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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE SECOND DERIVATIVES ABSOLUTE VALUES ARE QUASI-CONVEX

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Abstract. In this note we obtain some inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex. Applications for special means are also provided.

1. Introduction

Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following two inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

hold. This double inequality is known in the literature as the Hermite–Hadamard inequality for convex functions.

In recent years many authors established several inequalities connected to this fact. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard's-type inequalities see [1]–[18].

We recall that the notion of quasi-convex function generalizes the notion of convex function. More exactly, a function $f : [a, b] \to \mathbf{R}$ is said to be *quasi-convex* on $[a, b]$ if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \quad \forall x, y \in [a, b]. \tag{1.1}$$

Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex, (see for instance [1]–[5] and [12]).

Recently, D.A. Ion [12] obtained two inequalities of the right hand side of Hermite-Hadamard's type for functions whose derivatives in absolute values are quasi-convex functions, as follow:

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Theorem 1. *Let $f : I^{\circ} \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\max\left\{|f'(a)|,|f'(b)|\right\}.$$

Theorem 2. *Let $f : I^{\circ} \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $|f'|^{p/(p-1)}$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le \frac{(b-a)}{2(p+1)^{1/p}}\left(\max\left\{|f'(a)|^{p/(p-1)},|f'(b)|^{p/(p-1)}\right\}\right)^{(p-1)/p}.$$

The main aim of this paper is to establish new refined inequalities of the right-hand side of Hermite-Hadamard result for the class of functions whose second derivatives at certain powers are quasi-convex functions.

2. Hermite-Hadamard Type Inequalities

In order to prove our main theorems, we need the following lemma [10], [16].

Lemma 1. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be twice differentiable mapping on I°, $a, b \in I$ with $a < b$ and f'' is integrable on $[a, b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)\,f''(ta+(1-t)b)\,dt.$$

A simple proof of this equality can be also done integrating by parts twice in the right hand side. The details are left to the interested reader.

The next theorem gives a new result of the upper Hermite-Hadamard inequality for quasi-convex functions.

Theorem 3. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be twice differentiable mapping on I°, $a, b \in I$ with $a < b$ and f'' is integrable on $[a, b]$. If $|f''|$ is an quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{12}\max\left\{|f''(a)|,|f''(b)|\right\}.$$

Proof. From Lemma 1, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$
\begin{aligned}
&\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|dt \\
&\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)\max\left\{|f''(a)|,|f''(b)|\right\}dt \\
&\leq \frac{(b-a)^2}{2}\max\left\{|f''(a)|,|f''(b)|\right\}\int_0^1 t(1-t)\,dt \\
&= \frac{(b-a)^2}{12}\max\left\{|f''(a)|,|f''(b)|\right\}
\end{aligned}
$$

which completes the proof. □

The corresponding version for powers of the absolute value of the second derivative is incorporated in the following result:

Theorem 4. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be twice differentiable mapping on I°, $a, b \in I$ with $a < b$ and f'' is integrable on $[a,b]$. If $|f''|^{p/(p-1)}$ is quasi-convex on $[a,b]$, for $p > 1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\leq \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{1/q}
\end{aligned}
$$

where $q = p/(p-1)$.

Proof. From Lemma 1 and using the well known Hölder integral inequality, we have successively

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|dt \\
&\quad\leq \frac{(b-a)^2}{2}\left(\int_0^1\left(t-t^2\right)^p dt\right)^{1/p}\left(\int_0^1\left|f''(ta+(1-t)b)\right|^q dt\right)^{1/q} \\
&\quad\leq \frac{(b-a)^2}{2}\cdot\left(\frac{2^{-1-2p}\sqrt{\pi}\,\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\cdot\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{1/q} \\
&\quad= \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{1/q},
\end{aligned}
$$

where $1/p+1/q=1$. We note that, the Beta and Gamma functions (see [7], pp 908–910), are defined respectively, as follows:

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\ \ x,y>0$$

and

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt,\ \ x>0$$

are used to evaluate the integral

$$\int_0^1 \left(t-t^2\right)^p dt=\int_0^1 t^p(1-t)^p\,dt=\beta(p+1,p+1)$$

Using the properties of Beta function, that is, $\beta(x,x)=2^{1-2x}\beta\left(\frac{1}{2},x\right)$ and $\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we can obtain that

$$\beta(p+1,p+1)=2^{1-2(p+1)}\beta\left(\frac{1}{2},p+1\right)=2^{-2p-1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{\Gamma\left(\frac{3}{2}+p\right)},$$

where $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, which completes the proof. □

A more general inequality is given using Lemma 1, as follows:

Theorem 5. *Let $f:I\subset\mathbf{R}\to\mathbf{R}$ be twice differentiable mapping on I°, $a,b\in I$ with $a<b$ and f'' is integrable on $[a,b]$. If $|f''|^q$ is an quasi-convex on $[a,b]$, $q\geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{12}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{1/q}$$

Proof. From Lemma 1 and using well known power mean inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
&\leq\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|dt\\
&\leq\frac{(b-a)^2}{2}\left(\int_0^1\left(t-t^2\right)dt\right)^{1-1/q}\left(\int_0^1\left(t-t^2\right)\left|f''(ta+(1-t)b)\right|^q dt\right)^{1/q}\\
&\leq\frac{(b-a)^2}{2}\cdot\left(\frac{1}{6}\right)^{1-1/q}\cdot\left(\frac{1}{6}\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{1/q}\\
&=\frac{(b-a)^2}{12}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{1/q}
\end{aligned}$$

which completes the proof. □

3. Applications to special means

We consider the means for arbitrary real numbers α, β ($\alpha \neq \beta$). We take

1. *Arithmetic mean*:

$$A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbf{R}.$$

2. *Logarithmic mean*:

$$L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \ |\alpha| \neq |\beta|, \ \alpha, \beta \neq 0, \ \alpha, \beta \in \mathbf{R}.$$

3. *Generalized log-mean*:

$$L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)}\right]^{\frac{1}{n}}, \ n \in \mathbf{Z} \backslash \{-1, 0\}, \ \alpha, \beta \in \mathbf{R}, \ \alpha \neq \beta.$$

Now, using the results of Section 2, we give some applications for special means of real numbers.

Proposition 1. *Let $a, b \in \mathbf{R}$, $a < b$ and $n \in \mathbf{N}$, $n \geq 2$. Then, we have*

$$|L_n^n(a, b) - A(a^n, b^n)| \leq \frac{n(n-1)}{12}(b-a)^2 \max\left\{|a|^{n-2}, |b|^{n-2}\right\}.$$

Proof. The assertion follows from Theorem 3 applied to the quasi-convex mapping $f(x) = x^n$, $x \in \mathbf{R}$. □

Proposition 2. *Let $a, b \in \mathbf{R}$, $a < b$ and $0 \notin [a, b]$. Then, for all $p > 1$, we have*

$$\left|L^{-1}(a, b) - A\left(a^{-1}, b^{-1}\right)\right|$$

$$\leq \frac{(b-a)^2}{4}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\left(\max\left\{|a|^{-3q}, |b|^{-3q}\right\}\right)^{1/q}.$$

Proof. The assertion follows from Theorem 4 applied to the quasi-convex mapping $f(x) = 1/x$, $x \in [a, b]$. □

Proposition 3. *Let $a, b \in \mathbf{R}$, $a < b$ and $n \in \mathbf{N}$, $n \geq 2$. Then, for all $q \geq 1$, we have*

$$|L_n^n(a, b) - A^n(a, b)| \leq \frac{n(n-1)}{12}(b-a)^2\left(\max\left\{|a|^{(n-2)q}, |b|^{(n-2)q}\right\}\right)^{1/q}.$$

Proof. The assertion follows from Theorem 5 applied to the quasi-convex mapping $f(x) = x^n$, $x \in \mathbf{R}$. □

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