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Quenching estimates for a non-Newtonian filtration equation with singular boundary conditions

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Abstract. This study concerns with the quenching features of solutions of the non-Newtonian filtration equation. Various conditions on the initial condition are shown to guarantee quenching at either the left or right boundary. Theoretical quenching rates and lower bounds to the quenching time are determined for certain cases. Numerical experiments are provided to illustrate and provide additional validation of the theoretical predictions to the quenching rates and times.

Keywords. non-Newtonian filtration equation, singular boundary condition, quenching, finite differences

1 Introduction

Nonlinear evolution equations are ubiquitous in mathematical models describing various scientific phenomena. Evolution equations that form a singularity in finite time only within a temporal or spatial derivative are said to *quench*. This is in contrast to blow-up phenomena where a singularity forms in the solution itself. As an example, in solid-fuel combustion, a finite time singularity occurs in the rate of change of temperature or pressure reaches a critical, yet finite, threshold that results in ignition. Determining the time for which quenching may occur is both a difficult numerical and theoretical question. In [10], Kawarada introduced the quenching problem to the literature in the study of a one-dimensional heat equation with a nonlinear source term and Dirichlet boundary conditions. The equations proposed have become known as the Kawarada equations and its extensions have been a point of interest of both numerically [1, 8, 16] and theoretically [6, 7, 5, 14, 18, 20]. The equations and its extensions serve as fruitful arena to explore numerical and theoretical constructs that aid in deepening understanding of nonlinear evolution equations in totality. In this paper, the effect of a singular boundary condition is analyzed. Theoretical estimates to the quenching time and location can be determined based on basic requirements on the initial conditions.

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Consider the nonlinear diffusion equation with singular boundary conditions:

$$\begin{cases} (\phi(u))_t = (|u_x|^{r-2}\, u_x)_x, \ 0 < x < a, \ 0 < t < T, \\ u_x\,(0,t) = u^{-p}(0,t), \ u_x\,(a,t) = (1-u(a,t))^{-q}, \ 0 < t < T, \\ u\,(x,0) = u_0\,(x)\,, \ 0 \leq x \leq a, \end{cases} \tag{1.1}$$

where $\phi(s)$ is a properly smooth and strictly monotone increasing function with $\phi(0) = 0, \ \phi(1) = 1$ and $\phi'(s) \leq 0$. p, q are positive constants, $r \geq 2$ and $T \leq \infty$ and the initial function $u_0(x)$ is a non-negative smooth function providing the compatibility conditions:

$$u_0'\,(0) = u_0^{-p}(0), \ u_0'\,(a) = (1-u_0(a))^{-q}.$$

In the situation, $\phi(u) = u^{1/m}$ $(0 < m < 1)$, (1.1) is well-known as the standard non-Newtonian filtration equation that attempts to model non-stationary fluid flow through a porous medium where the tangential stress of the fluid's displacement velocity, u, has a power dependence under thermodynamic expansion and compression as a conclusion of heat transfer [12, 13, 19]. The singular boundary conditions model a nonlinear radiation law at the boundary and is prevalent to polytropic filtration equations [11, 12, 13, 19]. Notice that if $u(a,t)1$ then a singularity occurs at the right boundary condition. More precisely, we say that $u(x,t)$ *quenches* if and only if we have:

$$\lim_{t\to T^-} \min\{u(x,t) : 0 \leq x \leq a\} \to 0 \text{ or } \lim_{t\to T^-} \max\{u(x,t) : 0 \leq x \leq a\} \to 1.$$

In the rest of the study, the quenching time of (1.1) is demonstrated as T.

As is well known, when $\phi(u) = u$ and $r = 2$, the equations turn into the heat equation. In [15] Selcuk and Ozalp examined the following problem to determine quenching criteria:

$$\begin{cases} u_t = u_{xx}, \ 0 < x < a, \ 0 < t < T, \\ u_x\,(0,t) = u^{-p}(0,t), \ u_x\,(a,t) = (1-u(a,t))^{-q}, \ 0 < t < T, \\ u\,(x,0) = u_0\,(x)\,, \ 0 \leq x \leq a, \end{cases} \tag{1.2}$$

In [15], it was shown that:

1. If $u_0(x)$ satisfies $u_{xx}(x) \leq 0$, then $\lim_{t\to T^-} u(0,t) \to 0$ and $u_t(0,t)$ blows up in finite time and the quenching location is at $x = 0$;
2. If $u_0(x)$ satisfies $u_{xx}(x) \geq 0$ then quenching will occur at $x = a$.

In this paper, new theoretical estimates are derived for quenching rates for (1.2). In addition, we provide necessary conditions that guarantee quenching at a boundary location for a more general $\phi(u)$ and $r \geq 2$ for (1.1).

