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The evolution of the electric field along optical fiber with respect to the type-2 and 3 PAFs in Minkowski 3-space

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Abstract. In this paper, we introduce the type-2 and the type-3 Positional Adapted Frame(PAF) of spacelike curve and timelike curve in Minkowski 3-space. We study the evolutions of the electric vector fields with respect to the type-2 and type-3 PAFs. As a result, we also investigate the Fermi-Walker parallel and the Lorentz force equation of the electric vector fields for the type-2 and type-3 PAFs in Minkowski 3-space.

Keywords. Positional adapted frame, electric field vector, magnetic field vector, Lorentz force equation

1 Introduction

It is well known that the Frenet frame of a space curve plays an important role in the study of curve and surface theory, and this frame is the most well-known frame along a space curve. However, the Frenet frame is undefined wherever the curvature vanishes, such as at points of inflection or along straight sections of the curve. In order to solve this problem, Bishop [3] introduced a new frame along a space curve which is more suitable for applications, which is called Bishop frame or parallel transport frame. After that, many mathematicians studied various alternative methods of frame of a space curve. For example, Arbind et al. [1] studied a general 1-dimensional higher-order theory for tubes and rods in terms of the hybrid frame of a space curve. In [15] the authors discussed hybrid optical magnetic Lorentz flux by using hybrid frame. Also, Gürbüz et al. [10] presented three formulations associated with the modified nonlinear Schrödinger equation with respect to the hybrid frame in Minkowski 3-space. Recently, in [16], [17] Özen and Tosun introduced the type-1 and the type-2 Positional Adapted Frames (PAF) as another frames for the trajectories with non-vanishing angular momentum in the Euclidean 3-space. This frame is used to investigate the kinematics of moving particles.

On the other hand, Berry's geometric phasec is related to the time evolution of a space curve. The geometric phase of linearly polarized light defines its angle of rotation. The evolution of an electric vector field is connected with the geometric phase topic. Also this topic have numerous applications in modern optic. In the last years, many mathematicians have been studying the geometric phase and the evolution of the electric field vector [4]-[8], [11]-[14] etc.

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In this paper, we discuss the type-2 and the type-3 PAFs in Minkowski 3-space. In [9] the first author studied the type-1 PAF in Minkowski space and obtained the evolution of the electric field with respect to the type-1 PAF. Therefore, we want to get two other classes of the evolutions of the electric vector field according to the type-2 and the type-3 PAFs in Minkowski 3-space as natural extensions of Özen and Tosun's formulation

2 Construction of the type-2 and 3 PAFs

In this section, we construct the new frames in terms of the Frenet frame of the non-null curve in the Minkowski 3-space.

The Minkowski 3-space $\mathbb{R}_1^3$ is a real space $\mathbb{R}^3$ with the indefinite inner product $\langle \cdot\, , \cdot \rangle_L$ defined on each tangent space by

$$\langle \mathbf{x}, \mathbf{y} \rangle_L = x_1 y_1 + x_2 y_2 - x_3 y_3,$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are vectors in $\mathbb{R}_1^3$.

A nonzero vector $\mathbf{x}$ in $\mathbb{R}_1^3$ is said to be spacelike, timelike or null if $\langle \mathbf{x}, \mathbf{x} \rangle_L > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle_L < 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle_L = 0$, respectively.

Let $\beta : I \to \mathbb{R}_1^3$ be a non-null curve parametrized by the arc-length s in the Minkowski 3-space $\mathbb{R}_1^3$. Derivative formulae for the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ are given by

$$\begin{pmatrix} \mathbf{T}_s \\ \mathbf{N}_s \\ \mathbf{B}_s \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_2 \kappa & 0 \\ -\varepsilon_1 \kappa & 0 & \varepsilon_3 \tau \\ 0 & -\varepsilon_2 \tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{2.1}$$

where $\langle \mathbf{T}, \mathbf{T} \rangle_L = \varepsilon_1$, $\langle \mathbf{N}, \mathbf{N} \rangle_L = \varepsilon_2$, $\langle \mathbf{B}, \mathbf{B} \rangle_L = \varepsilon_3$,$\varepsilon_i = \pm 1$. Here κ and τ are the curvature and the torsion of the non-null curve β. On the other hand, the Lorentz cross product implies

$$\mathbf{T} \times_L \mathbf{N} = \varepsilon_3 \mathbf{B}, \quad \mathbf{N} \times_L \mathbf{B} = \varepsilon_1 \mathbf{T}, \quad \mathbf{B} \times_L \mathbf{T} = \varepsilon_2 \mathbf{N}.$$

