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Viscosity solutions for the relativistic inhomogeneous Vlasov equation in Schwarzschild outer space-time in the presence of the Yang-Mills field

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Abstract. This paper provides an existence and uniqueness result for the viscosity solution of the relativistic inhomogeneous Vlasov equation with a given Yang-Mills field in temporal jauge. The geometric background used here is the Schwarzschild outer space-time.

Keywords. Relativistic Vlasov equation, viscosity solution, partial differential equation, spherically symmetric

1 Introduction

The notion of viscosity solutions was introduced in 1981 by M. G. Crandall and P. L. Lions for first order Hamilton-Jacobi equations. The main interest of this theory, applied to certains partial differential equations is based on the fact that it allows merely continuous functions to be solutions of these equations. It provides a very general existence and uniqueness result and precise formulations of the boundary conditions [5]. For more details about the notion of viscosity solutions, we refer interested readers to ([3], [5], [6], [14], [15]) and references therein. The method has not yet been widely used in kinetic theory where one of the main equations is that of Vlasov, more specifically in the framework of general relativity, excluding works ([2],[9]).
The relativistic Vlasov equation describes the collision free evolution of massives particles where speeds are relatively high. It is distingusted from other equations of kinetic theory by the fact that there is no direct interactions between particles, solely by the field which are collectively generated by the motion of all particles. The characteristic feature of kinetic theory is that, its models are statistical and particles system are described by a distribution function $f = f(t, x, p, q)$; which represents the density of particles at a given space-time position $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, with momentum $p \in \mathbb{R}^4$ and non-Abelian charges of particles $q \in \mathbb{R}^N$. Note that, here we are working in the presence of Yang-Mills particles, see ([2],[9],[11], [16]) and references therein for the notion on Yang-Mills charges. Many authors have already studied the relativistic Vlasov equation, taking it alone or in association with other field equations, see ([2],[7],[9],[10],[11], [13],[16],[18],[20]).
In the work carried out in ([9]) by two of authors, the study of the Vlasov equation was made in a homogeneous background and they obtained a Cauchy problem for the Hamilton-Jacobi equation. One of the first simplest ways to describe the universe, when taking in consideration the inhomogeneity is to use the spherically symmetric space-times. We consider here the particular

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case of Schwarzschild outer space-time for which we need the Dirichlet condition on the sphere which is the boundary of the domain.
This work is therefore an extension of work [9] for two reasons: we work here in a non-homogeneous framework and the Hamilton-Jacobi problem that we obtain is a Cauchy-Dirichlet problem. The rest of the paper is organized as follows:
In section 2, following Etienne Takou and Fidèle L. Ciake Ciake [20], we provide the correct formulation of the relativistic Vlasov equation in Cartesian coordinates, taking as the background space-time a Schwarzschild outer space-time in presence of a given Yang-Mills field.
In section 3, we give the relativistic Vlasov equation in the new variables following [20], next we give the Hamilton-Jacobi equation equivalent of the relativistic Vlasov equation and we prove energy estimates.
In section 4, we prove existence and uniqueness theorems for the relativistic Vlasov equation.

2 The equation and the space-time

In this paper, greek indices $(\alpha, \beta, \gamma, ...)$ vary from 0 to 3 and latin indices $(i, j, k, l, ...)$ from 1 to 3. Unless otherwise specified, we adopt the Einstein summation convention

$$a_\alpha b^\alpha = \sum_\alpha a_\alpha b^\alpha.$$

The general form of Vlasov equation is as follows:

$$p^\alpha \frac{\partial f}{\partial x^\alpha} + P^\alpha \frac{\partial f}{\partial p^\alpha} + Q^a \frac{\partial f}{\partial q^a} = 0, \tag{2.1}$$

where

$$P^\alpha = -\Gamma^\alpha_{\lambda\gamma} p^\lambda p^\gamma + p^\beta q.F^\alpha_\beta, \qquad Q^a = -C^a_{bc} p^\alpha A^b_\alpha q^c. \tag{2.2}$$

Equation (2.1), which governs collisionless evolution of charged particles with non-zero mass m in the presence of a given Yang-Mills field, is considered here in the Schwarzschild outer space-time. The momentum of particles, denoted $p = (p^\alpha) = (p^0, p^i)$, provides information on their speed; their non-Abelian charge is denoted q. The distribution function f, is then a function of $(x^\alpha, p^\alpha, q^a)$ where (x^α, p^α) denotes the usual coordinates of the tangent bundle $T(\mathbb{R}^4)$ of $\mathbb{R}^4$. Particles evolve without collisions in the space-time $(\mathbb{R}^4, h)$, on the one hand under the action of their own gravitational field represented by the given metric tensor $h = (h_{\alpha\beta})$ which informs on gravitational effects, and on the other hand under the non-Abelian force generated by the Yang-Mills field $F = (F_{\alpha\beta})$, itself deriving from a given Yang-Mills potential $A = (A_\alpha)$. Consider $\mathbb{V}$ a neighborhood of the origin $O_{\mathbb{R}^3}$, we consider space $(\mathbb{R}_+ \times (\mathbb{R}^3 \setminus \mathbb{V}), h)$ as Schwarzschild outer space-time whose symmetric metric tensor $h = (h_{\alpha\beta})$ is of Lorentzian signature $(-, +, +, +)$ and can be written in the following form

$$ds^2 = -e^{\lambda(t,r)} dt^2 + e^{\nu(t,r)} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2, \tag{2.3}$$

where λ, ν are given class two differentiables functions which depend on the t and r variables, $t \in \mathbb{R}$, $r \in]0, +\infty[$ and $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi]$. Such a metric is said to be Schwarzschild outer space-time since it admits an action of the group $SO(3)$ by isometries.
The trajectorie $s \longmapsto (x^\alpha(s), p^\alpha(s), q^a(s))$ of such particles are non-longer geodesics, but are solutions of differential system:

$$\frac{dp^\alpha}{ds} = P^\alpha; \quad \frac{dx^\alpha}{ds} = p^\alpha; \quad \frac{dq^a}{ds} = Q^a. \tag{2.4}$$

Since equation (2.1) is expressed in Cartesian coordinates while the Christofell coefficients are given in spherical coordinates, then in what follows we will give a transformation of the equation (2.1) into spherical coordinates
So, let:

$$x^1 = r\sin\theta\cos\varphi, x^2 = r\sin\theta\sin\varphi, x^3 = r\cos\theta \ \ and \ \ r = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2} = \sqrt{\delta_{ij}x^ix^j}.$$

Setting $x_i = \delta_{ij}x^j$, h can be written as:

$$h = h_{00}dt^2 + h_{ij}dx^idx^j, \tag{2.5}$$

with components h_{00} and h_{ij} as obtained in [20],

$$h_{00} = -e^{\lambda}, \qquad h_{ij} = \delta_{ij} + (e^{\nu}-1)\frac{x_ix_j}{r^2}, \qquad h_{0i} = 0. \tag{2.6}$$

we obtain the determinant of metric h using relations (2.5) and (2.6). Its expression is given by:

$$det(h_{\alpha\beta}) = -e^{\lambda+\nu}.$$

After computation, the reverse of (h) is given by

$$h^{-1} = h^{00}dt^2 + h^{ij}dx_idx_j \tag{2.7}$$

with

$$h^{00} = -e^{-\Phi}, \quad h^{ij} = \delta^{ij} + (e^{-\nu}-1)\frac{x^ix^j}{r^2}, \quad h^{0j} = 0.$$

The rest mass of particles is normalized to the unity, that is $m = 1$. Actually, the collisionless charged particles move on the future sheet of the mass hyperboloid $P(\mathbb{R}^4) \subset T(\mathbb{R}^4)$, whose equation is $P_{t,x}(p)\colon g(p,p) = -1$ or equivalently, using expression (2.3) of g:

$$P_{t,x}\colon p^0 = e^{-\frac{\lambda}{2}}\sqrt{1 + h_{ij}p^ip^j}, \tag{2.8}$$

where the choice $p^0 > 0$ symbolizes the fact that particles eject towards the future. We also suppose that the non-Abelian charge q of the Yang-Mills particles are functions of class C^∞ from $\mathbb{R}^4$ to G whose given norm are is $e > 0$. To clarify this idea, let us suppose that in fact G is the Euclidean space $\mathbb{R}^N$ embedded with an ad-invariant scalar product positive defined, which is denoted by the dot '$\cdot$'. Thus, q takes his value in an orbit of G, which is the sphere $\mathbb{S}$ described by

$$(\mathbb{S})\colon q\cdot q = e^2. \tag{2.9}$$

Notice additionally that, the ad-invariant property of this scalar product means that:

$$u\cdot[v,w] = [u,v]\cdot w, \quad u,v,w \in G. \tag{2.10}$$

The relation (2.9) allows to express the components q^N of q as functions of $q = (q^a)$, $a = 1,...,N-1$. Using relations (2.8), (2.9) and the fact that we study a inhomogeneous phenomenon, we obtain that the distribution function f of Yang-Mills particles is in fact a function of $(t,x,p^i,q^a) = (t,x,p,q)$, $i = 1,2,3;\ a = 1,2,...,N-1$. Thus, $f = f(t,x,p,q)$, $t \in \mathbb{R}_+$, $x \in (\mathbb{R}^3 \setminus \mathbb{V})$, $p \in \mathbb{R}^3$, $q \in \mathbb{R}^{N-1}$. Thus, for the transformation of the equation (2.1), we shall need expressions of the Christofell symbols $\Gamma^i_{\alpha\beta}$ which are given by (see [20]):

$$\Gamma^i_{00} = \frac{1}{2}\partial_r\lambda\frac{x^i}{r}e^{\lambda-\nu}, \qquad \Gamma^i_{0j} = -\frac{1}{2}\partial_t\nu\frac{x^ix_j}{r^2}, \tag{2.11}$$

$$\Gamma^i_{jk} = \frac{1-e^{-\nu}}{r}\left[\delta_{jk} - \frac{x_jx_k}{r^2}\right]\frac{x^i}{r} + \frac{1}{2}\partial_r\lambda\frac{x^ix_kx_j}{r^3}. \tag{2.12}$$

3 Hamilton-Jacobi equation and energy estimates

In order to find a more simplified form of equation (2.1), we consider the new momenta variables as in ([13], [20]). Before we define the geometric framework on $\mathbb{R}^3$ characterized by vectors defined in the following way:

$$e^i_d = \delta^i_d + (e^{-\frac{\nu}{2}} - 1)\delta_{dj}\frac{x^i x^j}{r^2} \ \textit{such that}\ h_{ij}e^i_d e^j_s = \delta_{sd}, \tag{3.1}$$

here, we take d as index in this new geometric framework while the index i is for natural coordinates. Thus the momentum p will be replaced by v and the p^i components of p will be replaced by the v^i components of v.