In the following, the initial condition may satisfy either of the two conditions:

$$u_{xx}(x,0) \ \geq \ 0, 0 < x < a, \text{ or} \tag{1.3}$$
$$u_{xx}(x,0) \ \leq \ 0, 0 < x < a. \tag{1.4}$$

Additionally, the initial condition is assumed to satisfy:

$$u_x(x,0) \geq 0, 0 < x < a. \tag{1.5}$$

In this paper, the combined assumptions on the initial conditions will be shown to guarantee that quenching occurs in finite time.

Chan and Yuen [5] investigated a comparable problem with a slight change in the boundary conditions:

$$\begin{aligned}
&u_t = u_{xx}, \text{ in } \Omega, \\
&u_x(0,t) = (1-u(0,t))^{-p}, \ u_x(a,t) = (1-u(a,t))^{-q}, \ 0<t<T, \\
&u(x,0) = u_0(x), \ 0 \le u_0(x) < 1, \text{ in } \bar{D},
\end{aligned}$$

where $a, \ p, \ q > 0, T \le \infty, \ D = (0,a), \ \Omega = D \times (0,T)$. In [5], they showed that if the initial condition is a lower solution then $u(x,t)$ quenches and $x = a$ is the unique quenching point. A bound to the quenching time was not determined.

In [14], Selcuk and Ozalp examined the equations:

$$\begin{aligned}
&u_t = u_{xx} + (1-u)^{-p}, \ 0<x<1, \ 0<t<T, \\
&u_x(0,t) = 0, \ u_x(1,t) = -u^{-q}(1,t), \ 0<t<T, \\
&u(x,0) = u_0(x), \ 0 < u_0(x) < 1, \ 0 \le x \le 1.
\end{aligned}$$

It was shown that if $u(x,0)$ satisfies $u_{xx}(x,0) + (1-u(x,0))^{-p} \ge 0$ and $u_x(x,0) \le 0$ then $x = 0$ is the quenching point and that $\lim_{t\to T^-} u(0,t) \to 1$ for finite T. Moreover, Selcuk and Ozalp were able to determine a theoretical estimate to the quenching rate, $u_t(x,t)$, as the quenching time is approached. A lower bound for the quenching time was also determined.

In [12], Li and et.al. focused the quenching problem for non-Newtonian filtration equation with a singular boundary condition:

$$\begin{cases}
(\psi(u))_t = (|u_x|^{r-2} u_x)_x, \ 0<x<1, \ 0<t<T, \\
u_x(0,t) = 0, \ u_x(1,t) = -g(u(1,t)), \ 0<t<T, \\
u(x,0) = u_0(x), \ 0 \le x \le 1,
\end{cases} \tag{1.6}$$

where $\psi(u)$ is a monotone increasing function with $\psi(0) = 0, \ p > 1, \ g(u) > 0, \ g'(u) < 0$ for $k > 0$, and $\lim_{u\to 0^+} g(u) = \infty$. They showed that $x = 1$ is the only quenching point in finite time under proper conditions, Further, they obtained a quenching rate and gave an example of an application of their results.

In this paper, the quenching problem, (1.1), exhibits two types of singularity terms: the boundary outflux sources u^{-p} and $(1-u)^{-q}$. Motivated by problems (1.2) and (1.6), we investigate the quenching behavior of (1.1). Building on the research in [15], several open questions are further addressed, in particular:

1. What are the sufficient criteria that guarantees quenching?
2. What are sharp estimates to the quenching rate?
3. What are the estimated quenching times?
4. Where in the domain is quenching guaranteed to occur?

This paper is arranged as follows. In Section 2, it is shown that the solution quenches in finite time T and $\lim\limits_{t\to T^-} |u_t(a,t)| \to \infty$ or $\lim\limits_{t\to T^-} u(a,t) \to 1$ and $x = a$ is the only quenching point. This is shown to occur when (1.3) or (1.4), respectively, for $r > 2$. In Section 3, estimates based on lower bounds to the quenching rates are obtained for u_t near the quenching time for $\phi(u) = u$ and $r = 2$. Section 4 details the development of the finite difference numerical approximation. The numerical experiments provide experimental validation to our theoretical results shown in Section 3. We highlight our main results in our conclusions in Section 5.

2 Quenching for the non-Newtonian filtration equation

For clarity, we rewrite (1.1) into the following form:

$$\begin{cases} u_t = B(u)(|u_x|^{r-2} u_x)_x, \ 0 < x < a, \ 0 < t < T, \\ u_x(0,t) = u^{-p}(0,t), \ u_x(a,t) = (1-u(a,t))^{-q}, \ 0 < t < T, \\ u(x,0) = u_0(x), \ 0 \le x \le a, \end{cases} \tag{2.1}$$

where $r \geq 2$, $B(u) = 1/\phi'(u)$ and $\phi'(u) \neq 0$ for $u > 0$.