Suppose that a point particle moves along the non-null curve β with the arc-length s and the time t in the Minkowski 3-space $\mathbb{R}_1^3$. Then the non-null tangent vector $\mathbf{T}$, the velocity vector $\mathbf{v}$ and the linear momentum vector $\mathbf{M}_l$ are given by

$$\mathbf{T}(s) = \frac{d\mathbf{z}}{ds}, \quad \mathbf{v}(t) = \frac{ds}{dt} \mathbf{T}(s), \quad \mathbf{M}_l(t) = m \frac{ds}{dt} \mathbf{T}(s),$$

where m is a constant mass and $\mathbf{z}$ is the position vector of the particle as

$$\begin{aligned} \mathbf{z} \quad = \quad & \varepsilon_1 \langle \beta(s), \mathbf{T}(s) \rangle_L \mathbf{T}(s) + \varepsilon_2 \langle \beta(s), \mathbf{N}(s) \rangle_L \mathbf{N}(s) \\ & + \varepsilon_3 \langle \beta(s), \mathbf{B}(s) \rangle_L \mathbf{B}(s). \end{aligned}$$

The angular momentum $\mathbf{M}_a$ of the particle is the Lorentz cross product of the position vector $\mathbf{z}$ and the linear momentum vector $\mathbf{M}_l$ at the time t, and it is expressed as

$$\begin{aligned} \mathbf{M}_a &= \mathbf{z} \times_L \mathbf{M}_l \\ &= -\varepsilon_2 \varepsilon_3 m \frac{ds}{dt} \langle \beta(s), \mathbf{N}(s) \rangle_L \mathbf{B}(s) \\ &\quad + \varepsilon_2 \varepsilon_3 m \frac{ds}{dt} \langle \beta(s), \mathbf{B}(s) \rangle_L \mathbf{N}(s). \end{aligned}$$

Suppose that the normal component of the angular momentum is not zero, and consider

$$\begin{aligned} -\mathbf{z} &= \varepsilon_1 \langle -\beta(s), \mathbf{T}(s)\rangle_L \mathbf{T}(s) \\ &\quad + \varepsilon_2 \langle -\beta(s), \mathbf{N}(s)\rangle_L \mathbf{N}(s) + \varepsilon_3 \langle -\beta(s), \mathbf{B}(s)\rangle_L \mathbf{B}(s). \end{aligned} \tag{2.2}$$

Consider the projections $\mathbf{w}_1$ and $\mathbf{w}_2$ on Span $\{\mathbf{N}, \mathbf{B}\}$ and Span $\{\mathbf{T}, \mathbf{N}\}$ of the vector $-\mathbf{z}$. Then these vectors become

$$\begin{aligned} \mathbf{w}_1 &= \varepsilon_2 \langle -\beta(s), \mathbf{N}(s)\rangle_L \mathbf{N}(s) + \varepsilon_3 \langle -\beta(s), \mathbf{B}(s)\rangle_L \mathbf{B}(s), \\ \mathbf{w}_2 &= \varepsilon_1 \langle -\beta(s), \mathbf{T}(s)\rangle_L \mathbf{T}(s) + \varepsilon_2 \langle -\beta(s), \mathbf{N}(s)\rangle_L \mathbf{N}(s), \end{aligned}$$

respectively. It follows that

$$\mathbf{w}_1 - \mathbf{w}_2 = \varepsilon_3 \langle -\beta(s), \mathbf{B}(s)\rangle_L \mathbf{B}(s) + \varepsilon_1 \langle \beta(s), \mathbf{T}(s)\rangle_L \mathbf{T}(s). \tag{2.3}$$