$$p^i = v^d e^i_d \quad \textit{with} \quad v^i = p^i + (e^{\frac{\nu}{2}} - 1)\frac{x\cdot p x^i}{r^2}, \tag{3.2}$$

Using $x \cdot p = e^{-\frac{\nu}{2}} x \cdot v$, we have:

$$p^i = v^i + (e^{-\frac{\nu}{2}} - 1)\frac{x\cdot v x^i}{r^2}. \tag{3.3}$$

New coordinates in this change of variables (3.2) and (3.3) has the advantage that by the mass-shell condition (2.8), we have

$$p^0 = e^{-\frac{\lambda}{2}}\sqrt{1+|v|^2}. \tag{3.4}$$

In the next, we will use following notations:

$$v^0 = \sqrt{1+|v|^2}, \quad p^0 = e^{-\frac{\lambda}{2}}v^0, \quad v_i = \delta_{ij}v^j \quad . \tag{3.5}$$

By this change of variables, instead of (t, x, p, q), we will use (t, x, v, q) as new variables. We henceforth set:

$$f(t,x,v,q) = \bar{f}(t,x,p(t,x,v),q) = \bar{f}(t,x,v + (e^{-\frac{\nu}{2}} - 1)\frac{x\cdot vx}{r^2}, q). \tag{3.6}$$

Now, we look the computation of the partial derivatives of f that appear in equation (2.1) to better reduce it. After calculation we get (see [20]):

$$-p^0\frac{\partial f}{\partial t} = -e^{-\frac{\lambda}{2}}v^0\left[\frac{\partial \bar{f}}{\partial t} + \frac{1}{2}\partial_t\nu\frac{x\cdot vx^i}{r^2}\frac{\partial \bar{f}}{\partial v^i}\right], \tag{3.7}$$

$$p^i\frac{\partial f}{\partial x^i} = p^i\left(\frac{\partial \bar{f}}{\partial x^i} + \gamma^l_i\frac{\partial \bar{f}}{\partial v^l}\right), \tag{3.8}$$

where

$$\begin{aligned}\gamma^l_i = \frac{1}{2}\partial_r\nu e^{\frac{\nu}{2}}x^l x_i\frac{x\cdot v}{r^3} + (1-e^{-\frac{\nu}{2}})\delta^l_i\frac{v\cdot x}{r^2} + (e^{\frac{\nu}{2}}-1)\frac{v_i x^l}{r^2} \\ +(e^{-\frac{\nu}{2}} - e^{\frac{\nu}{2}})\frac{x_i x^l}{r^2}\frac{x\cdot v}{r^2},\end{aligned} \tag{3.9}$$

and

$$\left(\Gamma^i_{00}(p^0)^2 + 2\Gamma^i_{0j}p^0p^j + 2\Gamma^i_{jk}p^jp^k - p^0 q\cdot F^i_0 - p^l q\cdot F^i_l\right)\frac{\partial f}{\partial p^i} =$$

$$\left(\Gamma^i_{00}(p^0)^2 + 2\Gamma^i_{0j}p^0p^j + 2\Gamma^i_{jk}p^jp^k - p^0 q\cdot F^i_0 - p^l q\cdot F^i_l\right)\frac{\partial \bar{f}}{\partial v^i}e^{\frac{\nu}{2}}, \tag{3.10}$$

$$C^a_{bc}A^b_k q^c \frac{\partial f}{\partial q^a} = C^a_{bc}A^b_k q^c \frac{\partial \bar{f}}{\partial q^a}. \tag{3.11}$$

(3.7),(3.8), (3.10) and (3.11) into the relativistic Vlasov equation (2.1), and still denote the unknown function f instead of $\bar{f}$, we obtain after using Einstein summation convention, some rearrangements and by dividing the resulting equation by second relation of (3.5)

$$\frac{\partial f}{\partial t} + A^i \frac{\partial f}{\partial x^i} + B^i \frac{\partial f}{\partial v^i} + C^a \frac{\partial f}{\partial q^a} = 0, \tag{3.12}$$

where

$$A^i = \frac{e^{\frac{\lambda}{2}}}{v^0}\left[v^i + (e^{-\frac{\nu}{2}} - 1)\frac{x^i x\cdot v}{r^2}\right], \tag{3.13}$$

$$\begin{aligned} B^i = {} & \frac{e^{\frac{\lambda}{2}}}{v^0}\frac{e^{-\frac{\nu}{2}} - 1}{r^2}\delta_{jk}v^l(x^i v^k - x^k v^i) - \frac{\partial_r\lambda}{2}e^{\frac{(\lambda-\nu)}{2}}\frac{x^i}{r}v^0 - \frac{\partial_t\nu}{2}\frac{x^i x\cdot v}{r^2} - q\cdot F^i_0 e^{\frac{\nu}{2}} \\ & - \left(e^{\frac{\nu+\lambda}{2}}\frac{v^l}{v^0} + (e^{\frac{\lambda}{2}} - e^{\frac{\nu+\lambda}{2}})\frac{x\cdot v x^l}{r^2 v^0}\right) q\cdot F^i_l, \end{aligned} \tag{3.14}$$

and

$$C^a = \left(e^{\frac{\lambda}{2}}\frac{v^k}{v^0} + (e^{\frac{\lambda-\nu}{2}} - e^{\frac{\lambda}{2}})\frac{x\cdot v x^k}{v^0 r^2}\right) C^a_{bc}A^b_k q^c. \tag{3.15}$$

Assumption 1. The following assumptions have been adopted.
A1. The distribution function is spherically symmetric i.e. f is assumed to be invariant under simultaneous rotations of x and v. As a consequence, a direct computation leads to the following condition (see [13] and [20]):

$$\delta_{lk}v^l(v^k x^i - v^i x^k)\frac{\partial f}{\partial v^i} = \delta_{lk}x^l(x^k v^i - x^i v^k)\frac{\partial f}{\partial x^i} \tag{3.16}$$

A2. We require the assumption that all partial derivatives of λ and ν are bounded i.e. there exists $C > 0$ such that:

$$|\partial_t\nu| \le C, \qquad |\partial_t\lambda| \le C, \qquad |\partial_r\nu| \le C, \qquad |\partial_r\lambda| \le C, \tag{3.17}$$

and we take λ and ν such that there exists $\Pi \in C_b([0,T])$ which satisfy:

$$e^\lambda \le \Pi(t), \qquad e^\nu \le \Pi(t). \tag{3.18}$$

Assumption **A1** in (3.12) gives:

$$\begin{aligned} & \left[\frac{\partial_r\lambda}{2}e^{\frac{(\lambda-\nu)}{2}}\frac{x^i}{r}v^0 + \frac{\partial_t\nu}{2}e^{\frac{\nu}{2}}\frac{x^i x\cdot v}{r^2} - q\cdot F^i_0 e^{\frac{\nu}{2}} - \left(e^{\frac{\nu+\lambda}{2}}\frac{v^l}{v^0} + (e^{\frac{\lambda}{2}} - e^{\frac{\nu+\lambda}{2}})\frac{x\cdot v x^l}{r^2 v^0}\right) q\cdot F^i_l\right]\frac{\partial f}{\partial v^i} \\ & + \left(e^{\frac{\lambda}{2}}\frac{v^k}{v^0} + (e^{\frac{\lambda-\nu}{2}} - e^{\frac{\lambda}{2}})\frac{x\cdot v x^k}{v^0 r^2}\right) C^a_{bc}A^b_k q^c\frac{\partial f}{\partial q^a} - e^{\frac{\lambda-\mu}{2}}\frac{v^i}{v^0}\frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial t}. \end{aligned} \tag{3.19}$$

Now, we consider the function H defined by the left hand side of the relativistic Vlasov equation (3.19) as follows:

$$H(t,x,v,q,\nabla_{x,v,q}f(t,x,v,q)) = e^{\frac{\lambda-\mu}{2}}\frac{v^i}{v^0}\frac{\partial f}{\partial x^i} - \left(e^{\frac{\lambda}{2}}\frac{v^k}{v^0} + (e^{\frac{\lambda-\nu}{2}} - e^{\frac{\lambda}{2}})\frac{x\cdot v x^k}{v^0 r^2}\right) C^a_{bc}A^b_k q^c\frac{\partial f}{\partial q^a}$$

$$- \left[\frac{\partial_r \lambda}{2} e^{\frac{(\lambda-\nu)}{2}} \frac{x^i}{r} v^0 + \frac{\partial_t \nu}{2} e^{\frac{\nu}{2}} \frac{x^i x \cdot v}{r^2} - q \cdot F_0^i e^{\frac{\nu}{2}} - \left(e^{\frac{\nu+\lambda}{2}} \frac{v^l}{v^0} + (e^{\frac{\lambda}{2}} - e^{\frac{\nu+\lambda}{2}}) \frac{x \cdot v x^l}{r^2 v^0} \right) q \cdot F_l^i \right] \frac{\partial f}{\partial v^i}. \tag{3.20}$$

We set $u_i = \dfrac{\partial f}{\partial x^i}$, $z_i = \dfrac{\partial f}{\partial v^i}$ and $w_a = \dfrac{\partial f}{\partial q^a}$ in the relation (3.20).
Let $\Omega = \mathbb{R}^3 \backslash \mathbb{V}$. We will consider $\rho > 0$ as the radius of the neighborhood $\mathbb{V}$ in the sequel. The Hamiltonian H can be rewritten as a function:

$$\begin{aligned} &H : [0; +\infty[\times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \longrightarrow \mathbb{R} \\ &(t, x, v, q, u, z, w) \longmapsto H(t, x, v, q, u, z, w), \end{aligned}$$

where

$$H(t, x, v, q, u, z, w) = \mathcal{A}^i u_i + \mathcal{B}^i z_i + \mathcal{D}^a w_a, \tag{3.21}$$

with

$$\mathcal{A}^i = e^{\frac{\lambda-\mu}{2}} \frac{v^i}{v^0}, \tag{3.22}$$

$$\mathcal{B}^i = -\frac{\partial_r \lambda}{2} e^{\frac{(\lambda-\nu)}{2}} \frac{x^i}{r} v^0 - \frac{\partial_t \nu}{2} e^{\frac{\nu}{2}} \frac{x^i x \cdot v}{r^2} + q \cdot F_0^i e^{\frac{\nu}{2}} + \left(e^{\frac{\nu+\lambda}{2}} \frac{v^l}{v^0} + (e^{\frac{\lambda}{2}} - e^{\frac{\nu+\lambda}{2}}) \frac{x \cdot v x^l}{r^2 v^0} \right) q \cdot F_l^i \tag{3.23}$$

and

$$\mathcal{D}^a = - \left(e^{\frac{\lambda}{2}} \frac{v^k}{v^0} + (e^{\frac{\lambda-\nu}{2}} - e^{\frac{\lambda}{2}}) \frac{x \cdot v x^k}{v^0 r^2} \right) C_{bc}^a A_k^b q^c. \tag{3.24}$$

