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# Adjoint relations between the category of poset acts and some other categories

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Dedicated to my mother Malak

**Abstract**. In this paper, first the congruences in the category **PosAct**-$S$ of all poset acts over a pomonoid $S$; an $S$-act in the category **Pos** of all posets, with action preserving monotone maps between them, are introduced. Then, we study the existence of the free and cofree objects in the category **PosAct**-$S$. More precisely, we consider all forgetful functors between this category and the categories **Pos**-$S$ of all $S$-posets, **Pos** of all posets, **Act**-$S$ of all $S$-acts, and **Set** of all sets, and we study the existence of their left and right adjoints. It is shown that the category **Pos**-$S$ is a full reflective and coreflective subcategory of **PosAct**-$S$.

***Keywords.*** Poset act, free object, cofree object, adjoint pair.

## 1 Introduction

The action of a monoid $S$ on a set, namely $S$-act, is an important algebraic structure in mathematics and other mathematical areas such as graph theory and algebraic automata theory as well as in computer science. For example, computer scientists use the notion of a projection algebra (sets with an action of the monoid $(\mathbb{N}^{\infty}, min)$) as a convenient means of algebraic specification of process algebras (see [7, 10]). Combining the notions of a poset and an act, many algebraic and categorical properties of the category of actions of a pomonoid on a poset, namely $S$-poset, have been studied. In fact, $S$-posets appear naturally in the study of mappings between posets (see [6]). More precisely, as $S$-acts correspond to representations of monoids by transformations of sets, $S$-posets correspond to order preserving representations of pomonoids by order preserving transformations of posets. Preliminary work on properties of $S$-posets was done by Fakhruddin in the 1980s (see [8] and [9]), and was continued in recent papers [2, 3, 4, 5, 11, 12, 13, 14]. In the present paper, actions of a pomonoid $S$ on a set, $S$-acts as unary algebraic structures, are investigated as algebras in the category **Pos**. Even when $S$ is a pogroup the notions of $S$-poset and poset act are not the same and this motivates the author to study poset acts as a generalization of $S$-posets. The category of poset acts with action preserving monotone maps between

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them, first has been introduced and studied by Skornyakov in [15] where it is shown that the category of posets is fundamental and fundamentality is defined via the category **PosAct**-$S$. In [16], it is shown that the category of poset acts has enough regular injectives. Probably this is the first paper where regular injectivity of poset acts has been considered. Finally in [17], it is proved that every regular injective poset act is complete as a poset. Also, it is proved that all complete poset acts over a monoid $S$ are injective if and only if $S$ is a group. In Section 2 of this paper, congruences in the category of poset acts are introduced. In Section 3, it is shown that the category **Pos**-$S$ is a full reflective and coreflective subcategory of **PosAct**-$S$. In Section 4, the existence of the free and cofree objects in the category **PosAct**-$S$ are studied. More precisely, we consider all forgetful functors between this category and the categories **Pos** of posets, **Act**-$S$ of $S$-acts, and **Set** of sets, and study the existence of their left and right adjoints.

In the rest of this section, we briefly recall the preliminary notions about the actions of a monoid on a set and a pomonoid on a poset. For more information, see [5, 10].

The category of all partially ordered sets (posets) with order preserving (monotone) maps between them is denoted by **Pos**. A poset is said to be *complete* if each of its subsets has an infimum and a supremum.

Let $S$ be a monoid with 1 as its identity. A *right $S$-act* is a set $A$ equipped with an action $\lambda : A \times S \to A, (\lambda(a,s)$ is denoted by $as)$ such that $a1 = a$ and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. An *$S$-map* $f : A \to B$ between $S$-acts is an action preserving map, that is $f(as) = f(a)s$ for each $a \in A, s \in S$. The category of all $S$-acts and $S$-maps between them is denoted by **Act**-$S$.

Recall that a monoid (semigroup) $S$ is said to be a *pomonoid* (*posemigroup*) if it is also a poset whose partial order $\leq$ is compatible with its binary operation (that is, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$).

A *right $S$-poset* over a pomonoid $S$ is a poset $A$ which is also an $S$-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. An *$S$-poset map* (or *morphism*) is an action preserving monotone map between $S$-posets. Moreover, regular monomorphisms (equalizers) are exactly *order-embeddings*; that is, morphisms $f : A \to B$ for which $f(a) \leq f(a')$ if and only if $a \leq a'$, for all $a, a' \in A$. The category of all $S$-posets and $S$-poset maps between them is denoted by **Pos**-$S$.

