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A NOTE ON SOLVABILITY OF THREE-POINT BOUNDARY VALUE PROBLEMS FOR THIRD-ORDER DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN

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Abstract. Third-point boundary value problems for third-order differential equation

$$\begin{cases} [q(t)\phi(x''(t))]' + kx'(t) + g(t, x(t), x'(t)) = p(t), \quad t \in (0,1), \\ x'(0) = x'(1) = x(\eta) = 0 \end{cases}$$

is considered. Sufficient conditions for the existence of at least one solution of above problem are established. Some known results are improved.

1. Introduction

In [1], the solvability of problem

$$\begin{cases} x'''(t) + k^2 x'(t) + g(x, x') = p(t), \quad t \in (0,\pi), \\ x'(0) = x'(\pi) = x(\eta) = 0 \end{cases} \tag{1}$$

was studied, where $0 < \eta < \pi$ and

(B_1). $g(u,v)$ is continuous in R^2, has continuous partial derivatives g_u and g_v, bounded in R^2;

(B_2). $|1-g_v|+\sqrt{\pi|g_u|} \le 2k$ for $k \in N\backslash\{1\}$, $|1-g_v|+\sqrt{\pi}|g_u| \le 2$ for $k = 1$, and $\lim_{|v|\to\infty} vg(u,v) = \mu \ne 0$ uniformly for $u \in R$.

Problem (1) when $k = 1$ was also studied by Nagle and Pothoven [2] and by Rovderova [4] under the condition

(B_3). g is bounded on one side and continuous.

In [3], Gupta studied the existence of solutions to boundary value problems similar to (1) of the type

$$\begin{cases} x'''(t) + \pi^2 x'(t) + g(t, x, x', x'') = p(t), \quad t \in (0,1), \\ x'(0) = x'(1) = x(\eta) = 0, \end{cases} \tag{2}$$

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where g is a Caratheodory function and $p \in L^1[0,1]$, under the conditions

(C). $\int_0^1 p(t)\sin \pi t dt = 0$, and g satisfies

$$g(t,u,v,w)v \geq 0, \text{ for } t \in [0,1], \ u,v,w \in R,$$

and

$$\lim_{|v|\to\infty} \frac{g(t,u,v,w)}{v} < 3\pi^2 \text{ uniformly for } (t,u,w) \in [0,1] \times R^2.$$

The purpose of this paper is to establish new sufficient conditions for the existence of at least one solution of problem

$$\begin{cases} [q(t)\phi(x''(t))]' + kx'(t) + g(t,x(t),x'(t)) = p(t), \ t \in (0,1), \\ x'(0) = x'(1) = x(\eta) = 0, \end{cases} \tag{3}$$

where $k \in R$, $q, p \in L^1[0,1]$ and g is a Caratheodory function, $\phi(x) = |x|^{p-2}x$ for $x \neq 0$ and $\phi(0) = 0$, which is called p-Laplacian.

In Section 2, we establish existence results for problem (3). The examples, which can not be solved by result in [1-3], to illustrate the main theorems will be given in Section 4.

2. Main Result

Let $X = C^1[0,1] \times C^0[0,1]$, $Y = L^1[0,1] \times L^1[0,1]$, their norms are defined by $||(x,y)|| = \max\{||x||_\infty, ||x'||_\infty, ||y||_\infty\}$ for $(x,y) \in X$ and $||(u,v)|| = \max\{\int_0^1 |u(s)|ds, \int_0^1 |v(s)|ds\}$ for $(u,v) \in Y$, respectively. Then X and Y are Banach spaces.

Let $D(L) = \{(x,y) \in X : x'' \in L^1(0,1), (qy)' \in L^1(0,1) \text{ with } x'(0) = x'(1) = x(\eta) = 0\}$. Define the linear operator $L : D(L) \cap X \to Y$ by

$$L\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x''(t) \\ (q(t)y(t))' \end{pmatrix} \text{ for all } (x,y) \in D(L) \cap X.$$

Define the nonlinear operator $N : X \to Y$ by

$$N\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \phi^{-1}(y(t)) \\ -kx'(t) - g(t,x(t),x'(t)) + p(t) \end{pmatrix} \text{ for all } (x,y) \in X.$$

It is easy to show that $\text{Ker}L = \{(0, c/q(t)) : c \in R\}$; $\text{Im}L = \{(u,v) \in Y : \int_0^1 u(s)ds = 0\}$; L is a Fredholm operator of index zero; There are projectors $P : X \to X$ and $Q : Y \to Y$ such that $\text{Ker}L = \text{Im}P$ and $\text{Ker}Q = \text{Im}L$. In fact, we have

$$P\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ y(0)/q(t) \end{pmatrix} \text{ for } (x,y) \in X,$$

$$Q\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \int_0^1 u(t)dt \\ 0 \end{pmatrix} \text{ for } (u,v) \in Y.$$

Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is L-compact on $\overline{\Omega}$; $(x, y) \in D(L)$ is a solution of the operator equation $L(x, y) = N(x, y)$ implies that x is a solution of problem (3).