The relativistic Vlasov equation (2.1), in the Schwarzschild outer space-time has been transformed into the following Hamilton-Jacobi equation:

$$f_t(t, x, v, q) + H(t, x, v, q, u, z, w) = 0. \tag{3.25}$$

Let us assume: $f_0 : \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ and $g : [0, T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ two Lipschitz continuous and bounded functions and $T > 0$ a given real number.
We consider the following Cauchy-Dirichlet problem

$$\begin{cases} (a) \;\; f_t(t, x, v, q) + H(t, x, v, q, u, z, w) = 0 \quad in \quad [0; T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}, \\ (b) \;\; f(0, x, v, q) = f_0(x, v, q) \quad in \quad \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}, \\ (c) \;\; f(t, x, v, q) = g(t, x, v, q) \quad in \quad [0, T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}, \end{cases} \tag{3.26}$$

where $u = \nabla_x f, \quad z = \nabla_v f, \quad w = \nabla_q f$.
Our goal in the next, will be to prove that the Cauchy-Dirichlet problem (3.26) has a unique viscosity solution $f \in C([0, T] \times \bar{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^{N-1}, \mathbb{R})$.
The formulation of the Cauchy-Dirichlet problem being completed, we will first of all establish estimates which will be useful to us in the proofs of existence and uniqueness results. Firstly, let us state the following proposition:

Proposition 3.1. *Under Assumption **A1**, we have the following statments:*

1) $\dfrac{v^i}{v^0}$, $\dfrac{1}{v^0}$, $\Gamma^\alpha_{\beta\gamma}$ *are bounded.*

2) *For all $t \in [0, T]$, mapping $t \longmapsto v(t)$ is uniformly bounded.*

Proof. **1)** • Firstly, using relations (3.5), and some majorations we obtain:

$$\left|\frac{v^i}{v^0}\right| \leq 1 \quad and \quad \left|\frac{1}{v^0}\right| \leq 1. \tag{3.27}$$

• (2.11) and (2.12) show that:

$$\left|\Gamma^i_{00}\right| \leq \frac{1}{2}CC_1, \quad \left|\Gamma^i_{0j}\right| \leq \frac{1}{2}C \quad and \quad \left|\Gamma^i_{jk}\right| \leq \frac{2}{\rho} + \frac{1}{2}C, \tag{3.28}$$

where $|\Pi(t)| \leq C_1$.
2) By relation (3.2), we get:

$$|v(t)| \leq |p(t)| + C_1|p(t)|. \tag{3.29}$$

Now, it is sufficient to show that $|p(t)|$ is uniformly bounded. From equations (2.2) and (2.4), we have:

$$\frac{dp^i}{ds} = -\left(\Gamma^i_{00}p^0 + 2\Gamma^i_{0j}p^j + 2\Gamma^i_{jk}\frac{p^j}{p^0}p^k\right) + q.(F^i_0 + \frac{p^j}{p^0}F^i_j). \tag{3.30}$$

Since,

$$\frac{p^i}{p^0} = \frac{v^i + (e^{-\frac{\nu}{2}} - 1)\frac{x \cdot vx^i}{r^2}}{e^{-\frac{\lambda}{2}}v^0} \leq \frac{v^i}{v^0}e^{\frac{\lambda}{2}} + e^{\frac{\lambda}{2}}(e^{-\frac{\nu}{2}} - 1)\frac{x \cdot vx^i}{v^0r^2},$$

we then have

$$\left|\frac{p^i}{p^0}\right| \leq \left|\frac{v^i}{v^0}\right||e^{\frac{\lambda}{2}}| + |e^{\frac{\lambda}{2}}||e^{-\frac{\nu}{2}} - 1|\left|\frac{x \cdot vx^i}{v^0r^2}\right| \leq 2C_1, \tag{3.31}$$

$$|p^0| \leq \sqrt{1 + C_1|p|^2} \leq 1 + C_1|p|^2. \tag{3.32}$$

Using equation (2.9), inequalities (3.28), (3.31), (3.32) and the fact that
$F \in C^\infty_0([0, +\infty[\times\Omega, \mathbb{R})$, we get:

$$\left|\frac{dp^i}{ds}\right| \leq \left(\left|\Gamma^i_{00}\right||p^0| + 2\left|\Gamma^i_{0j}\right||p^j| + 2\left|\Gamma^i_{jk}\right|\left|\frac{p^j}{p^0}\right||p^k|\right) + |q|.\left(\left|F^i_0\right| + \left|\frac{p^i}{p^0}\right|\left|F^i_i\right|\right), \tag{3.33}$$

$$\leq \frac{1}{2}C(1 + C_1)C_1|p|^2 + \left[C + \left(\frac{4}{\rho} + C\right)C_1\right]|p| + e\left|F\right|\left(1 + 2C_1 + \frac{1}{2}CC_1\right). \tag{3.34}$$

So we obtain

$$\left|\dot{p}\right| \leq B\left|p\right|^2 + A\left|p\right| + D, \tag{3.35}$$

where $A = C + \left(\frac{8}{\rho} + 2C\right)C_1$, $B = \frac{1}{2}C\left(1 + C_1\right)C_1$ and $D = e\left|F\right|\left(1 + 2C_1\right) + \frac{1}{2}CC_1$.
Integrating the relation (3.35) over $[0, t]$, $t \in [0, T[$, we obtain

$$|p(t)| \leq E(t) + A\int_0^t |p(s)|\,ds + B\int_0^t |p(s)|^2\,ds, \tag{3.36}$$

in which $E(t) = tD + |p(0)|$.
Applying the Gronwall Lemma of nonlinear integral inequatities (see [21] Theorem 2.4), we obtain the following relation:

$$|p| \leq \frac{e^{AT}}{\dfrac{1}{E(T)} - \dfrac{Be^{AT}}{A}} \leq \frac{AE(T)e^{AT}}{A - E(T)Be^{AT}}, \quad t \in [0, T[,$$

ρ being chosen such that $A - E(T)Be^{AT}$ is non-negative. Using relation (3.29), we see that $v(t)$ is bounded. Which end the proof of the proposition. ∎

Proposition 3.2. *Let $T > 0$ and $R > 0$ two reals numbers, the Hamiltonian H*

$$H : [0;T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$$
$$(t, x, v, q, u, z, w) \longmapsto H(t, x, v, q, u, z, w)$$

given by relation (3.21) satisfies following properties:

(i) H is continuous in (x, v, q, u, z, w).

(ii) H is Lipschitz with respect to (t, x, v, q).

(iii) H is Lipschitz with respect to (u, z, w).

(iv) There exists one modulus M_R such that
$\left|H(t, x, v, q, u, z, w) - H(t', x', v', q', u', z', w')\right| \leq M_R(\left|t - t'\right| + \left|x - x'\right| + \left|v - v'\right| + \left|u - u'\right| + \left|q - q'\right| + \left|z - z'\right| + \left|w - w'\right|)$
$\forall\ (t, x, v, q, u, z, w), (t', x', v', q', u', z', w') \in [0, T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$, *where* $|u|, |z|, |w|, \left|u'\right|, \left|z'\right|, \left|w'\right| \leq R$

The following remark will be very useful

Remark 1. R1. Using the inequality of finite increments, we obtain following inequalities in which y is an element on the segment $[x, x']$ and t_1 is also an element between t and t'.

$$\left|e^{\frac{\Lambda'}{2}} - e^{\frac{\Lambda}{2}}\right| \leq \frac{1}{2} e^{\frac{\Lambda(t_1, y)}{2}} \left|\nabla\Lambda(t_1, y)\right| \left(\left|x - x'\right| + \left|t - t'\right|\right).$$

R2. We have the following estimations

$$\frac{x^i}{r^2} - \frac{(x')^i}{(r')^2} = \frac{x^i}{r^2 (r')^2}\left(r^2 - (r')^2\right) + \frac{1}{(r')^2}\left(x^i - (x')^i\right);$$
$$\frac{(x')^k (x')^l}{(r')^2} - \frac{x^k x^l}{r^2} = \frac{(x')^k (x')^l}{r^2 (r')^2}(r^2 - (r')^2) + \frac{1}{r^2}\left((x')^k (x')^l - x^k x^l\right);$$
$$\frac{(v')^i}{(v')^0} - \frac{v^i}{v^0} = \frac{(v')^i}{(v')^0 v^0}((v')^0 - v^0) + \frac{1}{v^0}((v')^i - v^i);$$
$$(x')^k (x') - x^k x = (x')^k ((x') - x) + x((x')^k - x^k).$$
$$|r^2 - (r')^2| \leq 3(r + r')\left|x - x'\right|, \quad |r - r'| \leq 3\left|x - x'\right|,$$
$$|v^0 - (v')^0| \leq (|v| + |v'|))\left|v - v'\right|.$$