Lemma 2.1.

(a) Assume that (1.5) holds. Then, $u_x(x,t) > 0$ in $(0,a) \times (0,T_0)$.

(b) Assume that (1.4) holds. Then, $u_t(x,t) < 0$ in $(0,a) \times (0,T_0)$.

(c) Assume that (1.3) holds. Then, $u_t(x,t) > 0$ in $(0,a) \times (0,T_0)$.

Proof.

(a) Let $z(x,t) = u_x(x,t)$. Then, $z(x,t)$ satisfies

$$\begin{aligned} &z_t = B(u)(|z|^{r-2} z)_{xx} + B'(u)z(|z|^{r-2} z)_x, \ 0 < x < a, \ 0 < t < T_0, \\ &z(0,t) = u^{-p}(0,t), \ z(a,t) = (1-u(a,t))^{-q}, \ 0 < t < T_0, \\ &z(x,0) = u_0'(x). \end{aligned}$$

With the help of the Maximum Principle, we have $z > 0$ and for this reason $u_x(x,t) > 0$ in $(0,a) \times (0,T_0)$.

(b) Let $w(x,t) = u_t(x,t)$. Then, $w(x,t)$ satisfies on $0 < x < a$ and $0 < t < T_0$:

$$w_t = B'(u)(|u_x|^{r-2} u_x)_x w + (r-1)B(u)(|u_x|^{r-2} w_x)_x,$$

and

$$\begin{aligned} &w_x(0,t) = -pu^{-p-1}(0,t)w(0,t), \ 0 < t < T_0, \\ &w_x(a,t) = q(1-u(a,t))^{-q-1} w(a,t), \ 0 < t < T_0, \\ &w(x,0) = B(u_0(x)) \left(\left| u_0'(x) \right|^{r-2} u_0'(x) \right)_x, 0 \le x \le a. \end{aligned}$$

With the help of the Maximum Principle, we have $w < 0$ and for this reason $u_t(x,t) < 0$ in $(0,a) \times (0,T_0)$.

(c) In like manner, $u_0(x)$ supposes (1.3), then using the above proof we obtain $u_t(x,t) > 0$ in $(0,a) \times (0,T_0)$.

□

Theorem 2.1.

(a) The solution u of (2.1) quenches at a finite time (T), quenching phenomenon occurs $x = 0$ point and $u_t(0,t)$ blows up at T with the help of (1.4) and (1.5).

(b) The solution u of (2.1) quenches at a finite time (T), quenching phenomenon occurs $x = a$ point and $u_t(a,t)$ blows up at T with the help of (1.3) and (1.5).

Proof.

(a) Suppose that (1.4) is provided.We have $u_t(x,t) < 0$ in $(0,a) \times (0,T_0)$ with the help of Lemma (2.1(b)). Furthermore, by (1.4):

$$\omega = -(1-u(a,0))^{-q(r-1)} + u^{-p(r-1)}(0,0) > 0.$$

We use the following auxiliary function to prove the theorem:

$$A(t) = \int_0^a \phi(u(x,t))dx, 0 < t < T.$$

Then

$$A'(t) = (1-u(a,t))^{-q(r-1)} - u^{-p(r-1)}(0,t) \leq -\omega,$$

by $u_t(x,t) < 0$ in $(0,a)\times(0,T_0)$. Hence, $A(t) \leq A(0) - \omega t$; which signifies that $A(T_0) = 0$ for some $T_0, (0 < T \leq T_0)$ which signifies u quenches in finite time.

Now, from our assumption of $r \geq 2$ and $\phi(u)$ is an increasing function, and Theorem 2.1 (a) and b, we obtain

$$\begin{aligned}(\phi(u))_t = (|u_x|^{r-2} u_x)_x \quad &\rightarrow \quad \phi'(u)u_t = (r-1)u_x^{r-2}u_{xx} \\ &\rightarrow \quad u_{xx} = \frac{\phi'(u)u_t}{(r-1)u_x^{r-2}} < 0.\end{aligned}$$

Thus, we get u_x is a decreasing function and since $u_x(a,t) = (1-u(a,t))^{-q} > 1$, $u_x(x,t) > 1$ in $(0,a) \times (0,T)$. If we integrate the above inequality, we have

$$u(\eta,t) > u(0,t) + \eta > 0.$$

where $\eta \in (0,a)$. So u does not quench in $(0,a]$.