Define a new vector $\mathbf{H}$ as follows:

$$\begin{aligned} \mathbf{H} &= \frac{\mathbf{w}_1 - \mathbf{w}_2}{\sqrt{|\langle \mathbf{w}_1 - \mathbf{w}_2, \mathbf{w}_1 - \mathbf{w}_2\rangle_L|}} \\ &= \varepsilon_1 \frac{\langle \beta(s), \mathbf{T}(s)\rangle_M}{\sqrt{\left|\varepsilon_1 \langle \beta(s), \mathbf{T}(s)\rangle_L^2 + \varepsilon_3 \langle \beta(s), \mathbf{B}(s)\rangle_L^2\right|}} \mathbf{T}(s) \\ &\quad + \varepsilon_3 \frac{\langle -\beta(s), \mathbf{B}(s)\rangle_L}{\sqrt{\left|\varepsilon_1 \langle \beta(s), \mathbf{T}(s)\rangle_L^2 + \varepsilon_3 \langle \beta(s), \mathbf{B}(s)\rangle_L^2\right|}} \mathbf{B}(s) \end{aligned}$$

and take another vector $\mathbf{D} = \mathbf{H} \times_L \mathbf{N}$ given by

$$\begin{aligned} \mathbf{D} &= \varepsilon_1\varepsilon_3 \frac{\langle \beta(s), \mathbf{T}(s)\rangle_L}{\sqrt{\left|\varepsilon_1 \langle \beta(s), \mathbf{T}(s)\rangle_L^2 + \varepsilon_3 \langle \beta(s), \mathbf{B}(s)\rangle_L^2\right|}} \mathbf{B}(s) \\ &\quad + \varepsilon_1\varepsilon_3 \frac{\langle \beta(s), \mathbf{B}(s)\rangle_L}{\sqrt{\left|\varepsilon_1 \langle \beta(s), \mathbf{T}(s)\rangle_L^2 + \varepsilon_3 \langle \beta(s), \mathbf{B}(s)\rangle_L^2\right|}} \mathbf{T}(s). \end{aligned}$$

Assume that $\varepsilon_1 \langle \beta(s), \mathbf{T}(s)\rangle_L^2 + \varepsilon_3 \langle \beta(s), \mathbf{B}(s)\rangle_L^2 > 0$. In this case the moving frame $\{\mathbf{H}, \mathbf{N}, \mathbf{D}\}$ along the non-null curve β is called the type-2 Positional Adapted Frame (type-2 PAF) in the Minkowski 3-space.

Now, we give the relationship between the Frenet frame and the type-2 PAF frame.

First of all, if $\mathbf{N}$ is a timelike vector and $\mathbf{H}$, $\mathbf{D}$ are spacelike vectors ($\mathbf{T}$ and $\mathbf{B}$ are spacelike vectors), then it can be written by

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{N} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix},$$

where ϕ is an angle between $\mathbf{D}$ and $\mathbf{B}$. On the other hand, the type-2 PAF frame apparatus p_1, p_2, p_3 are given by

$$p_1 = \kappa(s)\cos\phi + \tau(s)\sin\phi,$$

$$\begin{aligned} p_2 &= -\phi', \\ p_3 &= -\kappa(s)\sin\phi + \tau(s)\cos\phi. \end{aligned}$$

Secondly, if $\mathbf{D}$ is a timelike vector and $\mathbf{H}$, $\mathbf{N}$ are spacelike vectors, ($\mathbf{B}$ is a timelike vector), then we have the relationship as follows:

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{N} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} \cosh\phi & 0 & \sinh\phi \\ 0 & 1 & 0 \\ \sinh\phi & 0 & \cosh\phi \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix},$$

where ϕ is the angle between the vectors $\mathbf{D}$ and $\mathbf{B}$. The type-2 PAF frame apparatus p_1, p_2, p_3 are given by

$$\begin{aligned} p_1 &= \kappa(s)\cosh\phi(s) - \tau(s)\sinh\phi(s), \\ p_2 &= -\phi'(s), \\ p_3 &= -\kappa(s)\sinh\phi(s) + \tau(s)\cosh\phi(s). \end{aligned}$$