Proof. Let $T > 0$ be given:
i) $H(t, x, v, q, u, z, w)$ is naturally defined and continuous in (t, x, v, q, u, z, w) since $v^0 > 1$.
ii) Let $(u, z, w) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$ be fixed and $(t, x, v, q), (t^{'}, x^{'}, v^{'}, q^{'}) \in [0, T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$. Using relation (3.21) of H, we obtain:

$$\begin{aligned} H(t, x, v, q, u, z, w) - H(t^{'}, x^{'}, v^{'}, q^{'}, u, z, w) &= N_1 u_i + C^a_{bc} A^b_k \left[N_1^{'} + N_5\right] w_a \\ + \left[(se^{\frac{\nu^{'}}{2}} - qe^{\frac{\nu}{2}})F^i_0 + N_1^{''} \cdot F^i_l + N_2 \cdot F^i_l + \frac{1}{2}N_3 + \frac{1}{2}N_4\right] z_i \end{aligned} \tag{3.37}$$

where

$$\begin{aligned} N_1 &= \left(\frac{(v^{'})^i}{(v^{'})^0} e^{\frac{\lambda^{'}-\mu^{'}}{2}} - \frac{v^i}{v^0} e^{\frac{\lambda-\mu}{2}}\right), \quad N_1^{''} = \left(\frac{(v^{'})^l}{(v^{'})^0}(q^{'}) e^{\frac{\lambda^{'}+\nu^{'}}{2}} - \frac{v^l}{v^0} q e^{\frac{\lambda+\nu}{2}}\right), \\ N_2 &= \left((e^{\frac{\lambda^{'}}{2}} - e^{\frac{\nu^{'}+\lambda^{'}}{2}}) \frac{(x^{'}) \cdot (v^{'})(x^{'})^k}{r^2 (v^{'})^0}(q^{'}) - (e^{\frac{\lambda}{2}} - e^{\frac{(\nu+\lambda)}{2}}) \frac{x \cdot v x^k}{r^2 v^0} q\right), \\ N_1^{'} &= \left(\frac{(v^{'})^k}{(v^{'})^0}(q^{'})^c e^{\frac{\lambda^{'}}{2}} - \frac{v^k}{v^0} q^c e^{\frac{\lambda}{2}}\right), \quad N_3 = \partial_r \lambda e^{\frac{\lambda-\nu}{2}} \frac{x^i}{r} v^0 - \partial_{r^{'}} \lambda^{'} e^{\frac{\lambda^{'}-\nu^{'}}{2}} \frac{(x^{'})^i}{r^{'}} (v^{'})^0, \\ N_4 &= \left(\partial_t \nu e^{\frac{\nu}{2}} \frac{x^i x \cdot v}{r^2} - \partial_t \nu^{'} e^{\frac{\nu^{'}}{2}} \frac{(x^{'})^i (x^{'}) \cdot (v^{'})}{(r^{'})^2}\right), \\ N_5 &= \frac{1}{(v^{'})^0} \frac{(x^{'}) \cdot (v^{'})}{(r^{'})^2} (x^{'})^k (q^{'})^c (e^{\frac{\lambda^{'}-\nu^{'}}{2}} - e^{\frac{\lambda^{'}}{2}}) - \frac{1}{v^0} \frac{x \cdot v}{r^2} x^k q^c (e^{\frac{\lambda-\nu}{2}} - e^{\frac{\lambda}{2}}). \end{aligned} \tag{3.38}$$

Now we have:

$$N_1^{''} = (q^{'}) e^{(\Phi^{'}+\Lambda^{'})/2} \left(\frac{(v^{'})^l}{(v^{'})^0} - \frac{v^l}{v^0}\right) + \frac{v^l}{v^0} e^{(\Phi^{'}+\Lambda^{'})/2}((q^{'}) - q) + \frac{v^l}{v^0} s(e^{(\Phi^{'}+\Lambda^{'})/2} - e^{(\Phi+\Lambda)/2}). \tag{3.39}$$

$$\begin{aligned} N_1^{'} &= \frac{(v^{'})^k}{(v^{'})^0}(q^{'})^c \left(e^{\frac{\lambda}{2}} - e^{\frac{\lambda^{'}}{2}}\right) + \frac{(v^{'})^k}{(v^{'})^0} e^{\frac{\lambda}{2}} \left((q^{'})^c - q^c\right) + q^c e^{\frac{\lambda}{2}} \left(\frac{(v^{'})^k}{(v^{'})^0} - \frac{v^k}{v^0}\right), \\ N_1 &= \frac{(v^{'})^i}{(v^{'})^0}(e^{\frac{\lambda^{'}-\mu^{'}}{2}} - e^{\frac{\lambda-\mu}{2}}) + e^{\frac{\lambda-\mu}{2}} \left(\frac{(v^{'})^i}{(v^{'})^0} - \frac{v^i}{v^0}\right). \end{aligned} \tag{3.40}$$

$$\begin{aligned} N_2 &= \frac{(x^{'}) \cdot (v^{'})(x^{'})^k}{(r^{'})^2 (v^{'})^0}(q^{'}) \left(e^{\frac{\lambda^{'}}{2}} - e^{\frac{\lambda}{2}} + e^{\frac{\nu+\lambda}{2}} - e^{\frac{\nu^{'}+\lambda^{'}}{2}}\right) + (e^{\frac{\lambda}{2}} - e^{\frac{\nu+\lambda}{2}}) \frac{x \cdot v x^k}{r^2 v^0}((q^{'}) - q) \\ &+ (e^{\frac{\lambda}{2}} - e^{\frac{\nu+\lambda}{2}}) \left((x^{'})^k (x^{'}) - x^k x\right) \cdot (v^{'}) \frac{(q^{'})}{r r^{'} (v^{'})^0} + (e^{\frac{\lambda}{2}} - e^{\frac{\nu+\lambda}{2}}) s \frac{x^k x^i}{r^2} \delta_{ij} \cdot \left(\frac{(v^{'})^j}{(v^{'})^0} - \frac{v^j}{v^0}\right). \end{aligned} \tag{3.41}$$

$$N_3 = \partial_r \lambda e^{\frac{\lambda-\nu}{2}} \frac{x^i}{r}(v^0 - (v^{'})^0) + \partial_r \lambda e^{\frac{\lambda-\nu}{2}} (v^{'})^0 \left(\frac{x^i}{r} - \frac{(x^{'})^i}{r^{'}}\right) \tag{3.42}$$

$$+ e^{\frac{\lambda^{'}-\nu^{'}}{2}} \frac{(x^{'})^i}{r^{'}}(v^{'})^0 \left(\partial_r\lambda - \partial_{r^{'}}\lambda^{'}\right) + \partial_r\lambda \frac{(x^{'})^i}{r^{'}}(v^{'})^0 \left(e^{\frac{\lambda-\nu}{2}} - e^{\frac{\lambda^{'}-\nu^{'}}{2}}\right).$$

$$N_4 = \partial_t\nu e^{\frac{\nu}{2}} \frac{x^i}{r^2} x^j (v^k - (v^{'})^k) + \partial_t\nu e^{\frac{\nu}{2}} \frac{x^i}{r^2}(v^{'})^k(x^j - (x^{'})^k) + e^{\frac{\nu^{'}}{2}} \frac{(x^{'})^i}{(r^{'})^2}(x^{'})^j(v^{'})^k(\partial_t\nu - \partial_t\nu^{'})$$
$$+ \partial_t\nu e^{\frac{\nu}{2}}(x^{'})^j(v^{'})^k \left(\frac{x^i}{r^2} - \frac{(x^{'})^i}{(r^{'})^2}\right) + \partial_t\nu \frac{(x^{'})^i}{(r^{'})^2}(x^{'})^j(v^{'})^k \left(e^{\frac{\nu}{2}} - e^{\frac{\nu^{'}}{2}}\right). \tag{3.43}$$

$$N_5 = \delta_{jl} \frac{(x^{'})^l}{(r^{'})^2}(x^{'})^k(q^{'})^c(e^{\frac{\lambda^{'}-\nu^{'}}{2}} - e^{\frac{\lambda^{'}}{2}}) \left(\frac{(v^{'})^j}{(v^{'})^0} - \frac{v^j}{v^0}\right) + \delta_{jl}\frac{v^j}{v^0}\frac{x^l}{r^2}x^k(e^{\frac{\lambda^{'}-\nu^{'}}{2}} - e^{\frac{\lambda^{'}}{2}})((q^{'})^c - q^c)$$
$$+ \delta_{jl}\frac{v^j}{v^0}(q^{'})^c \left[(e^{\frac{\lambda^{'}-\nu^{'}}{2}} - e^{\frac{\lambda^{'}}{2}}) \left(\frac{(x^{'})^l(x^{'})^k}{(r^{'})^2} - \frac{x^l x^k}{r^2}\right) + \frac{x^l x^k}{r^2}\left(e^{\frac{\lambda^{'}-\nu^{'}}{2}} - e^{\frac{\lambda}{2}} - e^{\frac{\lambda-\nu}{2}} + e^{\frac{\lambda}{2}}\right)\right]. \tag{3.44}$$

We then deduce, using Proposition 3.1, $|q| = e$, $A, F \in C_0^{+\infty}([0;T]\times\Omega)$ and Remark 1, that:

$$|N_1| \leq \left|\nabla_{(t_1,y)}\lambda(t_1,y)\right| C_1 \left(|t-t^{'}| + |x-x^{'}|\right) + C_1\left[|v| + |v^{'}| + 1\right]\left|v - v^{'}\right|, \tag{3.45}$$

$$|N_1^{'}| \leq \frac{e}{2}C_1\left|\nabla_{(t,x)}\lambda(t_1,y)\right| \left(|t-t^{'}| + |x-x^{'}|\right)$$
$$+ C_1\left[|q^{'} - q| + e\left(|v| + |v^{'}| + 1\right)|v - v^{'}|\right], \tag{3.46}$$

$$|N_1^{''}| \leq \frac{e}{2}C_1\left|\nabla_{(t,x)}(\lambda(t_1,y) + \nu(t_1,y)\right| \left(|t-t^{'}| + |x-x^{'}|\right) \tag{3.47}$$
$$+ C_1^2\left[|q^{'} - q| + e\left(|v| + |v^{'}| + 1\right)|v - v^{'}|\right],$$

$$|N_2| \leq 2C_1|q^{'} - q| + 2eC_1\left[\frac{2}{\rho}|x - x^{'}| + \left(|v| + |v^{'}| + 1\right)|v - v^{'}|\right]$$
$$\frac{e}{2}\left(|t-t^{'}| + |x-x^{'}|\right)\left|\nabla_{(t_1,y)}\lambda(t_1,y)\right|\left[C_1 + \left|\nabla_{(t,x)}\nu(t_1,y)\right|C_1\right], \tag{3.48}$$

$$|N_3| \leq CC_1|v - v^{'}| + CC_1\left[1 + (1 + C_1)|v|^2\right]\left[\left(\frac{3}{\rho} + \frac{1}{\rho^2}\right)|x - x^{'}|\right.$$
$$\left. +2 + \frac{1}{2}\left|\nabla_{t,x}\lambda(t_1,y)\right|\left(|t-t^{'}| + |x-x^{'}|\right)\right], \tag{3.49}$$

$$|N_4| \leq CC_1|v - v^{'}| + +CC_1|v^{'}|\left[\frac{1}{\rho}|x - x^{'}| + 2 + \frac{7}{\rho}|x - x^{'}|\right], \tag{3.50}$$

$$|N_5| \leq 2eC_1\left[|v| + |v^{'}| + 1\right]|v - v^{'}| + 2C_1|q - q^{'}| + e\left[2C_1\left(\frac{6}{\rho} + \frac{1}{\rho}\right)|x - x^{'}|\right.$$
$$\left. +2\left|\nabla_{(t_1,y)}\lambda(t_1,y)\right|C_1\left(|t-t^{'}| + |x-x^{'}|\right)\right]. \tag{3.51}$$