Suppose

(A_1)**.** There are bounded functions $a \in L^1(0,1)$, bounded function $b \in L^1(0,1)$ and function $r \in L^1(0,1)$ such that $g(t,x,y) = h(t,x,y) + f(t,x,y)$, and there are positive constants β, θ such that $h(t,x,y)y \le -\beta|y|^{\theta+1}$, and $|f(t,x,y)| \le a(t)|x|^\theta + b(t)|y|^\theta + r(t)$, for all $(x,y) \in R^2$ and a.e. $t \in [0,1]$.

(A_2)**.** If $\theta > 1$, then

$$||b||_\infty + \left(\int_0^1 |a(t)|^{\frac{\theta+1}{\theta}} dt\right)^{\frac{\theta}{\theta+1}} < \beta;$$

if $\theta = 1$, then

$$k + ||b||_\infty + \left(\int_0^1 |a(t)|^2 dt\right)^{\frac{1}{2}} < \beta.$$

(A_3)**.** There is a constant $\delta > 0$ such that $q(t) \ge \delta$ for all $t \in [0,1]$.

Now, consider $L(x, y) = \lambda N(x, y)$ with $\lambda \in (0,1)$. We have

$$\begin{cases} x''(t) = \lambda \phi^{-1}(y(t)), \\ (q(t)y(t))' = \lambda(-kx'(t) - g(t,x,x') + p(t)). \end{cases}$$

It follows that

$$[q(t)\phi(x''(t))]' = \phi(\lambda)\lambda(-kx'(t) - g(t,x,x') + p(t)). \tag{4}$$

Thus

$$[q(t)\phi(x''(t))]x'(t) = \phi(\lambda)\lambda(-kx'(t) - g(t,x,x') + p(t))x'(t).$$

Since $x'(0) = x'(1) = 0$ implies that

$$\int_0^1 [q(t)\phi(x''(t))]'x'(t)dt = -\int_0^1 q(t)\phi(x''(t))x''(t)dt \le 0,$$

we get

$$-k\int_0^1 [x'(t)]^2 dt + \int_0^1 p(t)x'(t)dt \le \int_0^1 g(t,x(t),x'(t))x'(t)dt.$$

(A_1) implies that

$$\begin{aligned}
\beta\int_0^1 |x'(t)|^{\theta+1}dt &\le k\int_0^1 [x'(t)]^2dt - \int_0^1 p(t))x'(t)dt + \int_0^1 f(t,x(t),x'(t))x'(t)dt \\
&\le k\int_0^1 [x'(t)]^2dt + \int_0^1 |p(t)||x'(t)|dt + \int_0^1 |a(t)||x(t)|^\theta|x'(t)|dt \\
&\quad + \int_0^1 |b(t)||x'(t)|^{\theta+1}dt + \int_0^1 |r(t)||x'(t)|dt \\
&\le k\int_0^1 [x'(t)]^2dt
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1 (|p(t)|+|r(t)|)^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^1 |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}\\
&+||b||_\infty\int_0^1 |x'(t)|^{\theta+1}dt+\int_0^1 |a(t)||x'(t)|dt\left(\int_0^1 |x'(t)|dt\right)^{\theta}\\
\le\ & k\int_0^1 [x'(t)]^2dt\\
&+\left(\int_0^1 (|p(t)|+|r(t)|)^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^1 |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}\\
&+||b||_\infty\int_0^1 |x'(t)|^{\theta+1}dt\\
&+\left(\int_0^1 |a(t)|^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^1 |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}\left(\int_0^1 |x'(t)|^{\theta+1}dt\right)^{\frac{\theta}{\theta+1}}\\
\le\ & k\int_0^1 [x'(t)]^2dt\\
&+\left(\int_0^1 (|p(t)|+|r(t)|)^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^1 |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}\\
&+||b||_\infty\int_0^1 |x'(t)|^{\theta+1}dt\\
&+\left(\int_0^1 |a(t)|^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\int_0^1 |x'(t)|^{\theta+1}dt.
\end{aligned}
$$