Also, the derivative formulas of the type-2 PAF frame $\{\mathbf{H}, \mathbf{N}, \mathbf{D}\}$ of the non-null curve β in the Minkowski 3-space are expressed as

$$\begin{pmatrix} \mathbf{H}_s \\ \mathbf{N}_s \\ \mathbf{D}_s \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_2 p_1 & \epsilon_3 p_2 \\ -p_1 & 0 & \epsilon_3 p_3 \\ -p_2 & -\epsilon_2 p_3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{H} \\ \mathbf{N} \\ \mathbf{D} \end{pmatrix}, \tag{2.4}$$

where

$$\begin{aligned} \mathbf{H} \times_L \mathbf{D} &= \mathbf{N}, \quad \mathbf{N} \times_L \mathbf{D} = -\varepsilon_2 \mathbf{H}, \quad \mathbf{H} \times_L \mathbf{N} = \mathbf{D}, \\ \langle \mathbf{H}, \mathbf{H} \rangle_L &= 1, \quad \langle \mathbf{N}, \mathbf{N} \rangle_L = \varepsilon_2, \quad \langle \mathbf{D}, \mathbf{D} \rangle_L = \varepsilon_1 \varepsilon_3, \\ \langle \mathbf{H}, \mathbf{N} \rangle_L &= \langle \mathbf{N}, \mathbf{D} \rangle_L = \langle \mathbf{H}, \mathbf{D} \rangle_L = 0. \end{aligned}$$

In a similar way, we can define the new frame in terms of the Frenet frame.

Suppose that the binormal component of the angular momentum is not zero. The projections $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ on Span $\{\mathbf{N}, \mathbf{B}\}$ and Span $\{\mathbf{T}, \mathbf{B}\}$ of the vector $-\mathbf{z}$ are given by, respectively

$$\begin{aligned} \mathbf{\Gamma}_1 &= \varepsilon_2 \langle -\beta(s), \mathbf{N}(s) \rangle_L \mathbf{N}(s) + \varepsilon_3 \langle -\beta(s), \mathbf{B}(s) \rangle_L \mathbf{B}(s), \\ \mathbf{\Gamma}_2 &= \varepsilon_1 \langle -\beta(s), \mathbf{T}(s) \rangle_L \mathbf{T}(s) + \varepsilon_3 \langle -\beta(s), \mathbf{N}(s) \rangle_L \mathbf{B}(s). \end{aligned}$$

From this,

$$\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2 = \varepsilon_2 \langle -\beta(s), \mathbf{N}(s) \rangle_L \mathbf{N}(s) + \varepsilon_1 \langle \beta(s), \mathbf{T}(s) \rangle_L \mathbf{T}(s).$$

It follows that we define the new vector $\mathbf{F}$ as follows:

$$\begin{aligned} \mathbf{F} &= \frac{\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2}{\sqrt{|\langle \mathbf{\Gamma}_1 - \mathbf{\Gamma}_2, \mathbf{\Gamma}_1 - \mathbf{\Gamma}_2 \rangle_L|}} \\ &= \varepsilon_1 \frac{\langle \beta(s), \mathbf{T}(s) \rangle_L}{\sqrt{\left| \varepsilon_1 \langle \beta(s), \mathbf{T}(s) \rangle_L^2 + \varepsilon_2 \langle \beta(s), \mathbf{N}(s) \rangle_L^2 \right|}} \mathbf{T}(s) \\ &\quad + \varepsilon_2 \frac{\langle -\beta(s), \mathbf{N}(s) \rangle_L}{\sqrt{\left| \varepsilon_1 \langle \beta(s), \mathbf{T}(s) \rangle_L^2 + \varepsilon_2 \langle \beta(s), \mathbf{N}(s) \rangle_L^2 \right|}} \mathbf{N}(s). \end{aligned}$$

Also, the binormal vector $\mathbf{B}$ and the vector $\mathbf{H}$ lead to another vector $\mathbf{P}$:

$$\begin{aligned}\mathbf{P} &= \mathbf{F}\times_L\mathbf{B}\\ &= -\varepsilon_1\varepsilon_2\frac{\langle\beta(s),\mathbf{T}(s)\rangle_L}{\sqrt{\left|\varepsilon_1\langle\beta(s),\mathbf{T}(s)\rangle_L^2+\varepsilon_2\langle\beta(s),\mathbf{N}(s)\rangle_L^2\right|}}\mathbf{N}(s)\\ &\quad -\varepsilon_1\varepsilon_2\frac{\langle\beta(s),\mathbf{N}(s)\rangle_M}{\sqrt{\left|\varepsilon_1\langle\beta(s),\mathbf{T}(s)\rangle_L^2+\varepsilon_2\langle\beta(s),\mathbf{N}(s)\rangle_L^2\right|}}\mathbf{T}(s).\end{aligned}$$