Assume that u_t is bounded in $[0,a) \times [0,T)$ and M is a positive constant. Hence, we obtain $u_t > -M$, that is

$$B(u)(|u_x|^{r-2} u_x)_x > -M.$$

$\phi'(s)$ is not increasing since $\phi''(s) < 0$. Further, let σ and τ, which supply $0 < \tau \leq v < 1$ in $[0,\sigma] \times [0,T)$, then, $B(u) = \frac{1}{\phi'(u)} \geq B(\tau)$. Then the inequality becomes,

$$\begin{aligned}(|u_x|^{r-2} u_x)_x \quad &> \quad \frac{-M}{B(u)} \geq \frac{-M}{B(\tau)}, \\ (u_x^{r-1})_x \quad &> \quad \frac{-M}{B(\tau)},\end{aligned}$$

since $u_x(x,t) > 0$ in $(0,a) \times (0,T_0)$. If we integrate the above inequality, we have

$$(1-u(a,t))^{-(r-1)q} - u^{-(r-1)p}(0,t) > \frac{-Ma}{B(\tau)}.$$

Of course, the left-hand side tends to negative infinity, while the right-hand side is finite where $t \rightarrow T^-$. Hence, a contradiction persists in the assumption that u_t is bounded. Therefore, u_t blows up at the quenching time T and the quenching point $x = 0$.

(b) A similar proof as in part (a) can be established to show that quenching occurs only at the boundary $x = a$ and u_t blows up at the quenching time given that (1.3) and (1.5).

□

3 Quenching rates of the heat equation

In this section, theoretical estimates to the quenching rates and lower bounds to the quenching time in (1.2) are established. Presently, we consider the case $\phi(u) = u$ and $r = 2$ in (1.1). Let us assume one of the following conditions on the initial condition's spatial derivative:

$$u_x(x,0) \geq \frac{x}{a}(1-u(x,0))^{-q}, \qquad 0<x<a, \quad \text{or} \tag{3.1}$$

$$u_x(x,0) \geq \frac{(a-x)}{a}u^{-p}(x,0), \qquad 0<x<a. \tag{3.2}$$

Theorem 3.1. *If $u_0(x)$ satisfies condition (1.3), that is, the initial condition is not concave down, then there exists a positive constant C_1 such that*

$$u(a,t) \leq 1 - C_1(T-t)^{1/(2q+2)},$$

for t sufficiently close to the quenching time T.

Proof. Let us define an auxiliary function:

$$M(x,t) = u_t - \delta q(1-u)^{-q-1}u_x,$$

in $[0,a]\times[\tau,T)$ where $\tau \in (0,T)$ and δ is a positive constant to be specified later. It was proven in [15] that $u_t > 0$ and $u_x > 0$ in $(0,a)\times(0,T)$. $M(x,t)$ supplies

$$M_t - M_{xx} = \delta q(q+1)(q+2)(1-u)^{-q-3}u_x^3 + 2\delta q(q+1)(1-u)^{-q-2}u_x u_t > 0,$$

where $(x,t) \in (0,a)\times(\tau,T)$. Also, if we choose δ a small enough then $M(x,\tau) \geq 0$ for $x \in [0,a]$, and $M(0,t) > 0, M(a,t) > 0$ for $t \in [\tau,T)$. Hence, we get $M(x,t) \geq 0$ for $(x,t) \in [0,a]\times[\tau,T)$ with the help of the maximum principle. From here, the following inequality is obtained

$$u_t(x,t) \geq \delta q(1-u)^{-q-1}u_x(x,t), \quad (x,t) \in [0,a]\times[\tau,T)$$

Putting $x = a$, we get

$$u_t(a,t) \geq \delta q(1-u(a,t))^{-2q-1}.$$

Integrating over t from t to T gives,

$$u(a,t) \leq 1 - C_1(T-t)^{1/(2q+2)},$$

where $C_1 = (2\delta q(q+1))^{1/(2q+2)}$. □

If we provide the additional condition on the spatial derivative of the initial condition then we can obtain a lower bound to the value at the right boundary. This is encapsulated in the following theorem.

Theorem 3.2. *If $u_0(x)$ satisfies conditions (1.3) and (3.1) then there exists a positive constant C_2 such that*

$$u(a,t) \geq 1 - C_2(T-t)^{1/(2q+2)},$$

for t sufficiently close to the quenching time T.

Proof. Let us define an auxiliary function:

$$J(x,t) = u_x - \frac{x}{a}(1-u)^{-q}, \quad (x,t) \in [0,a] \times [0,T).$$

Then, $J(x,t)$ supplies

$$J_t - J_{xx} = \frac{1}{a}\left(2q(1-u)^{-q-1}u_x + xq(q+1)(1-u)^{-q-2}u_x^2\right).$$

$J(x,t)$ cannot acquire a negative interior minimum since $u_x(x,t) > 0$. On the other hand, by our condition (3.1) we have $J(x,0) \geq 0$ and

$$J(0,t) = u^{-p}(0,t) > 0, \quad J(a,t) = 0,$$

for $a \leq 1$ and $t \in (0,T)$. By the maximum principle, we obtain that $J(x,t) \geq 0$ for $(x,t) \in [0,1] \times [0,T)$. Therefore,

$$J_x(a,t) = \lim_{h \to 0^+} \frac{J(a,t) - J(a-h,t)}{h} = \lim_{h \to 0^+} \frac{-J(a-h,t)}{h} \leq 0.$$