If $\theta=1$, then

$$
\begin{aligned}
&\left(\beta-k-||b||_\infty-\left(\int_0^1 |a(t)|^2dt\right)^{\frac{1}{2}}\right)\int_0^1 |x'(t)|^2dt\\
&\quad\le\left(\int_0^1 (|p(t)|+|r(t)|)^2dt\right)^{\frac{1}{2}}\left(\int_0^1 |x'(t)|^2dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

If $\theta>1$ and $k\le 0$, then

$$
\begin{aligned}
&\left(\beta-||b||_\infty-\left(\int_0^1 |a(t)|^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\right)\int_0^1 |x'(t)|^{\theta+1}dt\\
&\quad\le\left(\int_0^1 (|p(t)|+|r(t)|)^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^1 |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}.
\end{aligned}
$$

If $\theta>1$ and $k>0$, then

$$
\begin{aligned}
&\left(\beta-||b||_\infty-\left(\int_0^1 |a(t)|^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\right)\int_0^1 |x'(t)|^{\theta+1}dt\\
&\quad\le\left(\int_0^1 (|p(t)|+|r(t)|)^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^1 |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}+k\left(\int_0^1 |x'(t)|^{\theta+1}dt\right)^{\frac{2}{\theta+1}}.
\end{aligned}
$$

It follows from (A_2) that there is a constant $M > 0$ such that $\int_0^1 |x'(t)|^{\theta+1} dt \le M$.

It is easy from (A_3) to get

$$\begin{aligned}
\delta\int_0^1 |x''(t)|^p dt &\le \int_0^1 q(t)\phi(x''(t))x''(t)dt \\
&= \phi(\lambda)\lambda\left(k\int_0^1 [x'(t)]^2 dt + \int_0^1 g(t,x(t),x'(t))x'(t)dt - \int_0^1 p(t)x'(t)dt\right) \\
&\le |k|\int_0^1 [x'(t)]^2 dt + \int_0^1 h(t,x(t),x'(t))x'(t)dt + \int_0^1 |f(t,x(t),x'(t))||x'(t)|dt \\
&\quad + \int_0^1 |p(t)||x'(t)|dt \\
&\le |k|\int_0^1 [x'(t)]^2 dt + \int_0^1 |f(t,x(t),x'(t))||x'(t)|dt + \int_0^1 |p(t)||x'(t)|dt \\
&\le |k|\int_0^1 [x'(t)]^2 dt + \int_0^1 |p(t)||x'(t)|dt \\
&\quad + \int_0^1 |a(t)||x(t)|^\theta |x'(t)|dt + \int_0^1 |b(t)||x'(t)|^{\theta+1} dt + \int_0^1 |r(t)||x'(t)|dt \\
&\le \begin{cases} \left[|k| + \left(\int_0^1 |a(t)|^2 dt\right)^{\frac{1}{2}} + ||b||_\infty\right]\int_0^1 |x'(t)|^2 dt \\ \qquad + \left(\int_0^1 (|p(t)|+|r(t)|)^2 dt\right)^{\frac{1}{2}}\left(\int_0^1 |x'(t)|^2 dt\right)^{\frac{1}{2}}, \\ |k|\left(\int_0^1 |x'(t)|^{\theta+1} dt\right)^{\frac{2}{\theta+1}} + \left(\int_0^1 (|p(t)|+|r(t)|)^{\frac{\theta+1}{\theta}} dt\right)^{\frac{\theta}{\theta+1}} \int_0^1 |x'(t)|^{\theta+1} dt \\ \qquad + \left(\int_0^1 |a(t)|^{\frac{\theta+1}{\theta}} dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^1 |x'(t)|^{\theta+1} dt\right)^{\frac{1}{\theta+1}} \\ \qquad + ||b||_\infty \int_0^1 |x'(t)|^{\theta+1} dt. \end{cases} \\
&\le \begin{cases} \left[|k| + \left(\int_0^1 |a(t)|^2 dt\right)^{\frac{1}{2}} + ||b||_\infty\right] M + \left(\int_0^1 (|p(t)|+|r(t)|)^2 dt\right)^{\frac{1}{2}} M^{\frac{1}{2}}, \\ |k| M^{\frac{2}{\theta+1}} + \left(\int_0^1 (|p(t)|+|r(t)|)^{\frac{\theta+1}{\theta}} dt\right)^{\frac{\theta}{\theta+1}} M^{\frac{1}{\theta+1}} \\ \qquad + \left(\int_0^1 |a(t)|^{\frac{\theta+1}{\theta}} dt\right)^{\frac{\theta}{\theta+1}} M + ||b||_\infty M. \end{cases}
\end{aligned}$$