## 2 The category PosAct-$S$

In the following, we introduce the category of acts in **Pos** and then the congruences in this category are introduced.

**Definition 1.** A *poset act* over a pomonoid $S$ is a poset $A$ together with a mapping $A \times S \to A, (a,s) \mapsto as$ such that
1. $a(st) = (as)t$,
2. $a1 = a$,
3. $a \leq a'$ implies $as \leq a's$ for every $a, a' \in A$ and $s, t \in S$.

This makes a poset act an ordered algebra in the sense of [1], where all operations $R_s$ are unary.

By a *poset act map* between poset acts, we mean an order preserving map which is also an $S$-map.

In Skornyakov's definition of poset acts in [15], $S$ is a monoid, but since if $S$ is a monoid, then the notions of $S$-poset and poset act coincide, we suppose that $S$ to be a pomonoid and we study and compare the categories **Pos**-$S$ of $S$-posets and **PosAct**-$S$ of poset acts.

The category of all poset acts with action-preserving monotone maps between them is denoted by **PosAct**-$S$. It is easily seen that the category **Pos**-$S$ is a full subcategory of **PosAct**-$S$.

As we mentioned above, each $S$-poset is a poset act but the converse is not true generally. For example, let $G = \{0, 1\}, 00 = 11 = 1, 01 = 10 = 0, 0 < 1$ be the two element pogroup and $A = \{a, b, c\}$ with the order $b < c$ be a poset. Then with the action $0a = a, 0b = b, 0c = c, 1a = 1b = 1c = b$, $A$ becomes a poset act which is not an $S$-poset. This example shows that even when $S$ is a pogroup the notions of $S$-poset and poset act are not the same.

**Definition 2.** If $A$ is a poset act, a *congruence* $\theta$ on $A$ is an equivalence relation on $A$ that is compatible with the $S$-action, and has the further property that $A/\theta$ can be equipped with a partial order so that $A/\theta$ is a poset act and the natural map $A \to A/\theta$ is a poset act morphism.

Recall that if $\theta$ is any binary relation on $A$, we write $a \leq_\theta a'$ if a so-called $\theta$-chain

$$a \leq a_1 \theta a'_1 \leq a_2 \theta a'_2 \leq \cdots \theta a'_m \leq a'$$

from $a$ to $a'$ exists in $A$.

In a similar way to [6] (Theorem 6.1, page 42 and Theorem 6.4, page 46) congruences on poset acts could be characterized.

**Theorem 2.1.** (i) *Let $A$ be a poset act and $\theta$ be an equivalence relation on $A$ which is compatible with the $S$-action. Then $\theta$ is a poset act congruence precisely when $a \leq_\theta a' \leq_\theta a$ implies $a\theta a'$, for all $a, a' \in A$. In this case, the induced order on $A/\theta$ is given by $[a] \leq [b]$ in $A/\theta$ if and only if there is a $\theta$-chain from $a$ to $b$ in $A$.*

(ii) *In general, let $A$ be a poset act and $\alpha$ be a binary relation on $A$ that is reflexive, transitive, and compatible with the $S$-action. Then the relation $\theta$ on $A$ given by*

$$a\theta a' \textit{ if and only if } a \leq_\alpha a' \leq_\alpha a$$

*describes the smallest poset act congruence on $A$ that contains $\alpha$, together with a suitable order on $A/\theta$ being*

$$[a]_\theta \leq [a']_\theta \textit{ if and only if } a \leq_\alpha a'.$$

*Proof.* (i) It can be easily checked that $\theta$ is an $S$-act congruence on $A$ and with the defined order relation on $A/\theta$, it becomes a poset act and the morphism $A \to A/\theta$ becomes a poset act map.
(ii) The proof of the first part is similar to the proof of case (i). Furthermore, if $\eta$ is any poset act congruence on $A$ such that $\alpha \subseteq \eta$, and $a \leq_\alpha a'$ then an $\alpha$-chain

$$a \leq a_1 \alpha a'_1 \leq a_2 \alpha a'_2 \leq \cdots \leq a_n \alpha a'_n \leq a'$$

exists, from which $[a]_\eta \leq [a']_\eta$. Similarly, $a' \leq_\alpha a$ implies $[a']_\eta \leq [a]_\eta$. The result $\theta \subseteq \eta$ follows. □

We will call $\theta$ in part (ii) of the above theorem the *poset act congruence generated by* $\alpha$. In particular, if $H \subseteq A \times A$ and $\alpha$ is the $S$-act congruence on $A$ generated by $H$, the corresponding poset act congruence $\theta$ will be denoted $\theta(H)$ and will be called the *poset act congruence generated by* $H$.