Assume that $\varepsilon_1\langle\beta(s),\mathbf{T}(s)\rangle_L+\varepsilon_3\langle\beta(s),\mathbf{B}(s)\rangle_L>0$. In this case, the moving frame $\{\mathbf{P},\mathbf{F},\mathbf{B}\}$ along the non-null curve β is called the type-3 Positional Adapted Frame (type-3 PAF) in the Minkowski 3-space.

If $\mathbf{B}$ is a timelike vector and $\mathbf{P}$, $\mathbf{F}$ are spacelike vectors, then the relationship between the Frenet frame and the type-3 PAF frame are expressed by

$$\begin{pmatrix}\mathbf{P}\\ \mathbf{F}\\ \mathbf{B}\end{pmatrix}=\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix},$$

where ϕ is the angle between the vectors $\mathbf{F}$ and $\mathbf{N}$. Also, the type-3 PAF frame apparatus n_1, n_2, n_3 are given by

$$\begin{aligned}n_1 &= \kappa-\phi_s,\\ n_2 &= -\tau\sin\phi,\\ n_3 &= \tau\cos\phi.\end{aligned}$$

If $\mathbf{P}$ is a timelike vector ($\mathbf{T}$ is timelike) and $\mathbf{F}$, $\mathbf{B}$ are spacelike vectors, then we have

$$\begin{pmatrix}\mathbf{P}\\ \mathbf{F}\\ \mathbf{B}\end{pmatrix}=\begin{pmatrix}\cosh\phi & \sinh\phi & 0\\ \sinh\phi & \cosh\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix},$$

it follows that the type-3 PAF frame apparatus n_1, n_2, n_3 are given by

$$\begin{aligned}n_1 &= \kappa+\phi_s,\\ n_2 &= \tau\sinh\phi,\\ n_3 &= \tau\cosh\phi.\end{aligned}$$

On the other hand, the derivative formulas of the type-3 PAF frame $\{\mathbf{P},\mathbf{F},\mathbf{B}\}$ of the non-null curve β in Minkowski 3-space become

$$\begin{pmatrix}\mathbf{P}_s\\ \mathbf{F}_s\\ \mathbf{B}_s\end{pmatrix}=\begin{pmatrix}0 & n_1 & \epsilon_3 n_2\\ -\epsilon_1 n_1 & 0 & \epsilon_3 n_3\\ -\epsilon_1 n_2 & -n_3 & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}\\ \mathbf{F}\\ \mathbf{B}\end{pmatrix}, \tag{2.5}$$

where

$$\begin{aligned}\langle\mathbf{P},\mathbf{P}\rangle_L &= \varepsilon_1\varepsilon_2,\quad \langle\mathbf{F},\mathbf{F}\rangle_L=1,\quad \langle\mathbf{B},\mathbf{B}\rangle_L=\varepsilon_3,\\ \langle\mathbf{P},\mathbf{F}\rangle_L &= \langle\mathbf{F},\mathbf{B}\rangle_L=\langle\mathbf{P},\mathbf{B}\rangle_L=0,\\ \mathbf{B}\times_L\mathbf{P} &= \mathbf{F},\quad \mathbf{P}\times_L\mathbf{B}=\varepsilon_3\mathbf{F},\quad \mathbf{P}\times_L\mathbf{F}=-\mathbf{B}.\end{aligned}$$

3 The evolution of the electric field with respect to the type-2 PAF

In this section, we study the evolution of the electric vector field with respect to the type-2 PAF in the Minkowski 3-space. To get results, we split it into three cases according to the type-2 PAF.

Case I. Consider an optical fiber $\mathcal{O}$ described by the spacelike curve $\beta = \beta(s)$ and the timelike binormal vector of the type-2 PAF in the Minkowski 3-space $\mathbb{R}^3_1$.

