2-j^2} \right|^2 < \infty.$$

Obviously, all terms in the representation of the matrix $(B\Gamma B)_{sj}$ in our case are reduced to this form. Therefore, $B\Gamma B \in \mathfrak{S}_2(\mathcal{H})$. It follows from (3.3) that the operator $\Gamma_m B$ differs from ΓB by a finite-rank operator. Thus, $B\Gamma_m B \in \mathfrak{S}_2(\mathcal{H})$. Property (c) is proved.

Property (d) is proved by direct computations. Consider the matrix of the operator $(\Gamma B)(A_\theta - \psi I)^{-1} - (A_\theta - \psi I)^{-1}(\Gamma B)$ for $\psi \in \rho(A_\theta)$. We have

$$\frac{B_{sj}}{(\lambda_s-\lambda_j)(\lambda_j-\psi)} - \frac{B_{sj}}{(\lambda_s-\psi)(\lambda_s-\lambda_j)} = \frac{B_{sj}}{(\lambda_j-\psi)(\lambda_s-\psi)}.$$

Note that the last representation is the matrix of the operator $(A_\theta-\psi I)^{-1}(B-JB)(A_\theta-\psi I)^{-1}$. Therefore, the matrices of the operators $(\Gamma B)(A_\theta-\psi I)^{-1}-(A_\theta-\psi I)^{-1}(\Gamma B)$ and $(A_\theta-\psi I)^{-1}(B-JB)(A_\theta-\psi I)^{-1}$ coincide. Thus, $A_\theta(\Gamma B)x-(\Gamma B)A_\theta x=(B-JB)x$, $x\in D(A_\theta)$. Using this identity and the formula (3.3), we have

$$\begin{aligned}A_\theta(\Gamma_m B)x &= A_\theta(\Gamma B)x - A_\theta P_{(m)}(\Gamma B)P_{(m)}x = A_\theta(\Gamma B)x - P_{(m)}(A_\theta(\Gamma B))P_{(m)}x\\ &= (B-JB)x + (\Gamma B)A_\theta x - P_{(m)}(B-JB)P_{(m)}x - P_{(m)}((\Gamma B)A_\theta)P_{(m)}x\\ &= (B-J_mB)x + (\Gamma_m B)A_\theta x.\end{aligned}$$

Property (d) is proved.

Using similar arguments as [17, Lemma 8], we obtain Property (e). □

We establish that the operator S_θ is similar to the operator $A_\theta-\widetilde{B}$, where $\widetilde{B}$ is the Hilbert-Schmidt operator, i. e. $\widetilde{B}\in\mathfrak{U}$. Directly from Lemma 3.2 and [19, Theorem 3.3], we have the following result.

Theorem 3.1. *The operator $S_\theta=A_\theta-B$ is similar to $A_\theta-\widetilde{B}$, where $\widetilde{B}\in\mathfrak{U}$ and has the form*

$$\widetilde{B}=J_mB+(I+\Gamma_mB)^{-1}(B\Gamma_mB-(\Gamma_mB)J_mB). \tag{3.12}$$

The similarity transform is given by $I+\Gamma_mB$ such that

$$(A_\theta-B)(I+\Gamma_mB)=(I+\Gamma_mB)(A_\theta-\widetilde{B}).$$

4 The proof of the main result

In this section we prove Theorem 1.1. In order to obtain this result we use the scheme of Section 2.

Proof of Theorem 1.1. By Theorem 3.1 the operator S_θ is similar to the operator $A_\theta-\widetilde{B}$, where $\widetilde{B}\in\mathfrak{U}$ and has the form (3.12). Moreover, the similarity transform is given by $I+\Gamma_mB$.

Now we consider the operator $A_\theta-\widetilde{B}$. Applying Theorem 2.1 to this operator, we see that the operator $A_\theta-\widetilde{B}$ is similar to the operator $A-J_mX_*$, where X_* is a unique fixed point of the map $\Phi:\mathfrak{U}\to\mathfrak{U}$ given by (2.3). The similarity transformation is the operator $I+\Gamma_mX_*$. Then

$$\begin{aligned}S_\theta &= (I+\Gamma_mB)(A_\theta-\widetilde{B})(I+\Gamma_mB)^{-1}\\ &= (I+\Gamma_mB)(I+\Gamma_mX_*)(A_\theta-J_mX_*)(I+\Gamma_mX_*)^{-1}(I+\Gamma_mB)^{-1}\\ &= (I+V_m)(A_\theta-J_mX_*)(I+V_m)^{-1},\end{aligned} \tag{4.1}$$

where V_m is defined by

$$V_m=\Gamma_mB+\Gamma_mX_*+(\Gamma_mB)(\Gamma_mX_*). \tag{4.2}$$

Recall that the Riesz projections P_n, $n\in\mathbb{Z}$, are defined by (2.1). These projections are constructed for the set $\{\lambda_n\}$, where λ_n, $n\in\mathbb{Z}$, are eigenvalues of the operator A_θ. Consider the operator S_θ. It follows from [21] that the eigenvalues $\widetilde{\lambda}_n$, $n\in\mathbb{Z}$, of the operator S_θ are simple and labeled by

$$\dots\le \operatorname{Re}\widetilde{\lambda}_{-n-1}\le \operatorname{Re}\widetilde{\lambda}_{-n}\le\dots\le \operatorname{Re}\widetilde{\lambda}_{-1}\le \operatorname{Re}\widetilde{\lambda}_0\le \operatorname{Re}\widetilde{\lambda}_1\le\dots\le \operatorname{Re}\widetilde{\lambda}_n\le \operatorname{Re}\widetilde{\lambda}_{n+1}\le\dots.$$