The following is also important in this paper. Let $A$ be a poset act and let $H \subseteq A \times A$. Define a relation $\alpha(H)$ on $A$ by $a\alpha(H)a'$ if and only if $a = a'$ or

$$a = x_1s_1, y_1s_1 = x_2s_2, \cdots, y_{n-1}s_{n-1} = x_ns_n, y_ns_n = a'$$

for some $(x_i, y_i) \in H$ and $s_i \in S$. Note that the relation $\alpha(H)$ is transitive, reflexive, and compatible with the $S$-action. Therefore, the relation $\nu(H)$ defined on $A$ by

$$a\nu(H)a' \; if \; and \; only \; if \; a \leq_{\alpha(H)} a' \; and \; a' \leq_{\alpha(H)} a$$

is a poset act congruence on $A$, with the property $[a] \leq [a']$ in $A/\nu(H)$ if and only if $a \leq_{\alpha(H)} a'$. Furthermore, if $H \subseteq A \times A$ and $\beta$ is any poset act congruence on $A$ such that $[x]_\beta \leq [y]_\beta$ whenever $(x, y) \in H$, then $\nu(H) \subseteq \beta$.

**Definition 3.** Let $A$ be any poset act and $H \subseteq A \times A$. Then the relation $\nu(H)$ defined by

$$a\nu(H)a' \; if \; and \; only \; if \; a \leq_{\alpha(H)} a' \; and \; a' \leq_{\alpha(H)} a,$$

(where $\alpha(H)$ and $\leq_{\alpha(H)}$ are defined as above) is called *the poset act congruence on $A$ induced by $H$*. The order relation on $A/\nu(H)$ is given by

$$[a]_{\nu(H)} \leq [a']_{\nu(H)} \; \text{if} \; and \; only \; if \; a \leq_{\alpha(H)} a'.$$

We note in particular that $\theta(H) = \nu(H \cup H^{-1})$ for any $H \subseteq A \times A$.

**Remark 1.** Congruences on poset acts are characterized the same as congruences on $S$-posets.

We use the following Homomorphism Theorem for poset acts.

**Proposition 2.1.** *Let $f : A \to B$ be a surjective poset act map, and define $K_f = \{(a, b) \in A \times A : f(a) \leq f(b)\}$. Then*
(i) *The relations $\alpha(K_f)$ and $\leq_{\alpha(K_f)}$ both coincide with $K_f$.*
(ii) *$\nu(K_f) = \ker f$, and in $A/\ker f, [a]_{\ker f} \leq [a']_{\ker f}$ if and only if $f(a) \leq f(a')$.*
(iii) *The mapping $\bar{f} : A/\ker f \to B$ defined by $\bar{f}([a]_{\ker f}) = f(a)$ for $a \in A$ is a poset act isomorphism, and $\bar{f} \circ \pi = f$, where $\pi : A \to A/kerf$ is the canonical morphism.*

*Proof.* (i) Trivially $K_f \subseteq \alpha(K_f)$, and for the opposite inclusion, let $a\alpha(K_f)a'$. Then $a = a'$ or $a = x_1s_1, y_1s_1 = x_2s_2, \cdots, y_{n-1}s_{n-1} = x_ns_n, y_ns_n = a'$ for some $(x_i, y_i) \in K_f$ and $s_i \in S$. The result is obtained by applying $f$ to the above array. So $K_f = \alpha(K_f)$. If $(a, a') \in \alpha(K_f)$, then we may write $a \leq a\alpha(K_f)a' \leq a'$, showing $a \leq_{\alpha(K_f)} a'$. On the other hand, if $a \leq_{\alpha(K_f)} a'$, then applying $f$ to the sequence

$$a \leq a_1\alpha(K_f)a'_1 \leq a_2\alpha(K_f)a'_2 \leq \cdots \leq a_n\alpha(K_f)a'_n \leq a'$$

and using the fact that $\alpha(K_f) = K_f$, we easily obtain $f(a) \leq f(a')$, and so $(a, a') \in K_f$.
(ii) Using 1, one gets that $(a, a') \in \nu(K_f)$ if and only if $(a, a') \in K_f \cap K_f^{-1} = \ker f$.
The rest of the proof is routine. □

The set $K_f$ is called the *directed kernel of $f$*.