Hence there is a constant $M_1 > 0$ such that $\int_0^1 |x''(t)|^p dt \le M_1$. It follows that $|x'(t)| \le \int_0^1 |x''(t)|dt \le \left(\int_0^1 |x''(t)|^p dt\right)^{\frac{1}{p}} \le M_1^{\frac{1}{p}}$ for all $t \in [0,1]$. Thus $|x(t)| \le \int_0^1 |x'(t)|dt \le M_1^{\frac{1}{p}}$ for all $t \in [0,1]$. and $|x'(t)| \le M_1^{\frac{1}{p}}$ for all $t \in [0,1]$.

Since $x'(0) = x'(1) = 0$, there is $\xi \in [0,1]$ such that $y(\xi) = 0$. Then

$$\begin{aligned}
|q(t)y(t)| &= \left|q(\xi)y(\xi) + \int_\xi^t (q(s)y(s))' ds\right| \\
&\le |k|\int_0^1 |x'(t)|dt + \int_0^1 |g(t,x(t),x'(t))|dt + \int_0^1 |p(t)|dt
\end{aligned}$$

$$\le |k|M_1^{\frac{1}{p}} + \max_{t\in[0,1],|x|\le M_1^{\frac{1}{p}},|y|\le M_1^{\frac{1}{p}}} |g(t,x,y)| + \int_0^1 |p(t)|dt.$$

Hence

$$|y(t)| \le \frac{|k|M_1^{\frac{1}{p}} + \max_{t\in[0,1],|x|\le M_1^{\frac{1}{p}},|y|\le M_1^{\frac{1}{p}}} |g(t,x,y)| + \int_0^1 |p(t)|dt}{\delta} \text{ for all } t\in[0,1].$$

Let

$$\Omega_1 = \left\{(x,y)\in D(L)\bigcap X : L(x,y) = \lambda N(x,y),\ \lambda\in(0,1)\right\}.$$

It follows from above discussion that Ω_1 is bounded.

Now, for $(x,y) = (0, c/q(t)) \in \mathrm{Ker}L$, if $N(0,c/q(t)) \in \mathrm{Im}L$, then $\int_0^1 \phi^{-1}(c/q(t))dt = 0$, it follows that $c = 0$. Let

$$\Omega_2 = \left\{(x,y)\in \mathrm{Ker}L : N(x,y)\in \mathrm{Im}L\right\}.$$

Then $\Omega_2 = \{0\}$.

Let

$$\Omega_3 = \{x\in \mathrm{Ker}L :\ \lambda\wedge x + (1-\lambda)QNx = 0,\ \lambda\in[0,1]\},$$

where $\wedge : Y/\mathrm{Im}L \to \mathrm{Ker}L$ is given by $\wedge^{-1}(c,0) = (0,c/q(t))$. For $(0,c/q(t))\in\Omega_3$, and $\lambda\in[0,1]$, we have

$$-(1-\lambda)Q(\phi(c/q(t)), f(t,0,0)) = \lambda\wedge(0,c/q(t)).$$

It follows that

$$-(1-\lambda)\int_0^1 \phi(c/q(t))dt = \lambda c.$$

If $\lambda = 1$, then $c = 0$. If $\lambda\in[0,1)$, then

$$0 \le \lambda c^2 = -(1-\lambda)c\phi(c)\int_0^1 \phi(1/q(t))dt < 0,$$

a contradiction. So $\Omega_3 = \{0\}$.

Let X and Y be Banach spaces, $L:\ D(L)\subset X\to Y$ be a Fredholm operator of index zero, $P:\ X\to X$, $Q:\ Y\to Y$ be projectors such that

$$\mathrm{Im}\,P = \mathrm{Ker}\,L,\ \mathrm{Ker}\,Q = \mathrm{Im}\,L,\ \ X = \mathrm{Ker}\,L\oplus\mathrm{Ker}\,P,\ \ Y = \mathrm{Im}\,L\oplus\mathrm{Im}\,Q.$$

It follows that

$$L|_{D(L)\cap\mathrm{Ker}\,P}:\ D(L)\cap\mathrm{Ker}\,P\to\mathrm{Im}\,L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $D(L)\cap\overline{\Omega}\ne\emptyset$, the map $N:\ X\to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N:\ \overline{\Omega}\to X$ is compact.