Introduce the Riesz projections $\widetilde{P}_n$, $n \in \mathbb{Z}$, constructed for the sets $\{\widetilde{\lambda}_n\}$. The similarity of the operators S_θ and $A - J_m X_*$ and [18, Lemma 1] imply that the projections $\widetilde{P}_n$ and P_n are similar. Moreover, the relations (4.1) and [18, Lemma 1] yield the following identity

$$\widetilde{P}_n = (I + V_m)P_n(I + V_m)^{-1},$$

where V_m is defined by (4.2). Therefore,

$$\widetilde{P}_n - P_n = (I + V_m)P_n(I + V_m)^{-1} - P_n = (V_m P_n - P_n V_m)(I + V_m)^{-1}.$$

Now we estimate this representation. Using (4.2), we get

$$\begin{aligned} \|\widetilde{P}_n - P_n\|_2^2 &= \|(V_m P_n - P_n V_m)(I + V_m)^{-1}\|_2^2 \le C\|V_m P_n - P_n V_m\|_2^2 \\ &\le C\big(\|V_m P_n\|_2^2 + \|P_n V_m\|_2^2\big) \\ &\le C\Big(\|\big(\Gamma_m B + \Gamma_m X_* + (\Gamma_m B)(\Gamma_m X_*)\big)P_n\|_2^2 \\ &\qquad + \|P_n\big(\Gamma_m B + \Gamma_m X_* + (\Gamma_m B)(\Gamma_m X_*)\big)\|_2^2\Big) \\ &\le C\big(\|(\Gamma_m B)P_n\|_2^2 + \|(\Gamma_m X_*)P_n\|_2^2 + \|P_n(\Gamma_m B)\|_2^2 + \|P_n(\Gamma_m X_*)\|_2^2\big). \end{aligned} \tag{4.3}$$

Here we used that the operators $\Gamma_m X_*$ and $\Gamma_m B$ are bounded. Thus, we need to estimate $\|(\Gamma_m X_*)P_n\|_2$. We have

$$\begin{aligned} &\|(\Gamma_m X_*)P_n\|_2^2 \le \|(\Gamma X_*)P_n\|_2^2 \\ &= \left\| \sum_{j\in\mathbb{Z}} \frac{P_j X_* P_n}{\lambda_j - \lambda_n} \right\|_2^2 \le \sup_{j\in\mathbb{Z}} \frac{1}{|\lambda_j - \lambda_n|^2} \sum_{j\in\mathbb{Z}} \|P_j X_* P_n\|_2^2 \\ &\le C\|X_*\|_2^2 \sup_{j\ge 0} \frac{1}{|(2j+\theta)^{2k} - (2n+\theta)^{2k}|^2} \\ &\le C\|X_*\|_2^2 \sup_{j\ge 0} \frac{1}{\big((2j+\theta)^k - (2n+\theta)^k\big)^2\big((2j+\theta)^k + (2n+\theta)^k\big)^2} \\ &\le \frac{C\|X_*\|_2^2}{4(2n+\theta)^{2k}} \sup_{j\ge 0} \frac{1}{(j-n)^2((2j+\theta)^{k-1} + (2j+\theta)^{k-2}(2n+\theta) + \cdots + (2n+\theta)^{k-1})^2} \\ &\le \frac{C}{n^{4k-2}}. \end{aligned}$$

Using similar arguments, we obtain the same estimate for $\|P_n(\Gamma_m X_*)\|_2^2$. Therefore,

$$\|(\Gamma_m X_*)P_n\|_2^2 \le \frac{C}{n^{4k-2}}, \qquad \|P_n(\Gamma_m X_*)\|_2^2 \le \frac{C}{n^{4k-2}}. \tag{4.4}$$

Substituting (3.9) and (4.4) into (4.3), we obtain

$$\|\widetilde{P}_n - P_n\|_2^2 \le \frac{C}{n^{4k-2}}.$$

Therefore,

$$\sum_{|n|\ge m+1} \|\widetilde{P}_n - P_n\|_2^2 < \infty.$$

This estimate and the Bari-Markus theorem (see [20, Ch. VI, Theorem 5.2]) imply that the system of eigenfunctions and associated functions forms the Bari basis in $L^2(-1,1)$. □

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