The proof of the following theorem is routine.

**Theorem 2.2.** (Decomposition Theorem) *Let $g : A \to B$ be a surjective poset act morphism and $f : A \to C$ a poset act morphism with $K_g \subseteq K_f$. Then there exists a unique poset act morphism $h : B \to C$ such that $h \circ g = f$. Moreover, $K_g = K_f$ if and only if $h$ is an order-embedding, and $h$ is surjective if and only if $f$ is surjective.*

# 3 Pos-$S$ as a reflective and coreflective subcategory of PosAct-$S$

First, we give two adjoint pairs between **Pos**-$S$ and **PosAct**-$S$ which shows that **Pos**-$S$ is a reflective and coreflective subcategory of **PosAct**-$S$

**Theorem 3.1.** *The functor* $F' : \mathbf{PosAct} - S \longrightarrow \mathbf{Pos} - S$ *given by* $F'(A) = A/\nu(H)$*, where* $\nu(H)$ *is the poset act congruence induced by* $H = \{(as_1, as_2) : s_1 \leq s_2, a \in A\}$*, is a left adjoint to the inclusion functor* $i : \mathbf{Pos} - S \longrightarrow \mathbf{PosAct} - S$*.*

*Proof.* It is clear that for a given poset act $A$ over a pomonoid $S$, $A/\nu(H)$, where $\nu(H)$ is the poset act congruence induced by $H = \{(as_1, as_2) : s_1 \leq s_2, a \in A\}$, is an $S$-poset and the canonical epimorphism $\gamma : A \to A/\nu(H)$ is a universal poset act map. For, if $f : A \to B$ is any poset act map to an $S$-poset $B$ then since $\nu(H) \subseteq \ker f$, $\varphi : A/\nu(H) \to B$ defined by $\varphi([a]) = f(a)$ is the unique $S$-poset map with $\varphi \circ \gamma = f$. □

**Corollary 3.2.** *The category* **Pos**-$S$ *is reflective in* **PosAct**-$S$*.*

**Theorem 3.3.** *The functor* $K' : \mathbf{PosAct} - S \longrightarrow \mathbf{Pos} - S$*, given by* $K'(A) = A^{(S)}$ *of all monotone maps from* $S$ *to* $A$*, with pointwise order and action given by* $(fs)(t) = f(st)$ *for* $s, t \in S$ *and* $f \in A^{(S)}$*, is a right adjoint to the inclusion functor* $i : \mathbf{Pos} - S \longrightarrow \mathbf{PosAct} - S$*.*

*Proof.* First we show that the set $A^{(S)}$ of all monotone maps from $S$ to $A$, with pointwise order and action given by $(fs)(t) = f(st)$ for $s, t \in S$ and $f \in A^{(S)}$, is an $S$-poset. If $f_1, f_2 \in A^{(S)}$ with $f_1 \leq f_2$ and $s \in S$, then by the definition of the order on $A^{(S)}$, $f_1 s \leq f_2 s$. If $s_1 \leq s_2, s_1, s_2 \in S, f \in A^{(S)}$, then for $t \in S, s_1 t \leq s_2 t$, and so $fs_1(t) = f(s_1 t) \leq f(s_2 t) = fs_2(t)$ since $f$ is order preserving. Hence $fs_1 \leq fs_2$. The cofree map $\sigma : A^{(S)} \to A$ is defined by $\sigma(g) = g(1)$. This map is monotone, by the definition of the order on $A^{(S)}$. Further, given a poset act map $f : B \to A$ from a poset act $B$, the map $\bar{f} : B \to A^{(S)}$ given by $\bar{f}(a)(s) = f(as)$ is the unique $S$-poset map such that $\sigma \circ \bar{f} = f$. First, we show that $\bar{f}$ is order preserving. Let $b_1 \leq b_2$ and $s \in S$, then since $f$ is a poset act map one has $\bar{f}(b_1)(s) = f(b_1 s) \leq f(b_2 s) = \bar{f}(b_2)(s)$ and hence $\bar{f}(b_1) \leq \bar{f}(b_2)$. Secondly, $\bar{f}$ is action preserving, since for all $s, t \in S$ and $b \in B$ we have

$$\bar{f}(bs)(t) = f((bs)t) = f(b(st)) = \bar{f}(b)(st) = (\bar{f}(b)s)(t).$$

□

**Corollary 3.4.** *The category* **Pos**-$S$ *is coreflective in* **PosAct**-$S$*.*

# 4 Free and cofree objects in the category of poset acts

In this section, we study the existence of the free and cofree objects in the category **PosAct**-$S$. More precisely, we consider all forgetful functors between this category and the categories **Pos**, **Act**-$S$, and **Set** and study the existence of their left and right adjoints.