Using the previous inequalities in (3.38), we have:

$$\left|H(t,x,v,q,u,z,w) - H(t^{'},x^{'},v^{'},q^{'},u,z,w)\right| \leq$$
$$K\left(\left|t-t^{'}\right| + \left|x-x^{'}\right| + \left|v-v^{'}\right| + \left|q-q^{'}\right|\right) \tag{3.52}$$

where $K = K(C, e, \mathcal{M}, C_1, T, \mathcal{C}, |A|, |F|, \rho, R)$, $\mathcal{C} = \max_{a,b,c}|C^a_{bc}|$ and $|v(t)| \leq \mathcal{M}$.

iii) Let $(t, x, v, q) \in [0, T[\times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$ be fixed and $(u, z, w), (u^{'}, z^{'}, w^{'}) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$. Using relation (3.21) of H, we have:

$$H(t, x, v, q, u, z, w) - H(t, x, v, q, u^{'}, z^{'}, w^{'}) =$$
$$\mathcal{A}^i \left(u_i - (u^{'})_i\right) + \mathcal{B}^i \left(z_i - (z^{'})_i\right) + \mathcal{D}^a \left(w_a - (w^{'})_a\right). \tag{3.53}$$

Invoking Proposition 3.1, taking into account $|q| = e$, $A, F \in C_0^{+\infty}([0;T] \times \Omega)$ and Remark 1, we obtain following inequalities:

$$|\mathcal{A}^i| \leq C_1, \tag{3.54}$$

$$|\mathcal{B}^i| \leq \frac{C^{'}}{2} C_1 \left(1 + (1 + C_1)\right) \mathcal{M} + \frac{C}{2} C_1 \mathcal{M} + e|F|C_1 + 3eC_1^2|F| \tag{3.55}$$

$$|\mathcal{D}^a| \leq 3e\mathcal{C}|A|C_1. \tag{3.56}$$

$$\left|H(t, x, v, q, u, z, w) - H(t, x, v, q, u^{'}, z^{'}, w^{'})\right| \leq K^{'}(\left|u - u^{'}\right| + \left|z - z^{'}\right| + \left|w - w^{'}\right|), \tag{3.57}$$

where $K^{'} = K^{'}(C, e, \mathcal{M}, \mathcal{C}, C_1, T, |A|, |F|)$.
iV) Suppose $R > 0$ be given and taking
$(t, x, v, q, u, z, w), (t^{'}, x^{'}, v^{'}, q^{'}, u^{'}, z^{'}, w^{'}) \in [0, T[\times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$,
such that $|u|, |z|, |w|, |u^{'}|, |z^{'}|, |w^{'}| \leq R$. We use relations (3.52) and (3.57) to obtain:

$$\left|H(t, x, v, q, u, z, w) - H(t^{'}, x^{'}, v^{'}, q^{'}, u^{'}, z^{'}, w^{'})\right| \leq$$
$$K^{''} \left(\left|t - t^{'}\right| + \left|x - x^{'}\right| + \left|z - z^{'}\right| + \left|u - u^{'}\right| + \left|w - w^{'}\right|\right), \tag{3.58}$$

where $K^{''} = K + K^{'}$ and $M_R = K^{''}$ is a modulus. This ends the proof of our proposition. ■

4 Existence and uniqueness theorems

This section is devoted to the existence and uniqueness results for the Cauchy-Dirichlet problem (3.26). First of all, we will give the definition of viscosity solution.

Definition 1. Let f be a real values continuous function on $[0, T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$, H and f_0 be given.

(i) f is a viscosity subsolution of (3.26) if for all function $\varphi \in C^1([0, T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}, \mathbb{R})$ such that $f - \varphi$ has a local maximum point $(t_0, x_0, v_0, q_0) \in [0, T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$, then

$$\varphi_t(t_0, x_0, v_0, q_0) + H(t_0, x_0, v_0, q_0, u_0, z_0, w_0) \leq 0, \tag{4.1}$$

where $u = \nabla_x \varphi$, $z = \nabla_v \varphi$, $w = \nabla_q \varphi$.

(ii) f is a viscosity supersolution of (3.26) if for all function $\varphi \in C^1([0, T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}, \mathbb{R})$ such that $f - \varphi$ has a local minimum point $(t_0, x_0, v_0, q_0) \in [0, T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$, then

$$\varphi_t(t_0, x_0, v_0, q_0) + H(t_0, x_0, v_0, q_0, u_0, z_0, w_0) \geq 0, \tag{4.2}$$

where $u = \nabla_x \varphi$, $z = \nabla_v \varphi$, $w = \nabla_q \varphi$.

(iii) f is a viscosity solution of (3.26) if it is both a viscosity subsolution and a viscosity supersolution.

Now we consider the following problem

$$\begin{cases} (f_t)_\varepsilon + H(t,x,v,q,u_\varepsilon,z_\varepsilon,w_\varepsilon) - \varepsilon\Delta f_\varepsilon = 0 \ in\ [0,T]\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}, \\ f_\varepsilon = f_0 \ in\ \Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}, \\ f_\varepsilon = g_\varepsilon \ in\ [0,T]\times\partial\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}. \end{cases} \tag{4.3}$$

Before starting the resolution of problem (3.26), we summarize the "*vanishing viscosity*" method described by Evans in [12]. It consists in approximating problem (3.26) by the above problem (4.3), which is a Cauchy-Dirichlet problem for a quasi-linear parabolic PDE and which turns out to have smooth solutions (see [12] paragraph 7.3.2). $\varepsilon\Delta f_\varepsilon$ in (4.3) is the term which regularize the Hamilton-Jacobi equation. Thus we hope that as $\varepsilon \longrightarrow 0$, solution (f_ε) of (4.3) will converge to a weak solution of (3.26). We suppose that $f_0 : \Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ and $g : [0,T]\times\partial\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ are Lipschitz continuous bounded functions and $T > 0$ a real number are given. We need the following lemma.

Lemma 4.1. *Let I be a compact subset of $[0;T]\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}$. Let $f : I \longrightarrow \mathbb{R}$ a continuous function such that there exists $\varphi \in C^1(I,\mathbb{R})$ such that $f-\varphi$ has a strict local maximum at (t_0,x_0,v_0,q_0).*
If (f_n) is a sequence of functions which uniformly converge to f, then there exists a sequence of points $(t_n,x_n,v_n,q_n)_{n\in\mathbb{N}}$ such that :

$$\begin{cases} (t_n,x_n,v_n,q_n) \longrightarrow (t_0,x_0,v_0,q_n) \\ f_n(t_n,x_n,v_n,q_n) \underset{n\longrightarrow+\infty}{\longrightarrow} f(t_0,x_0,v_0,q_0) \\ f_n-\varphi \ \textit{has a local maximum at}\ (t_n,x_n,v_n,q_n),\ \textit{for all n at some range.} \end{cases}$$

<u>Proof</u>. Let I be a compact subset of $[0;T]\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}$, f a continuous function on I. There exists $\varphi\in C^1(I,\mathbb{R})$ such that $f-\varphi$ has a strict local maximum at (t_0,x_0,v_0,q_0). We assume that $(f_n)_n$ is a sequence of functions which uniformly converges to f and we show that there exists a sequence of points $(t_n,x_n,v_n,q_n)_{n\in\mathbb{N}}$ such that

$$\begin{cases} (t_n,x_n,v_n,q_n) \longrightarrow (t_0,x_0,v_0,q_0) \\ f_n(t_n,x_n,v_n,q_n) \underset{n\longrightarrow+\infty}{\longrightarrow} f(t_0,x_0,v_0,q_0) \\ f_n-\varphi \ \textit{has a local maximum at}\ (t_n,x_n,v_n,q_n),\ \textit{for all}\ \text{n}\ \textit{at some range.} \end{cases}$$

As $f-\varphi$ has a strict local maximum at (t_0,x_0,v_0,q_0), for $\rho = \dfrac{1}{n}$ enough small we can find ε_ρ such that if

$$|(t,x,v,q)-(t_0,x_0,v_0,q_0)| \le \rho,$$

then

$$f(t,x,v,q)-\varphi(t,x,v,q)+\varepsilon_\rho < f(t_0,x_0,v_0,q_0)-\varphi(t_0,x_0,v_0,q_0). \tag{4.4}$$

Since $(f_n)_n$ is uniformly convergent to f, then there exists $N_\rho\in\mathbb{N}$ such that:

$$\forall n > N_\rho,\ \forall z\in\Omega,\ |f_n(z)-f(z)| \le \frac{\varepsilon_\rho}{4}. \tag{4.5}$$

Thus we can write

$$f(t_0,x_0,v_0,q_0) - \frac{\varepsilon_\rho}{4} \leq f_n(t_0,x_0,v_0,q_0), \qquad f_n(t,x,v,q) \leq f(t,x,v,q) + \frac{\varepsilon_\rho}{4}. \tag{4.6}$$

Now relations (4.4) and (4.6), yield to the inequality:

$$f_n(t,x,v,q) - \varphi(t,x,v,q) + \frac{\varepsilon_\rho}{2} \leq f_n(t_0,x_0,v_0,q_0) - \varphi(t_0,x_0,v_0,q_0). \tag{4.7}$$

$f_n - \varphi$ is then bounded, hence there exists (t_n,x_n,v_n,q_n) such that $f_n - \varphi$ has a maximum at (t_n,x_n,v_n,q_n) in the ball $B((t_0,x_0,v_0,q_0),\rho)$ at some range $n > N_\rho$.
In the other hand, when $n \longrightarrow +\infty$, $\rho \longrightarrow 0$, then we deduce that

$$(t_n,x_n,v_n,q_n) \longrightarrow (t_0,x_0,v_0,q_0). \tag{4.8}$$

Since $(f_n)_n$ uniformly converges to f, then we have

$$f_n(t_n,x_n,v_n,q_n) \longrightarrow f(t_0,x_0,v_0,q_0). \tag{4.9}$$

So, relations (4.7), (4.8) and (4.9) conclude the proof. ■

Theorem 4.1. *(existence)*
Let $(u_n)_{n\in\mathbb{N}}$ be a non negative sequence of reals which converges to 0; $(f_{u_n})_{n\in\mathbb{N}}$ a class of two differentiable sequence functions, solutions of (4.3) for $\varepsilon = u_n$.

(a) There exists f such that:

$$f_{u_n} \underset{n\longrightarrow+\infty}{\longrightarrow} f \text{ in } C([0,T]\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1},\mathbb{R}) \text{ for the infinite norm on all compact},$$

(b) f is a viscosity solution of problem (3.26).

(c) If $f_\varepsilon = g_\varepsilon$ on $[0,T]\times\partial\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}$, $g_\varepsilon \longrightarrow g$ on $C([0,T]\times\partial\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1},\mathbb{R})$ and $f_{u_n} \longrightarrow f$ on $C([0,T]\times\bar\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1},\mathbb{R})$ then $f|_{[0,T]\times\partial\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}} = g$.