Suppose that the electric vector field $\mathbf{E}^{(2.Paf)}$ is perpendicular to the spacelike vector $\mathbf{H}$ with the timelike vector $\mathbf{D}$, that is,

$$\left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L = 0. \tag{3.1}$$

Theorem 3.1. *In Cace I the evolution of the electric vector field $\mathbf{E}^{(2.Paf)}$ with respect to the type-2 PAF is given by*

$$\begin{aligned} \mathbf{E}^{(2.Paf)}_s &= \left(-p_1 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L + p_2 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L \right) \mathbf{H} \\ &\quad + \sigma \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L \mathbf{N} - \sigma \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L \mathbf{D}. \end{aligned} \tag{3.2}$$

Proof. The general evolution of the electric vector field $\mathbf{E}^{(2.Paf)}$ with respect to the type-2 PAF in $\mathbb{R}^3_1$ is expressed by

$$\mathbf{E}^{(2.Paf)}_s = a_1 \mathbf{H} + a_2 \mathbf{N} + a_3 \mathbf{D}, \tag{3.3}$$

where a_1, a_2 and a_3 are arbitrary smooth functions. Consider no various loss mechanism along the optic fiber for the electric vector field $\mathbf{E}^{(2.Paf)}$ of the type-2 PAF in the Minkowski 3-space, then it can be written by

$$\left\langle \mathbf{E}^{(2.Paf)}, \mathbf{E}^{(2.Paf)} \right\rangle_L = \text{const}. \tag{3.4}$$

Using Eqs.(2.4), (3.1) and (3.3), we obtain

$$a_1 = -p_1 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L + p_2 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L, \tag{3.5}$$

where $\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \rangle_L \neq 0$ and $\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \rangle_L \neq 0$. Taking the derivative with respect to s of Eq.(3.4) and using Eq.(3.3), we also have

$$a_2 = \sigma \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L, \quad a_3 = -\sigma \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L, \tag{3.6}$$

where σ is a parameter. When Eqs.(3.5) and (3.6) are substituted in Eq.(3.3), we obtain Eq.(3.2). □

Theorem 3.2. *In Case I the polarization plane is rotated by an angle p_3 with respect to the type-2 PAF in $\mathbb{R}^3_1$.*

Proof. The Fermi-Walker derivative ${}^{FW}\mathbf{E}^{(2.Paf)}_s$ of the electric field $\mathbf{E}^{(2.Paf)}$ with respect to the type-2 PAF in $\mathbb{R}^3_1$ is given by

$${}^{FW}\mathbf{E}^{(2.Paf)}_s = \mathbf{E}^{(2.Paf)}_s - \left\langle \mathbf{H}, \mathbf{E}^{(2.Paf)} \right\rangle_L \mathbf{H}_s + \left\langle \mathbf{H}_s, \mathbf{E}^{(2.Paf)} \right\rangle_L \mathbf{H}. \tag{3.7}$$

The electric vector field $\mathbf{E}^{(2.Paf)}$ is Fermi-Walker parallel if and only if ${}^{FW}\mathbf{E}_s^{(2.Paf)} = 0$. If the electric vector field $\mathbf{E}^{(2.Paf)}$ is Fermi-Walker parallel, then Eq.(3.1) and Eq.(3.7) imply

$$\mathbf{E}_s^{(2.Paf)} = -\left\langle \mathbf{H}_s, \mathbf{E}^{(2.Paf)} \right\rangle_L \mathbf{H}. \tag{3.8}$$

The electric vector field $\mathbf{E}^{(2.Paf)}$ for the first case can be expressed by

$$\mathbf{E}^{(2.Paf)} = E^{(2.Paf)N}\mathbf{N} - E^{(2.Paf)D}\mathbf{D}, \tag{3.9}$$

where

$$E^{(2.Paf)N} = \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L, \quad E^{(2.Paf)D} = \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L.$$

Taking the derivative of Eq.(3.9) with respect to s and taking account of Eq. (2.4), the variation of the electric vector field $\mathbf{E}^{(2.Paf)}$ for the first case is obtained

$$\begin{aligned} \mathbf{E}_s^{(2.Paf)} = \; & (E_s^{(2.Paf)N} + p_3 E^{(2.Paf)D})\mathbf{N} \\ & -(p_3 E^{(2.Paf)N} + E_s^{(2.Paf)D})\mathbf{D} \\ & +(-p_1 E^{(2.Paf)N} + p_2 E^{(2.Paf)D})\mathbf{H}. \end{aligned} \tag{3.10}$$