## 4.1 Free objects in the category of poset acts

In this subsection, we find the free poset acts over posets, $S$-acts and sets.

**Free poset acts over posets.** By a free poset act on a poset $P$ we mean a poset act $F$ together with a monotone map $\iota : P \to F$ with the universal property that given any poset act $A$ and a monotone map $f : P \to A$ there exists a unique poset act map $\bar{f} : F \to A$ such that $\bar{f} \circ \iota = f$.

**Theorem 4.1.** *For a given poset $P$ and a pomonoid $S$, the free poset act on $P$ is given by $F = F_1'(P) = P \times S$, with the order $(x, s) \leq (y, t)$ if and only if $x \leq y, s = t$ and action $(x, s)t = (x, st)$, for $x \in P, s, t \in S$.*

*Proof.* With the order and action defined above, $P \times S$ is clearly a poset act and the map $\iota : P \to P \times S$ given by $x \mapsto (x, 1)$ is a universal monotone map. If $f : P \to A$ is any monotone map to a poset act $A$ then the map $\bar{f} : P \times S \to A$ defined by $\bar{f}(x, s) = f(x)s$ is the unique poset act map with $\bar{f} \circ \iota = f$. □

Now, the assignment $P \mapsto F_1'(P)$, which maps a poset $P$ to the free poset act, defines a left adjoint to the forgetful functor, and the free map is the unit of the adjunction.

**Corollary 4.2.** *The (free) functor $F_1' : \mathbf{Pos} \longrightarrow \mathbf{PosAct} - S$ given by $F_1'(P) = P \times S$ is a left adjoint to the forgetful functor $U_1' : \mathbf{PosAct} - S \longrightarrow \mathbf{Pos}$.*

**Corollary 4.3.** *In the category $\mathbf{Pos}-S$, $P \times S$, where $P$ is a poset, with componentwise order and action is isomorphic to $(P \times S)/\nu(H)$, an $S$-poset on $P \times S$, where $P \times S$ is a poset act with order given by $(x, s) \leq (y, t)$ if and only if $x \leq y, s = t$ and action $(x, s)t = (x, st)$, for $x \in P, s, t \in S$ and $\nu(H)$ is the poset act congruence induced by $H = \{(as_1, as_2) : s_1 \leq s_2, a \in P \times S\}$.*

*Proof.* The free functors $F_1 : \mathbf{Pos} \longrightarrow \mathbf{Pos} - S$ given by $F_1(P) = P \times S$, the free $S$-poset on $P$ with componentwise order and action $(x, s)t = (x, st)$, for $x \in P, s, t \in S$, obtained in [5], and $F' \circ F_1' : \mathbf{Pos} \longrightarrow \mathbf{Pos} - S$ introduced above, are both left adjoint to the forgetful functor $U_1 : \mathbf{Pos} - S \longrightarrow \mathbf{Pos}$. Then by the uniqueness of left adjoint $(P \times S)/\nu(H) \cong P \times S$. □

**Free poset acts over sets.** In this part, we describe free poset acts over sets and so give an adjoint pair between **Set** and **PosAct**-$S$. We show that the forgetful functor $U_2' : \mathbf{PosAct}-S \longrightarrow \mathbf{Set}$ has a left adjoint.

**Theorem 4.4.** *The (free) functor $F_2' : \mathbf{Set} \longrightarrow \mathbf{PosAct} - S$ given by $F_2'(X) = F_1'(X, \Delta)$ is a left adjoint to the forgetful functor $U_2' : \mathbf{PosAct} - S \longrightarrow \mathbf{Set}$.*

*Proof.* We know that the free poset over a set $X$ is the poset $(X, \Delta)$, where $\Delta$ denotes the equality relation (discrete order). By composing the two free functors, one from **Set** to **Pos** and the other from **Pos** to **PosAct**-$S$ we get the result. More explicitly, the free poset act on a set $X$ is $X \times S$ with the order given by $(x, s) \leq (y, t)$ if and only if $x = y$ and $s = t$, and the action given by $(x, s)t = (x, st)$, for $x \in X, s, t \in S$. □