Subsequently,

$$\begin{aligned} J_x(a,t) &= u_{xx}(a,t) - \frac{1}{a}(1-u(a,t))^{-q} - q(1-u(a,t))^{-2q-1} \\ &= u_t(a,t) - \frac{1}{a}(1-u(a,t))^{-q} - q(1-u(a,t))^{-2q-1} \leq 0 \end{aligned}$$

and

$$u_t(a,t) \leq \frac{(qa+1)}{a}(1-u(a,t))^{-2q-1}.$$

Integrating over t from t to T yields

$$u(a,t) \geq 1 - C_2(T-t)^{1/(2q+2)},$$

where $C_2 = \left[\frac{(qa+1)(2q+2)}{a}\right]^{1/(2q+2)}$. □

Corollary 3.3. *Given Theorems (3.1) and (3.2). Then as the quenching time is approached the quenching rate of the solution can be estimated as*

$$u(a,t) \sim 1 - (T-t)^{\frac{1}{2(q+1)}}.$$

Equivalently,

$$\frac{\ln(1-u(a,t))}{\ln(T-t)} \sim \frac{1}{2(q+1)}$$

In addition, a lower bound for the quenching time can be calculated. From Theorem (3.2), we have

$$T_q = \frac{a(1-u_0(a))^{2q+2}}{2(qa+1)(q+1)} \leq T.$$

In the following, we assume the initial condition satisfies condition (1.4). This condition guarantees quenching will occur at the left boundary, $x = 0$. Hence, we seek quenching estimates to the quenching rate of the solution.

Theorem 3.4. *If $u_0(x)$ satisfies condition (1.4), that is, the initial condition is not concave up, then there exists a positive constant C_3 such that*

$$u(0,t) \geq C_3(T-t)^{1/(2p+2)},$$

for t sufficiently close to the quenching time T.

Proof. Define

$$M(x,t) = u_t + \delta p u^{-p-1} u_x, \quad (x,t) \in [0,a] \times [\tau, T)$$

where $\tau \in (0,T)$ and δ is a positive constant to be specified later. It was shown in [15] that since $u_t < 0$ and $u_x > 0$ in $(0,a) \times (0,T)$ then $M(x,t)$ satisfies

$$M_t - M_{xx} = -\delta p(p+1)(p+2)u^{-p-3}u_x^3 + 2\delta p(p+1)u^{-p-2}u_x u_t < 0,$$

for $(x,t) \in (0,a) \times (\tau, T)$. Furthermore, if δ is small enough, then $M(x,\tau) \leq 0$ for $x \in [0,a]$ and $M(0,t) < 0$, $M(a,t) < 0$ for $t \in [\tau, T)$. Therefore, by the maximum principle, we obtain that $M(x,t) \leq 0$ for $(x,t) \in [0,a] \times [\tau,T)$. Subsequently, $u_t(x,t) \leq -\delta p u^{-p-1} u_x(x,t)$ for $(x,t) \in [0,a] \times [\tau, T)$. This means, at $x = 0$ we have:

$$u_t(0,t) \leq -\delta p u^{-2p-1}(0,t).$$

Integrating over t from t to T yields,

$$u(0,t) \geq C_3(T-t)^{1/(2p+2)},$$

where $C_3 = (2\delta p(p+1))^{1/(2p+2)}$. □

Theorem 3.5. *If $u_0(x)$ satisfies both (1.3) and (3.2) then there exists a positive constant C_4 such that*

$$u(0,t) \leq C_4(T-t)^{1/(2p+2)},$$

for t sufficiently close to the quenching time T.

Proof. Define

$$J(x,t) = u_x - \frac{(a-x)}{a} u^{-p}, \quad (x,t) \in [0,a] \times [0,T).$$

Then, $J(x,t)$ satisfies

$$J_t - J_{xx} = \frac{1}{a}\left(2pu^{-p-1}u_x + (a-x)p(p+1)(1-u)^{-p-2}u_x^2\right).$$

Since $u_x > 0$, then $J(x,t)$ cannot attain a negative interior minimum. On the other hand, by the assumed condition (3.2), then $J(x,0) \geq 0$ and

$$J(0,t) = 0, J(a,t) = (1-u(a,t))^{-q} > 0,$$

for $t \in (0,T)$. Therefore, by the maximum principle, we obtain that $J(x,t) \geq 0$ for $(x,t) \in [0,1] \times [0,T)$. As a result,

$$J_x(0,t) = \lim_{h \to 0^+} \frac{J(h,t) - J(0,t)}{h} = \lim_{h \to 0^+} \frac{J(h,t)}{h} \geq 0.$$