Proof. a) Let $(f_{u_n})_n$ be a smooth sequence of real-value functions of (4.3), I be a compact subset of $[0;T[\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}$. Thus $\forall n\in\mathbb{N}$, f_{u_n} is a uniformly continuous function on I, hence $(f_{u_n})_n$ is a uniformly equicontinuous familly of functions and bounded. Then by Arzela-Ascoli theorem, there exists a subsequence $(f_{u_{n_j}})_j \subset (f_{u_n})_n$ and a continuous function f, such that

$$(f_{u_{n_j}})_j \longrightarrow f \text{ uniformly on } I.$$

b) First of all, we show that f is a viscosity subsolution. We will then suppose that $f-\varphi$ has a strict local maximum at (t_0,x_0,v_0,q_0). Consider first that φ is a class two differentiable function. Applying the previous Lemma in $B((0_{\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}}),\varepsilon)_{\varepsilon>0}$ which is compact, there exists (t_n,x_n,v_n,q_n) such that

$$\begin{cases} (t_n,x_n,v_n,q_n) \longrightarrow (t_0,x_0,v_0,q_0), \qquad f_{u_n}(t_n,x_n,v_n,q_n) \underset{n\longrightarrow+\infty}{\longrightarrow} f(t_0,x_0,v_0,q_0) \\ f_{u_n} - \varphi \text{ has a local maximum at } (t_n,x_n,v_n,q_n), \text{ for all n at some range.} \end{cases}$$

Thus

$$\partial_t f_{u_n}(t_n,x_n,v_n,q_n) = \partial_t\varphi(t_n,x_n,v_n,q_n), \quad \partial_x f_{u_n}(t_n,x_n,v_n,q_n) = \partial_x\varphi(t_n,x_n,v_n,q_n),$$

$$\partial_v f_{u_n}(t_n, v_n, v_n, q_n) = \partial_v\varphi(t_n, x_n, v_n, q_n), \quad \partial_q f_{u_n}(t_n, v_n, v_n, q_n) = \partial_q\varphi(t_n, x_n, v_n, q_n), \tag{4.10}$$

and

$$\Delta f_{u_n}(t_n, x_n, v_n, q_n) \leq \Delta\varphi(t_n, x_n, v_n, q_n). \tag{4.11}$$

Since f_{u_n} is solution of (4.3) then

$$\varphi_t(t_n, x_n, v_n, q_n) + H_\varphi(t_n, x_n, v_n, q_n, \bar{u}_n, \bar{z}_n, \bar{w}_n) \leq u_n\Delta\varphi(t_n, x_n, v_n, q_n). \tag{4.12}$$

Thus, we obtain passing to the limit as $n \longrightarrow +\infty$ in (4.12), since φ and H are continuous, we obtain:

$$\varphi_t(t_0, x_0, v_0, q_0) + H_\varphi(t_0, x_0, v_0, q_0, \bar{u}_0, \bar{z}_0, \bar{w}_0) \leq 0. \tag{4.13}$$

For φ a class one differentiable function, we rather consider a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of class two differentiable functions which uniformly converges to φ on all compact and the sequence of their derivatives also converges uniformly on all compact to the derivative of φ. To show that f is a viscosity supersolution, we use the same previous argument with reverse inequalities (4.11) and (4.12), thus (i) and (ii) show that f is the viscosity solution of (3.26).

c) We suppose that

$$\begin{aligned} &f_\varepsilon = g_\varepsilon \textit{ on } [0,T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}, \quad g_\varepsilon \longrightarrow g \textit{ on } C([0,T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}, \mathbb{R}) \\ &f_{u_n} \longrightarrow f \textit{ on } C([0,T] \times \bar{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^{N-1}, \mathbb{R}) \end{aligned} \tag{4.14}$$

and we show that $\forall\ (t, x, v, q) \in [0, T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$, $f(t, x, v, q) = g(t, x, v, q)$.
Now $[0,T]\times\partial\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}$ being a closed subset, then there exists a subsequence $(t_k, x_k, v_k, q_k)_k \subset [0, T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$ such that $(t_k, x_k, v_k, q_k) \longrightarrow (t, x, v, q)$ on $[0, T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$. By continuity of f_{u_n}, f and g we have

$$f_{u_n}(t_k, x_k, v_k, q_k) \longrightarrow f(t_k, x_k, v_k, q_k) \longrightarrow f(t, x, v, q).$$

In addition, (4.14) gives:

$$f_{u_n}(t_k, x_k, v_k, q_k) \longrightarrow g(t_k, x_k, v_k, q_k) \longrightarrow g(t, x, v, q).$$

Since $C([0, T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1})$ is a locally convex separated space, we obtain

$$f|_{[0,T]\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}} = g$$

■

Theorem 4.2. *If H in (3.33) verifies estimates of the Proposition 3.2, then the problem (3.26) has one and only one uniformly continuous and bounded viscosity solution in $[0;T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$.*

A classical and very useful corollary of the previous theorem is:

Corollaire 1. Under assumptions of the previous theorem, if $f, \bar{f} \in C([0;T] \times \bar{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^{N-1})$ are respectivly viscosity sub and super-solutions of (3.26) then:

$$\underset{[0;T]\times\bar{\Omega}\times\mathbb{R}^3\times\mathbb{R}^{N-1}}{max}(f - \bar{f}) \leq \underset{[0;T]\times\partial\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}}{max}(f - \bar{f}) \tag{4.15}$$

Proof. See ([3] corollary 2.2) ■

Proof. (of theorem) We suppose that there exists two viscosities solutions f and $\bar{f}$ of problem (3.26), uniformly continuous and bounded in $[0;T]\times\bar{\Omega}\times\mathbb{R}^3\times\mathbb{R}^{N-1}$ with the same initial condition f_0.

We take f as viscosity sub-solution and $\bar{f}$ as viscosity supersolution which have same initial data. We are interested in the $\max\limits_{[0;T]\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}}(f-\bar{f})=M$.

Using the absurd reasoning, we suppose that $M>0$:

Step 1: Let $(\varepsilon,\alpha)\in]0,1[^2$, for all $(t,x,v,q),(t^{'},x^{'},v^{'},q^{'})\in[0;T]\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}$, we set:

$$\Phi_\varepsilon(t,t^{'},x,x^{'},v,v^{'},q,q^{'})=f(t,x,v,q)-\bar{f}(t^{'},x^{'},v^{'},q^{'})-\alpha(t+t^{'}) \tag{4.16}$$
$$-\varepsilon(|x|^2+|x^{'}|^2+|v|^2+|v^{'}|^2+|q|^2+|q^{'}|^2)-\frac{1}{\varepsilon^2}\left(\left|v-v^{'}\right|^2+\left|x-x^{'}\right|^2+\left|q-q^{'}\right|^2+|t-t^{'}|^2\right).$$

We choose in inequality (4.16)

$$(t,x,v,q)=(t^{'},x^{'},v^{'},q^{'})\in[0;T[\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1},$$

and we can fixe α and ε too small such that

$$\sup_{[0;T[\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}}\{f(t,x,v,q)-\bar{f}(t,x,v,q)-2\alpha t-2\varepsilon(|x|^2+|v|^2+|q|^2)\}\geq\frac{M}{2}.$$

Since $\Phi_\varepsilon\to-\infty$ when $|x|,|x^{'}|,|v|,|v^{'}|,|q|,|q^{'}|\to+\infty$ then the supremum of Φ_ε is reached at a certain point $(t_\varepsilon,t^{'}_\varepsilon,x_\varepsilon,x^{'}_\varepsilon,v_\varepsilon,v^{'}_\varepsilon,q_\varepsilon,q^{'}_\varepsilon)$.

In orther side, we have :

$$f(t_\varepsilon,x_\varepsilon,v_\varepsilon,q_\varepsilon)-\bar{f}(t^{'}_\varepsilon,x^{'}_\varepsilon,v^{'}_\varepsilon,v^{'}_\varepsilon)-\varepsilon(|x_\varepsilon|^2+|x^{'}_\varepsilon|^2+|v_\varepsilon|^2+|v^{'}_\varepsilon|^2+|q_\varepsilon|^2+|q^{'}_\varepsilon|^2)$$
$$-\frac{1}{\varepsilon^2}\left(\left|v_\varepsilon-v^{'}_\varepsilon\right|^2+\left|x_\varepsilon-x^{'}_\varepsilon\right|^2+|t_\varepsilon-t^{'}_\varepsilon|^2+|q_\varepsilon-q^{'}_\varepsilon|^2\right)-\alpha(t_\varepsilon+t_\varepsilon)\geq0,$$

hence,

$$f(t_\varepsilon,x_\varepsilon,v_\varepsilon,q_\varepsilon)-\bar{f}(t^{'}_\varepsilon,x^{'}_\varepsilon,v^{'}_\varepsilon,q^{'}_\varepsilon)\geq+\alpha(t_\varepsilon+t_\varepsilon)$$
$$+\varepsilon(|x_\varepsilon|^2+|x^{'}_\varepsilon|^2+|v_\varepsilon|^2+|v^{'}_\varepsilon|^2+|q_\varepsilon|^2+|q^{'}_\varepsilon|^2)$$
$$+\frac{1}{\varepsilon^2}\left(\left|v_\varepsilon-v^{'}_\varepsilon\right|^2+\left|x_\varepsilon-x^{'}_\varepsilon\right|^2+\left|q_\varepsilon-q^{'}_\varepsilon\right|^2+|t_\varepsilon-t^{'}_\varepsilon|^2\right).$$

Since f and $\bar{f}$ are bounded, the above inequality yields to:

$$\mathcal{R}+\bar{\mathcal{R}}\geq+\alpha(t_\varepsilon+t^{'}_\varepsilon)+\varepsilon(|x_\varepsilon|^2+|x^{'}_\varepsilon|^2+|v_\varepsilon|^2+|v^{'}_\varepsilon|^2+|q_\varepsilon|^2+|q^{'}_\varepsilon|^2)$$
$$+\frac{1}{\varepsilon^2}\left(\left|v_\varepsilon-v^{'}_\varepsilon\right|^2+\left|x_\varepsilon-x^{'}_\varepsilon\right|^2+\left|q_\varepsilon-q^{'}_\varepsilon\right|^2+|t_\varepsilon-t^{'}_\varepsilon|^2\right), \tag{4.17}$$

where $\mathcal{R}$ and $\bar{\mathcal{R}}$ are respectively bounded constants of f and $\bar{f}$.