**Proposition 4.1.** *Every poset act is isomorphic to the quotient of a free poset act.*

*Proof.* For a given poset act $A$, let $F = A \times S$ be the free poset act on the set $A$, as described above, and let $f : F \to A$ be the surjective poset act morphism defined by $f(a, s) = as$ for $a \in A, s \in S$. Then $F/\mathrm{ker}f \cong A$, where the order on $F/\mathrm{ker}f$ is given by $[(a, s)] \leq [(a', s')]$ if and only if $as \leq a's'$. □

**Corollary 4.5.** *For a given set $X$, an $S$-poset $X \times S$ on a set $X$ is isomorphic to $(X \times S)/\nu(H)$, an $S$-poset on a poset act $X \times S$, where $\nu(H)$ is the poset act congruence induced by $H = \{(xs_1, xs_2) : s_1 \leq s_2, x \in X \times S\}$.*

*Proof.* The free functors $F_1 \circ F_0 : \mathbf{Set} \longrightarrow \mathbf{Pos} - S$, where $F_0 : \mathbf{Set} \longrightarrow \mathbf{Pos}$ given by $F_0(X) = (X, \Delta)$ with $\Delta$ the discrete order and $F_1$ given by $F_1(P) = P \times S$, the free $S$-poset on $P$ with componentwise order and action $(x,s)t = (x, st)$, for $x \in P, s, t \in S$, obtained in [5], and $F' \circ F_1' \circ F_0 : \mathbf{Set} \longrightarrow \mathbf{Pos} - S$ are both left adjoint to the forgetful functor $U : \mathbf{Pos} - S \longrightarrow \mathbf{Set}$. Then by the uniqueness of left adjoint $(X \times S)/\nu(H) \cong X \times S$. □

**Free poset acts over acts.** Finally, we give an adjoint pair between **Act**-$S$ and **PosAct**-$S$. We show that the forgetful functor $U_3' : \mathbf{PosAct} - S \longrightarrow \mathbf{Act} - S$ has a left adjoint.

**Theorem 4.6.** *The free functor $F_3' : \mathbf{Act} - S \longrightarrow \mathbf{PosAct} - S$ given by $F_3'(A) = (A, \Delta)$, where $\Delta$ is the discrete (equality) order, is the left adjoint to the forgetful functor $U_3' : \mathbf{PosAct} - S \longrightarrow \mathbf{Act} - S$.*

## 4.2 Cofree objects in the category of poset acts

In this subsection, we find the cofree poset acts over posets, $S$-acts and sets.

**Cofree poset acts over posets.** By a cofree poset act on a poset $P$, we mean a poset act $C$ together with a monotone map $\sigma : C \to P$ with the universal property that given any poset act $A$ and a monotone map $f : A \to P$ there exists a unique poset act map $\bar{f} : A \to C$ such that $\sigma \circ \bar{f} = f$.

**Theorem 4.7.** *For a given poset $P$ and a pomonoid $S$, the cofree poset act on $P$ is the set $P^S$ of all maps from $S$ to $P$, with pointwise order and action given by $(fs)(t) = f(st)$ for $s, t \in S$ and $f \in P^S$.*

*Proof.* It is easily checked that the defined $P^S$ is a poset act. The cofree map $\sigma : P^S \to P$ is defined by $\sigma(g) = g(1), g \in P^S$. This map is monotone, by the definition of order on $P^S$. Further, given a monotone map $f : A \to P$ from a poset act $A$, the map $\bar{f} : A \to P^S$ given by $\bar{f}(a)(s) = f(as)$ is the unique poset act map such that $\sigma \circ \bar{f} = f$. □

**Corollary 4.8.** *The (cofree) functor $K_1' : \mathbf{Pos} \longrightarrow \mathbf{PosAct} - S$, given by $K_1'(P) = P^S$, is the right adjoint to the forgetful functor $U_1' : \mathbf{PosAct} - S \longrightarrow \mathbf{Pos}$.*

**Cofree poset acts over sets.** We note that the forgetful functor $U_2' : \mathbf{PosAct} - S \longrightarrow \mathbf{Set}$ does not have a right adjoint. Otherwise, if $K_2' : \mathbf{Set} \longrightarrow \mathbf{PosAct} - S$ is a right adjoint of the forgetful functor $U_2' : \mathbf{PosAct} - S \longrightarrow \mathbf{Set}$ then $\mathbf{Set} \xrightarrow{K_2'} \mathbf{PosAct} - S \xrightarrow{K} \mathbf{Pos} - S$ would be a right adjoint of the forgetful functor $U : \mathbf{Pos} - S \longrightarrow \mathbf{Set}$. But by Remark 16 of [5], the forgetful functor $U : \mathbf{Pos} - S \longrightarrow \mathbf{Set}$ does not have a right adjoint.