This yields

$$\begin{aligned} J_x(0,t) &= u_{xx}(0,t) + \frac{1}{a}u^{-p}(0,t) + pu^{-2p-1}(0,t) \\ &= u_t(0,t) + \frac{1}{a}u^{-p}(0,t) + pu^{-2p-1}(0,t) \geq 0 \end{aligned}$$

and

$$u_t(0,t) \geq -\frac{(pa+1)}{a}u^{-2p-1}(0,t).$$

Integrating from t from t to T gives

$$u(0,t) \leq C_4(T-t)^{1/(2p+2)},$$

where $C_4 = \left[\frac{(pa+1)(2p+2)}{a}\right]^{1/(2p+2)}$. □

Corollary 3.6. *Given Theorems (3.4) and (3.5). Then as the quenching time is approached the quenching rate of the solution is estimated as*

$$u(0,t) \sim (T-t)^{1/(2p+2)}$$

Equivalently,

$$\frac{\ln(u(0,t))}{\ln(T-t)} \sim \frac{1}{2(p+1)}$$

In addition, a lower bound for the quenching time is established from Theorem (3.5), namely,

$$T_p = \frac{au_0(0))^{2p+2}}{2(pa+1)(p+1)} \leq T.$$

for quenching time T.

3.1 Initial Conditions Examples

It is clear, that the estimates for the quenching rates and times rely heavily on properties of the initial condition. Here, we provide initial functions that satisfy the boundary conditions while simultaneously satisfying either conditions (1.3) and (3.1) or ((1.4) and (3.2).

Consider the initial condition,

$$u_0(x) = \frac{1}{4} + 4x + 4x^2, \quad 0 \leq x \leq a. \tag{3.3}$$

where $a = 1/8$. Let $p = 1$ and $q = \log_{16/3}(5)$. Since the initial condition is concave up throughout its entire domain then clearly condition (1.3) is satisfied. In addition, a straightforward calculation shows that the left boundary condition is satisfied, namely,

$$u_0'(0) = 4 = \frac{1}{u_0(0)^p}$$

At the right boundary we have $u_0'\left(\frac{1}{8}\right) = 5$ and

$$\frac{1}{\left(1 - u_0\left(\frac{1}{8}\right)\right)^q} = \left(\frac{16}{3}\right)^q = 5$$

In Fig. (1(a)) it is seen that the condition (3.1) is satisfied.

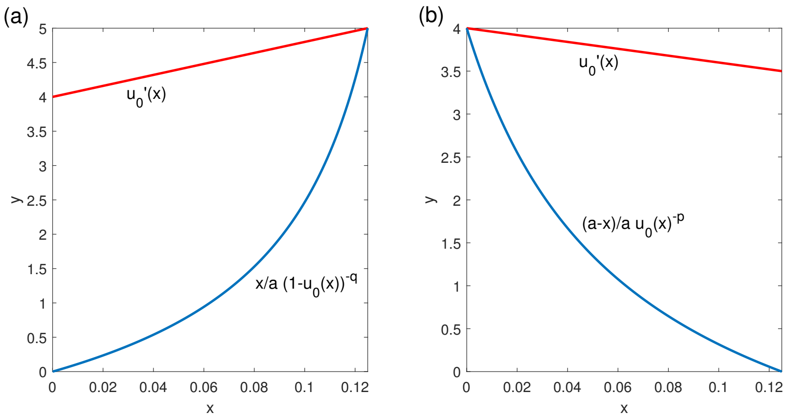

Figure 1: (a) A graph of $u_0'(x)$ (RED) and $\frac{x}{a}(1-u_0(x))^q$ (BLUE) for $u_0(x) = \frac{1}{4} - 4x - 4x^2$. It is clear that $u_0'(x) \geq \frac{x}{a}(1 - u_0(x))^{-q}$ is satisfied throughout the domain $0 \leq x \leq 1/8$. (b) A graph of $u_0'(x)$ (RED) and $\frac{a-x}{a}(u_0(x))^{-p}$ (BLUE) for $u_0(x) = \frac{1}{4} + 4x - 2x^2$. It is clear that $u_0'(x) \geq \frac{a-x}{a}(u_0(x))^{-p}$ is satisfied throughout the domain $0 \leq x \leq 1/8$.