We obtain:

$$|t_\varepsilon-t^{'}_\varepsilon|\leq\varepsilon\sqrt{2\mathbf{R}},\quad \left|x_\varepsilon-x^{'}_\varepsilon\right|\leq\varepsilon\sqrt{2\mathbf{R}},\quad \left|v_\varepsilon-v^{'}_\varepsilon\right|\leq\varepsilon\sqrt{2\mathbf{R}},\quad \left|q_\varepsilon-q^{'}_\varepsilon\right|\leq\varepsilon\sqrt{2\mathbf{R}}, \tag{4.18}$$

with $\mathbf{R}=\max\{\mathcal{R};\bar{\mathcal{R}}\}$.

Step 2: Since $\bar{f}$ is uniformly continuous on $[0;T[\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}$, then we introduce the modulus of continuity of $\bar{f}$ whose expression is given by:

$$m_{\bar{f}}(l) = \sup_{|t-t'|\leq l, |x-x'|\leq l, |v-v'|\leq l, |q-q'|\leq l} |\bar{f}(t,x,v,q) - \bar{f}(t',x',v',q')|,$$

which $m_{\bar{f}}(l) \longrightarrow 0$ when $l \longrightarrow 0$ or simply $m_{\bar{f}}(0) = 0$.
Furthermore, we obtain after substitutions:

$$\begin{aligned}
\frac{M}{2} &\leq \Phi(t'_\epsilon, t_\epsilon, x'_\epsilon, x_\epsilon, v'_\epsilon, v_\epsilon, q'_\epsilon, q_\epsilon) \\
&\leq f(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) - \bar{f}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, q'_\varepsilon) \\
&\leq f(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) - f(0, x_\varepsilon, v_\varepsilon, q_\varepsilon) + f(0, x_\varepsilon, v_\varepsilon, q_\varepsilon) - \bar{f}(0, x_\varepsilon, v_\varepsilon, q_\varepsilon) \\
&+ \bar{f}(0, x_\varepsilon, v_\varepsilon, q_\varepsilon) - \bar{f}(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) + \bar{f}(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) - \bar{f}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, q'_\varepsilon) \\
&\leq m_f(l) + 0 + m_{\bar{f}}(l) + m_{\bar{f}}(l);
\end{aligned}$$

finally, we obtain:

$$\frac{M}{2} \leq m_f(l) + m_{\bar{f}}(l) + m_{\bar{f}}(\varepsilon\sqrt{2\mathbf{R}}).$$

In particular, if $l = t_\varepsilon$ we have:

$$\frac{M}{2} \leq m_f(t_\varepsilon) + m_{\bar{f}}(t_\varepsilon) + m_{\bar{f}}(\varepsilon\sqrt{2\mathbf{R}}).$$

Moreover, if $t_\varepsilon = 0$, we get $M = 0$ when ε tends to 0, what is absurd. So there exists $\nu > 0$ such that $t_\varepsilon \geq \nu$. Similarly, following the same method as before with the variable $(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, q'_\varepsilon)$, it is shown that there exists $\nu > 0$ such that $t'_\varepsilon \geq \nu$.
Step 3: If $(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) \in [0,T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$ and $(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, q'_\varepsilon) \in [0,T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$, then we have

$$\begin{aligned}
f(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) - \bar{f}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, q'_\varepsilon) &\leq f(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) - \bar{f}(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) \\
&+ m_{\bar{f}}\left(\left|v_\varepsilon - v'_\varepsilon\right| + \left|x_\varepsilon - x'_\varepsilon\right| + \left|q_\varepsilon - q'_\varepsilon\right| + |t_\varepsilon - t'_\varepsilon|\right) \\
&\leq 0 + m_{\bar{f}}(4\varepsilon\sqrt{2\mathbf{R}}),
\end{aligned}$$

because $f \leq \bar{f}$ on $[0,T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$.
Now $f(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) - \bar{f}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, q'_\varepsilon) \geq \frac{M}{2}$, so $\frac{M}{2} \leq m_{\bar{f}}(4\varepsilon\sqrt{2\mathbf{R}})$.
For ε enough small, we obtain a contradiction, thus $(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) \in [0,T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$.
Similarly, we give proof for $(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, q'_\varepsilon) \in [0,T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$.
Step 4: We see that $(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon)$ is a maximum point of

$$\Theta(t,x,v,q) = f(t,x,v,q) - \varphi^1_\varepsilon(t,x,v,q),$$

where

$$\begin{aligned}
\varphi^1_\varepsilon(t,x,v,q) &= \bar{f}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, q'_\varepsilon) + \frac{1}{\varepsilon^2}\left(\left|v - v'_\varepsilon\right|^2 + \left|x - x'_\varepsilon\right|^2 + \left|q - q'_\varepsilon\right|^2 + |t - t'_\varepsilon|^2\right) \\
&+ \varepsilon(|x|^2 + |x'_\varepsilon|^2 + |v|^2 + |v'_\varepsilon|^2 + |q'_\varepsilon|^2 + |q|^2) + \alpha(t + t'_\varepsilon). \qquad (4.19)
\end{aligned}$$

$\Theta(t,x,v,q)$ is class $\mathcal{C}^1$ and thus $(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon)$ is a maximum point of
$\varphi^1_\varepsilon(t,x,v,q) = f(t,x,v,q) - \Theta(t,x,v,q)$.
f is a viscosity subsolution of (3.26) and $(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) \in [0;T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$, thus we obtain:

$$(\varphi^1_\varepsilon)_t(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) = \alpha + \frac{2}{\varepsilon^2}(t_\varepsilon - t'_\varepsilon), \quad (\nabla\varphi^1_\varepsilon)_x(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon) = \frac{2}{\varepsilon^2}(x_\varepsilon - x'_\varepsilon) + 2\varepsilon x_\varepsilon,$$

$$(\nabla\varphi_\varepsilon^1)_v(t_\varepsilon,x_\varepsilon,v_\varepsilon,q_\varepsilon)=\frac{2}{\varepsilon^2}(v_\varepsilon-v_\varepsilon^{'})+2\varepsilon v_\varepsilon,\quad (\nabla\varphi_\varepsilon^1)_q(t_\varepsilon,x_\varepsilon,v_\varepsilon,q_\varepsilon)=\frac{2}{\varepsilon^2}(q_\varepsilon-q_\varepsilon^{'})+2\varepsilon q_\varepsilon,$$

thus

$$\frac{\partial\varphi_\varepsilon^1}{\partial t_\varepsilon}+H\left(t_\varepsilon,x_\varepsilon,v_\varepsilon,q_\varepsilon,(\nabla\varphi_\varepsilon^1)_x,(\nabla\varphi_\varepsilon^1)_v,(\nabla\varphi_\varepsilon^1)_q\right)\le 0. \tag{4.20}$$

Finally

$$\begin{aligned}&\alpha+\frac{2}{\varepsilon^2}(t_\varepsilon-t_\varepsilon^{'})+\\ &H\left(t_\varepsilon,x_\varepsilon,v_\varepsilon,q_\varepsilon,\frac{2}{\varepsilon^2}(x_\varepsilon-x_\varepsilon^{'})+2\varepsilon x_\varepsilon,\frac{2}{\varepsilon^2}(v_\varepsilon-v_\varepsilon^{'})+2\varepsilon v_\varepsilon,\frac{2}{\varepsilon^2}(q_\varepsilon-q_\varepsilon^{'})+2\varepsilon q_\varepsilon\right)\le 0.\end{aligned} \tag{4.21}$$

Using the same skills, $(t_\varepsilon^{'},x_\varepsilon^{'},v_\varepsilon^{'},q_\varepsilon^{'})$ is a minimum point of

$$\Sigma(t^{'},x^{'},v^{'},q^{'})=\bar{f}(t^{'},x^{'},v^{'},q^{'})+\varphi_\varepsilon^2(t^{'},x^{'},v^{'},q^{'}),$$

with

$$\begin{aligned}\varphi_\varepsilon^2(t^{'},x^{'},v^{'},q^{'})=f(t_\varepsilon,x_\varepsilon,v_\varepsilon,q_\varepsilon)-\frac{1}{\varepsilon^2}\left(\left|v_\varepsilon-v^{'}\right|^2+\left|x_\varepsilon-x^{'}\right|^2+|t_\varepsilon-t^{'}|^2+\left|q_\varepsilon-q^{'}\right|^2\right)\\ -\varepsilon(|x_\varepsilon|^2+|x^{'}|^2+|v_\varepsilon|^2+|v^{'}|^2+|q_\varepsilon|^2+|q^{'}|^2)-\alpha(t+t_\varepsilon^{'}).\end{aligned}$$

Thus we see that $(t_\varepsilon^{'},x_\varepsilon^{'},v_\varepsilon^{'},q_\varepsilon^{'})$ is a minimum point of
$\varphi_\varepsilon^2(t^{'},x^{'},v^{'},q^{'})=\Sigma(t^{'},x^{'},v^{'},q^{'})-\bar{f}(t^{'},x^{'},v^{'},q^{'})$.
$\bar{f}$ is a viscosity super-solution of (3.26) and $(t_\varepsilon^{'},x_\varepsilon^{'},v_\varepsilon^{'},q_\varepsilon^{'})\in[0;T]\times\Omega\times\mathbb{R}^3\times\mathbb{R}^{N-1}$. we obtain:

$$(\varphi_\varepsilon^2)_t^{'}(t_\varepsilon^{'},x_\varepsilon^{'},v_\varepsilon^{'},q_\varepsilon^{'})=-\alpha+\frac{2}{\varepsilon^2}(t_\varepsilon^{'}-t),\quad (\nabla\varphi^2)_{x^{'}}(t_\varepsilon^{'},x_\varepsilon^{'},v_\varepsilon^{'},q_\varepsilon^{'})=\frac{2}{\varepsilon^2}(x_\varepsilon-x_\varepsilon^{'})-2\varepsilon x_\varepsilon^{'},$$

$$(\nabla\varphi^2)_{v^{'}}(t_\varepsilon^{'},x_\varepsilon^{'},v_\varepsilon^{'},q_\varepsilon^{'})=\frac{2}{\varepsilon^2}(v_\varepsilon-v_\varepsilon^{'})-2\varepsilon v_\varepsilon^{'},\quad (\nabla\varphi^2)_{q^{'}}(t_\varepsilon^{'},x_\varepsilon^{'},v_\varepsilon^{'},q_\varepsilon^{'})=\frac{2}{\varepsilon^2}(q_\varepsilon-q_\varepsilon^{'})-2\varepsilon q_\varepsilon^{'},$$

thus

$$\frac{\partial\varphi_\varepsilon^2}{\partial t_\varepsilon^{'}}+H\left(t_\varepsilon^{'},x_\varepsilon^{'},v_\varepsilon^{'},q_\varepsilon^{'},(\nabla\varphi^2)_{x^{'}},(\nabla\varphi^2)_{v^{'}},(\nabla\varphi^2)_{q^{'}}\right)\ge 0. \tag{4.22}$$