Theorem 2.1.([5]) *Let L be a Fredholm operator of index zero and let N be L-compact on Ω. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(domL \setminus KerL) \cap \partial\Omega] \times (0,1)$;*
(ii) *$Nx \notin ImL$ for every $x \in KerL \cap \partial\Omega$;*
(iii) *$deg(\wedge QN\big|_{KerL}, \Omega \cap KerL, 0) \neq 0$, where $\wedge : Y/ImL \to KerL$ is the isomorphism.*
Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

Theorem L. *Suppose that $(A_1) - (A_3)$ hold. Then problem (3) has at least one solution.*

Proof. Set Ω be a open bounded subset of X such that $(0,0) \in \Omega \supseteq \overline{\Omega_1}$. We know that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω, we have $Lx \neq \lambda Nx$ for $x \in (D(L)/\mathrm{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$; $Nx \notin \mathrm{Im}L$ for $x \in \mathrm{Ker}L \cap \partial\Omega$.

In fact, let $H(x,\lambda) = \lambda \wedge x + (1-\lambda)QNx$. According the definition of Ω, we know $\Omega \supset \overline{\Omega_3}$, thus $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \mathrm{Ker}L$, thus by homotopy property of degree,

$$\begin{aligned}\deg(QN|\mathrm{Ker}L, \Omega \cap \mathrm{Ker}L, 0) &= \deg(H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0)\\ &= \deg(H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0) = \deg(\wedge, \Omega \cap \mathrm{Ker}L, 0) \neq 0.\end{aligned}$$

Thus by Theorem 2.1, $L(x,y) = N(x,y)$ has at least one solution in $D(L)\cap\overline{\Omega}$, then x is a solution of problem (3). The proof is completed.

3. Examples

In this section, we present an example to illustrate the main result.

Example 3.1. Consider the problem

$$\begin{cases} [(t^2+2)x''(t)]' + kx'(t) = \beta\frac{3[x'(t)]^{2l+1}}{2+\sin[x(t)]^2} + a(t)[x(t)]^{2l+1} + b(t)[x'(t)]^{2l+1} + r(t),\ t \in (0,1),\\ x^{(i)}(0) = x^{(i)}(1), \quad i = 0,1,2, \end{cases} \tag{5}$$

where $\beta > 0$, $k \in R$, l is a positive integer, $a, b, r \in L^1[0,1]$. Corresponding to problem (3), let $\phi(x) = x$, $g(t,x,y) = -\beta\frac{3y^{2l+1}}{2+\sin x^2} - a(t)x^{2l+1} - b(t)y^{2l+1} - r(t)$, $p(t) = 0$ and set

$$h(t,x,y) = -\beta\frac{3y^{2l+1}}{2+\sin x^2},\ \ f(t,x,y) = -a(t)x^{2l+1} - b(t)y^{2l+1} - r(t).$$

It is easy to show from Theorem L that problem (5) has at least one solution if $l > 0$, and

$$||b||_\infty + \left(\int_0^1 |a(t)|^{\frac{\theta+1}{\theta}} dt\right)^{\frac{\theta}{\theta+1}} < \beta;$$

or if $l = 0$ and

$$k + ||b||_\infty + \left(\int_0^1 |a(t)|^2 dt\right)^{\frac{1}{2}} < \beta.$$

The equation in problem (5) can be transformed into

$$x'''(t) + \frac{2t}{t^2+2}x''(t) + \frac{k}{t^2+2}x'(t) = \frac{\beta[x'(t)]^{2l+1} + a(t)[x(t)]^{2l+1} + b(t)[x'(t)]^{2l+1} + r(t)}{t^2+2}.$$

It is easy to find that $(B_1),(B_2),(B_3),(C)$ are not satisfied. So Example 3.1 can not be solved by theorems in papers [1-4].

Example 3.2. Consider the problem

$$\begin{cases} [(x''(t))^3]' = \beta[x'(t)]^3 + a(t)[x(t)]^3 + b(t)[x'(t)]^3 + r(t),\ t \in (0,1), \\ x^{(i)}(0) = x^{(i)}(1), \quad i = 0,1,2, \end{cases} \tag{6}$$

where $\beta > 0$, l is a positive integer, $a,b,r \in L^1[0,1]$, $\phi(x) = |x|^2 x$. It is easy to find that problem (6) has at least one solution if

$$||b||_\infty + \left(\int_0^1 |a(t)|^{\frac{4}{3}} dt\right)^{\frac{3}{4}} < \beta.$$

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