**Cofree poset acts over acts.** We note that the forgetful functor $U_3' : \mathbf{PosAct} - S \longrightarrow \mathbf{Act} - S$ does not have a right adjoint. Otherwise, if $K_3' : \mathbf{Act} - S \longrightarrow \mathbf{PosAct} - S$ is a right adjoint of the forgetful functor $U_3' : \mathbf{PosAct} - S \longrightarrow \mathbf{Act} - S$ then $\mathbf{Act} - S \xrightarrow{K_3'} \mathbf{PosAct} - S \xrightarrow{K} \mathbf{Pos} - S$ would be a right adjoint of the forgetful functor $U_2 : \mathbf{Pos} - S \longrightarrow \mathbf{Act} - S$. But by the note after Theorem 17 of [5], the forgetful functor $U_2 : \mathbf{Pos} - S \longrightarrow \mathbf{Act} - S$ does not have a right adjoint.

### 4.3 Some other adjoint pair

Another way of constructing a poset act from a given poset $P$ and a pomonoid $S$ is to equip $P$ with the trivial action: $ps = p$ for all $p \in P, s \in S$. Denoting the obtained poset act by $G'(P)$, we see that the assignment $P \mapsto G'(P)$ is functorial. Moreover, this functor has a left adjoint, which we now construct.

**Theorem 4.9.** *The functor $H' : \mathbf{PosAct} - S \longrightarrow \mathbf{Pos}$ given by $H'(A) = A/\nu(W)$ where $\nu(W)$ is the poset congruence induced on the poset act $A$ by the set $W = \{(a, as) : a \in A, s \in S\}$ is the left adjoint of the functor $G' : \mathbf{Pos} \longrightarrow \mathbf{PosAct} - S$ (that equips a poset with the trivial action).*

*Proof.* If $g : A \to B$ is a poset act map, then $H'(g) : A/\nu(W) \to B/\nu(W)$ defined by $H'(g)[a] = [g(a)]$ is a well-defined monotone map, and $H'$ is a functor. The unit of this adjunction $\eta_A : A \to G'(A/\nu(W))$ for a poset act $A$ is the natural poset act epimorphism. It is a universal arrow to $G'$ because, for a given poset act map $g : A \to G'(P)$, where $P$ is a poset, we have a unique monotone $S$-map $\bar{g} : A/\nu(W) \to P$ given by $\bar{g}([a]) = g(a)$. To check that $\bar{g}$ is monotone, note that if $a \leq_{\nu(W)} a'$ as above, then because $g$ is monotone and satisfies $g(as) = g(a)$ for every $a \in A, s \in S$, we obtain $g(a) \leq g(a')$. □

**Corollary 4.10.** *For a given poset act $A$, the poset $A/\theta$ where $\theta$ is the poset congruence $\theta(W)$ induced on $A$ by the set $W = \{(a, as) : a \in A, s \in S\}$ is isomorphic to the poset $(A/\nu(W'))/\theta_1$, where $\nu(W')$ is the poset act congruence on $A$ induced by $W' = \{(as_1, as_2) : s_1 \leq s_2, a \in A\}$ and $\theta_1$ is the poset congruence $\theta_1(W'')$ induced on an $S$-poset $A/\nu(W')$ by the set $W'' = \{([a]_{\nu(W')}, [as]_{\nu(W')}) : a \in A, s \in S\}$.*

*Proof.* The functors $H' : \mathbf{PosAct} - S \longrightarrow \mathbf{Pos}$ and $H \circ F' : \mathbf{PosAct} - S \xrightarrow{F'} \mathbf{Pos} - S \xrightarrow{H} \mathbf{Pos}$, where $H$ obtained in [5], are both left adjoint to the functor $G' : \mathbf{PosAct} - S \longrightarrow \mathbf{Pos}$. Then by the uniqueness of left adjoint the desired result holds. □

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