Finally

$$\begin{aligned}&-\alpha+\frac{2}{\varepsilon^2}(t_\varepsilon-t_\varepsilon^{'})+\\ &H\left(t_\varepsilon^{'},x_\varepsilon^{'},v_\varepsilon^{'},q_\varepsilon^{'},\frac{2}{\varepsilon^2}(x_\varepsilon-x_\varepsilon^{'})-2\varepsilon x_\varepsilon^{'},\frac{2}{\varepsilon^2}(v_\varepsilon-v_\varepsilon^{'})-2\varepsilon v_\varepsilon^{'},\frac{2}{\varepsilon^2}(q_\varepsilon-q_\varepsilon^{'})-2\varepsilon q_\varepsilon^{'}\right)\ge 0.\end{aligned} \tag{4.23}$$

When we subtract inequality (4.23) of inequality (4.21), we have:

$$\begin{aligned}2\alpha\le H&\left(t_\varepsilon^{'},x_\varepsilon^{'},v_\varepsilon^{'},q_\varepsilon^{'},\frac{2}{\varepsilon^2}(x_\varepsilon-x_\varepsilon^{'})+2\varepsilon x_\varepsilon,\frac{2}{\varepsilon^2}(v_\varepsilon-v_\varepsilon^{'})+2\varepsilon v_\varepsilon,\frac{2}{\varepsilon^2}(q_\varepsilon-q_\varepsilon^{'})+2\varepsilon q_\varepsilon\right)\\ -H&\left(t_\varepsilon,x_\varepsilon,v_\varepsilon,q_\varepsilon,\frac{2}{\varepsilon^2}(x_\varepsilon-x_\varepsilon^{'})-2\varepsilon x_\varepsilon^{'},\frac{2}{\varepsilon^2}(v_\varepsilon-v_\varepsilon^{'})-2\varepsilon v_\varepsilon^{'},\frac{2}{\varepsilon^2}(q_\varepsilon-q_\varepsilon^{'})-2\varepsilon q_\varepsilon^{'}\right),\end{aligned}$$

in addition,

$$2\alpha\le H(t_\varepsilon^{'},x_\varepsilon^{'},v_\varepsilon^{'},q_\varepsilon^{'},\frac{2}{\varepsilon^2}(x_\varepsilon-x_\varepsilon^{'})+2\varepsilon x_\varepsilon,\frac{2}{\varepsilon^2}(v_\varepsilon-v_\varepsilon^{'})+2\varepsilon v_\varepsilon,\frac{2}{\varepsilon^2}(q_\varepsilon-q_\varepsilon^{'})+2\varepsilon q_\varepsilon)$$

$$
\begin{aligned}
&- H(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - x_\varepsilon^{'}) + 2\varepsilon x_\varepsilon, \frac{2}{\varepsilon^2}(v_\varepsilon - v_\varepsilon^{'}) + 2\varepsilon v_\varepsilon, \frac{2}{\varepsilon^2}(q_\varepsilon - q_\varepsilon^{'}) + 2\varepsilon q_\varepsilon)\\
&+ H(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - x_\varepsilon^{'}) + 2\varepsilon x_\varepsilon, \frac{2}{\varepsilon^2}(v_\varepsilon - v_\varepsilon^{'}) + 2\varepsilon v_\varepsilon, \frac{2}{\varepsilon^2}(q_\varepsilon - q_\varepsilon^{'}) + 2\varepsilon q_\varepsilon)\\
&- H(t_\varepsilon, x_\varepsilon, v_\varepsilon, q_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - x_\varepsilon^{'}) - 2\varepsilon x_\varepsilon^{'}, \frac{2}{\varepsilon^2}(v_\varepsilon - v_\varepsilon^{'}) - 2\varepsilon v_\varepsilon^{'}, \frac{2}{\varepsilon^2}(q_\varepsilon - q_\varepsilon^{'}) - 2\varepsilon q_\varepsilon^{'}).
\end{aligned}
$$

Using Proposition 3.2, we obtain:

$$
2\alpha \leq \varepsilon K \left(\left| x_\varepsilon + x_\varepsilon^{'} \right| + \left| v_\varepsilon + v_\varepsilon^{'} \right| + \left| q_\varepsilon + q_\varepsilon^{'} \right| \right) + K^{'} \left(\left| t_\varepsilon - t_\varepsilon^{'} \right| + \left| x_\varepsilon - x_\varepsilon^{'} \right| + \left| v_\varepsilon^{'} - v_\varepsilon \right| + \left| q_\varepsilon^{'} - q_\varepsilon \right| \right),
$$

so

$$
\alpha \leq \varepsilon \frac{K}{2} \left(\left| x_\varepsilon + x_\varepsilon^{'} \right| + \left| v_\varepsilon + v_\varepsilon^{'} \right| + \left| q_\varepsilon + q_\varepsilon^{'} \right| \right) + \frac{K^{'}}{2} \left(\left| t_\varepsilon - t_\varepsilon^{'} \right| + \left| x_\varepsilon - x_\varepsilon^{'} \right| + \left| v_\varepsilon^{'} - v_\varepsilon \right| + \left| q_\varepsilon^{'} - q_\varepsilon \right| \right), \tag{4.24}
$$

and using (4.18), we easily obtain $\alpha = 0$.
This is absurd because $\alpha > 0$ hence $M \leq 0$, thus $f \leq \bar{f}$.
ii) Reverse inequality is obtained by the same way, just interchange the role of f and $\bar{f}$.
So $f = \bar{f}$. ∎

Theorem 4.3. *(uniqueness)*
Let $T > 0$, $f_0 : \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ and $g : [0;T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ are given.

1. *The Cauchy-Dirichlet problem (3.26)*

$$
\begin{cases}
f_t(t,x,v,q) + H(t,x,v,q,u,z,w) = 0 \quad in \quad [0,T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}\\
f(0,x,v,q) = f_0(x,v,q) \quad in \quad \Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}\\
f(t,x,v,q) = g(x,v,q) \quad in \quad [0;T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}
\end{cases}
$$

   has one unique continuous viscosity solution.

2. *The relativistic Vlasov equation (2.1) in spherically symmetric space-time, admits a unique continuous viscosity solution $f = f(t,x,v,q)$ in $[0,+\infty[\times\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$ which satisfies the initial data $f(0,x,v,q) = f_0(x,v,q)$ in $\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$ and boundary data $f(t,x,v,q) = g(t,x,v,q)$ in $[0;T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$.*

Proof. 1) Let $R > 0$ and consider two continuous viscosities solutions f and $\bar{f}$ of problem (3.26) on $[0;T] \times \left(\overline{B}(0_{\mathbb{R}^3,R}) \backslash \overline{B}(0_{\mathbb{R}^3}, \rho) \times \overline{B}(0_{\mathbb{R}^3 \times \mathbb{R}^{N-1}}, R)\right)_{R>\rho}$, where they are uniformly continuous and bounded, next apply the previous theorem.
2) The conclusion comes from the equivalence between the relativistic Vlasov equation with the initial data $f(0,x,v,q) = f_0(x,v,q)$ in $\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$, boundary data $f(t,x,v,q) = g(t,x,v,q)$ in $[0;T] \times \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$ and the Cauchy-Dirichlet problem (3.26) ∎

References

[1] A. Alves, Equation de liouville pour les particules de masse nulle, C. R. Acad. Sci. paris, **278**, (1987), 1151-1154.

[2] R. D. Ayissi, R. M. Etoua and H. P. M. Tchagna, Viscosity solutions for the one-body liouville equation in Yang-Mills charged bianchi models with non-zero mass, Lett. Math. Phys., 105, (2015), 1289-1299.

[3] G. Barles, Solutions de viscosité des équations de Hamilton-Jacobi, Springer-Verlag Berlin Heidelberg, (1994).

[4] J. Binney and S. Tremaine, Galactic dynamics, Princeton University Press, (1987).

[5] A. Bressan, Viscosity solutions of hamilton-jacobi equations and optimal control problems, Adv. Pure Appl. Math., (2001).

[6] A. Briani and F. Rampazzo, A densty approach to hamilton-jacobi equation with t-measurable hamiltonians, Nonlinear differ. equa. appl., **12**, (2005).

[7] Y. Choquet-Bruhat, Existence and uniqueness for the Einstein-Maxwell-Vlasov system, Volume dedicated to Petrov, (1971).

[8] M. G. Crandall and H. Ishii and P. L. Lions, User's guide to viscosity solutions of second order partial differential equation, Amer. Math. Soc., **27**(1), (1992), 1-67.

[9] D. Dongo and M. K. Foko, Viscosity solutions for the Vlasov equation in the presence of a Yang-Mills Field in temporal gauge, Afr. Diaspora J. Math., **23**(1), (2020), 24-39.

[10] Y. Choquet-Bruhat et N. Noutchegueme, Solution globale des équations de Yang-Mills-Vlasov (masse nulle), C. R. Acad. Sci. Paris, **311**, (1991).

[11] Y. Choquet-Bruhat et N. Noutchegueme, Système de Yang-Mills-Vlasov en jauge temporelle, Ann. Inst. Henri Poincaré, **55**, (1991).

[12] L. C. Evans, Partial Differential Equations, A. M. S., (1997).

[13] G. Rein and A. D. Rendall, Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data commun, Math. Phys., **150**, (1992), 561-583.

[14] Shigeaki Koike, A Beginner's Guide to the Theory of viscosity solutions, Society of Japon, (2002).

[15] P. L. Lions, Generalized solutions of Hamilton-Jacobi equation, Research Notes in Mathematics, **69**, (1982).

[16] N. Noutchegueme and P. Noundjeu, Système de Yang-Mills-Valsov pour des particules avec des densités de charge de jauge non-abélienne sur un espace-temps courbe, Ann. Inst. Henri Pointcaré, (2000).

[17] A. D. Rendall, Global properties of locally spatially homogeneous cosmological models with matter, Math. Proc. Camb. Phil. Soc, **118**, (1995), 511-526.

[18] A. D. Rendall and C. Uggla, Dynamics of spatially homogeneous locally rotationally symmetric solutions of the Einsein-Vlasov equations, Class. Quantum Grav, **17**, (2000), 4697-4714.

[19] G. Börner, The early universe, Springer, 1971.

[20] E. Takou and F. L. C. Ciake, The relativistic Boltzmann equation on a spherically, symmetric gravitational Field Class. Quantum Grav., **34**(1), (2017), 33.

[21] Server S, Dragomir and Young-Ho Kim, On nonlinear integral inequalities of gronwall type in two variables, RGMIA research report collection, **6**(2), (2003).

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