In light of the initial condition (3.3) and by Corollary (3.3) we have a lower bound to quenching time. Namely:

$$T_q = \frac{(3/16)^{2q+2}}{16\left(\frac{1}{8}q+1\right)(q+1)} \approx 4.0002 \times 10^{-5}.$$

Similarly, if the initial condition is

$$u_0(x) = \frac{1}{4} + 4x - 2x^2, \quad 0 \leq x \leq a. \tag{3.4}$$

where $a = 1/8$. Let $p = 1$ and $q = \log_{32/9}\left(\frac{7}{2}\right)$. Since the initial condition is concave down throughout its entire domain then clearly condition (1.4) is satisfied. It is clear that the left boundary condition is satisfied. At the right boundary we have $u_0'\left(\frac{1}{8}\right) = \left(1 - u_0\left(\frac{1}{8}\right)\right)^{-q} = \frac{7}{2}$. In Fig. (1(b)), we see that condition (3.2) is satisfied. Furthermore, by Corollary (3.6) we have a lower bound to quenching time. Namely:

$$T_p = \frac{1}{9216} \approx 1.0851 \times 10^{-5}.$$

4 Numerical Approximation and Experiments

Let $x_j = jh$ for $j = 0, \ldots, N+1$ and $h = a/(N+1)$. Let $t_k = t_{k-1} + \tau_{k-1}$, where τ_{k-1} is the temporal step. Let $u_j(t)$ be the approximation to $u(x_j, t)$. Define the vector $\vec{u}(t) = (u_0(t), u_1(t), \ldots, u_N(t), u_{N+1}(t))^\top$, where $\vec{u}(0)$ is created from evaluating the initial condition at the grid points. Central difference approximations are utilized at each grid point to create the semidiscretized equations approximating (1.2), namely,

$$h^2\dot{\vec{u}}(t) = \vec{F}(\vec{u}(t)), \tag{4.1}$$

where $\vec{F} = (F_0, \dots, F_{N+1})$ with components defined as

$$F_k = \begin{cases} 2u_1 + \dfrac{2h}{(u_0)^p} - 2u_0 & k = 0 \\ u_{k-1} - 2u_k + u_{k+1} & k = 1, 2, \dots, N \\ 2u_N + \dfrac{2h}{(1 - u_{N+1})^q} - 2u_{N+1} & k = N + 1 \end{cases} \quad (4.2)$$

Define $\vec{v}_m$ as the approximation to $\vec{u}(t)$ at time $t = t_m$. Then, the solution is advanced through a second order accurate Crank-Nicolson scheme [17]:

$$\vec{v}_{m+1} = \vec{v}_m + \mu_m(\vec{F}(\vec{v}_{m+1}) + \vec{F}(\vec{v}_m)), \quad (4.3)$$

where $\mu_m = \tau_m/(2h^2)$. The scheme is overall second order accurate, however, due to the singular boundary conditions the equations are *stiff* and it is known that unless τ_k is sufficiently small then the method may manifest a reduction in the order of temporal convergence [9]. With this in mind, we expect the method to be overall first order accurate for modest temporal steps. It is common to approximate $\vec{v}_{m+1}$ in the right hand side by a first order Euler approximation, $\vec{v}_{m+1} \approx \vec{v}_m + 2\mu_m \vec{F}(\vec{v}_m)$. This maintains the overall accuracy of the scheme and creates a semi-explicit scheme for efficiency in computations [2]. The spatial grid is fixed throughout the computation, however, adaptation may occur in the temporal step. Temporal adaption for quenching problems is critical to ensure accuracy in the quenching time. An arc-length monitoring function for $\dot{\vec{u}}$ is used to adapt the temporal step. Define

$$m_i\left(\frac{\partial u_i}{\partial t}, t\right) = \sqrt{1 + \left(\frac{\partial^2 u_i}{\partial t^2}\right)^2}, \quad (x,t) \in [0, a] \times (0, T]$$

for $i = 0, \dots, N + 1$. The monitoring functions, m_i, monitor the arc-length of the characteristic at node x_i. Subsequently, as quenching is approached the temporal derivative grows beyond exponentially fast, therefore the arc-length will grow [3]. Therefore, we choose the temporal step such that the maximal arc-length between successive approximations at $[t_{k-2}, t_{k-1}]$ and $[t_{k-1}, t_k]$ are equivalent. Pragmatically, this leads to the equation for the temporal step:

$$\tau_k^2 = \tau_{k-1}^2 + \min_i \left\{ \left[\left(\frac{\partial u_i}{\partial t}\right)_{k-1} - \left(\frac{\partial u_i}{\partial t}\right)_{k-2} \right]^2 - \left[\left(\frac{\partial u_i}{\partial t}\right)_{k} - \left(\frac{\partial u_i}{\partial t}\right)_{k-1} \right]^2 \right\},$$

for $k = 2, \dots,$ and given the initial times steps of τ_0 and τ_1.

In the following experiments, we look to verify the second order convergence rate of the numerical routine. Assume that $t \ll T$. Let $\vec{v}_\tau$ be the approximation to $\vec{u}(\tau)$ for a fixed temporal step τ. Then, the maximum absolute difference between the numerical solution and $\vec{u}$ at time is $\max|\vec{v}_\tau - \vec{u}| \approx C\tau^p$, where C is some positive constant and p is the order of accuracy of the temporal scheme. Consider creating a new approximation with a temporal step $\tau/2$, then at each grid point,

$$\begin{aligned} |(\vec{v}_{\tau/2} - \vec{u})_i| &\approx C\left(\frac{h}{2}\right)^p = \frac{Ch^p}{2^p} \\ &\approx \frac{|(\vec{v}_\tau - \vec{u})_i|}{2^p} \end{aligned}$$

for $i = 0, \dots, N + 1$. Rearranging, yields an expression to estimate the order of accuracy,

$$p \approx \frac{1}{\ln(2)} \ln\left(\frac{|(\vec{v}_\tau - \vec{u})_i|}{|(\vec{v}_{\tau/2} - \vec{u})_i|}\right)$$

This generates an approximate convergence rate at each grid point x_i. In the majority of applications $\vec{u}$ is unknown. Hence, a numerical solution with a relatively fine temporal step is used to estimate the rate of the underlying cauchy sequence [4].

Consider the initial condition (3.3), where $a = 1/8$, $p = 1$, and $q = \log_{16/3}(5)$. We choose $\tau = 10^{-4}$ and $h = .01$. In such case, we estimate the convergence rate of 1.013. Therefore, a reduction in the temporal order of convergence is manifested. To estimate the quenching time and rates, we run the simulation with $h = .001$ and $\tau_0 = \tau_1 = 10^{-6}$. We adapt the temporal step but require $\tau_k \geq 10^{-9}$. The quenching time is numerically determined to be approximately $T \approx 1.9037 \times 10^{-3}$ which is greater than our estimated lower bound of 4×10^{-5}. A loglog plot of $1 - u(1/8, t)$ versus $T - t$ is shown in Fig. (2(a)). A least squares approximation suggests a slope of approximately 0.253286153170844. The theoretical estimate was predicted to be 0.255.

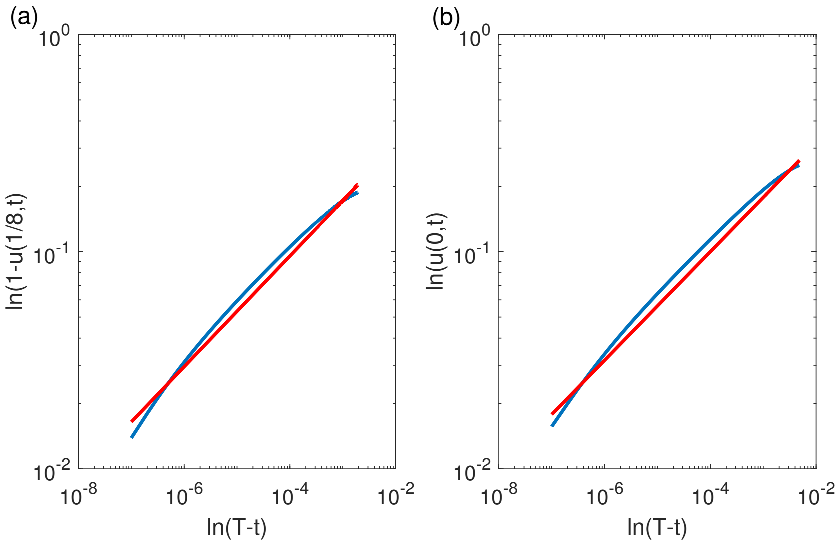

Figure 2: Loglog plots of the numerical observed (a) $1 - u(a, t)$ and (b) $u(0, t)$ versus $T - t$. The red curves in each subplot provide a loglog of (a) $(T - t)^{1/(2(q+1))}$ and (b) $(T - t)^{1/(2(p+1))}$.

Next, consider the initial condition (3.4), where $a = 1/8$, $p = 1$, and $q = \log_{32/9}(7/2)$. Again, we run the simulation with $h = .001$ and $\tau_0 = \tau_1 = 10^{-6}$. We adapt the temporal step but require $\tau_k \geq 10^{-9}$. The quenching time is numerically determined to be approximately $T \approx \times 10^{-3}$ which is greater than our estimated lower bound of 1.0851×10^{-5}. A loglog plot of $u(0, t)$ versus $T - t$ is shown in Fig. (2(b)). A least squares approximation suggests a slope of approximately 0.244301262418202. The theoretical estimate was predicted to be 0.25.

5 Conclusions

In this paper, a quenching problem with nonlinear boundary conditions are investigated. Certain conditions on the positivity, concavity, and the first derivative of the initial condition lead to theoretical lower bound to the quenching time, in addition to asymptotic estimates to the quenching rate. Numerical experiments provided additional validation of the pragmatic application of the theoretical analysis. We found that the experimental quenching time, T, was later than our predicted lower bound. Further, the experiments suggested quenching rates that were within 1% of the predicted asymptotic quenching rates.

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