From Eqs.(2.4), (3.8) and (3.10), we obtain

$$\begin{pmatrix} E_s^{(2.Paf)N} \\ E_s^{(2.Paf)D} \end{pmatrix} = \begin{pmatrix} 0 & -p_3 \\ -p_3 & 0 \end{pmatrix} \begin{pmatrix} E^{(2.Paf)N} \\ E^{(2.Paf)D} \end{pmatrix}, \tag{3.11}$$

it describes a rotation of the polarization plane by an angle p_3 with respect to the type-2 PAF on the fiber. Thus the proof is completed. □

A magnetic vector field is given by a closed 2-form $\mathcal{C}$ in a 3-dimensional Riemannian manifold M. The Lorentz force of a magnetic vector field $\mathbf{V}$ is described by skew-symmetric operator Φ, that is,

$$\langle \Phi\mathbf{X}, \mathbf{Y} \rangle = \mathcal{C}(\mathbf{X}, \mathbf{Y})$$

for all $\mathbf{X}$, $\mathbf{Y} \in \chi(M)$ and $\Phi(\mathbf{X}) = \mathbf{V} \times \mathbf{X}$ [18].

The magnetic curve produced by the type-2 PAF electric vector field $\mathbf{E}^{(2.Paf)}$ along the linearly polarized monochromatic light wave propogating in the optical fiber for the first case with respect to the type-2 PAF in the Minkowski 3-space is called a Lorentzian type-2 PAF electromagnetic curve for the first case. Therefore, the Lorentzian type-2 PAF electromagnetic curve of the spacelike curve $\beta = \beta(s)$ for the first case in $\mathbb{R}_1^3$ is described by (cf. [13])

$$\Phi^H \mathbf{E}^{(2.Paf)} = \mathbf{E}_s^{(2.Paf)} = \mathbf{V}^{(1)} \times_L \mathbf{E}^{(2.Paf)}, \tag{3.12}$$

where $\mathbf{V}^{(1)}$ is any divergence free vector field.

Theorem 3.3. *The Lorentz force equation according to the type 2-PAF of the Lorentzian type-2 Paf electromagnetic curve of the spacelike curve β for the first case is the following:*

$$\begin{bmatrix} \Phi^H(\mathbf{H}) \\ \Phi^H(\mathbf{N}) \\ \Phi^H(\mathbf{D}) \end{bmatrix} = \begin{bmatrix} 0 & p_1 & -p_2 \\ -p_1 & 0 & \sigma \\ -p_2 & \sigma & 0 \end{bmatrix} \begin{bmatrix} \mathbf{H} \\ \mathbf{N} \\ \mathbf{D} \end{bmatrix}. \tag{3.13}$$

Proof. From Eqs.(3.2) and (3.12), we obtain

$$\left\langle \Phi^H \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L = -\left\langle \mathbf{E}^{(2.Paf)}, \Phi^H \mathbf{H} \right\rangle_L, \tag{3.14}$$

$$\left\langle \Phi^H \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L = -\left\langle \mathbf{E}^{(2.Paf)}, \Phi^H \mathbf{N} \right\rangle_L, \tag{3.15}$$

$$\left\langle \Phi^H \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L = -\left\langle \mathbf{E}^{(2.Paf)}, \Phi^H \mathbf{D} \right\rangle_L. \tag{3.16}$$

Via Eqs.(3.14), (3.15) and (3.16), we can obtain Eq.(3.13). □

Theorem 3.4. *The magnetic vector field $\mathbf{V}^{(1)}$ with respect to the type-2 PAF in the first case is given by*

$$\mathbf{V}^{(1)} = -\sigma \mathbf{H} - p_2 \mathbf{N} + p_1 \mathbf{D}. \tag{3.17}$$

Proof. With the aid of Eq.(3.13), we derive Eq.(3.17). □

Case II. Suppose that an optical fiber $\mathcal{O}$ can be described by a spacelike curve β with the timelike normal vector for the type-2 PAF in the Minkowski 3-space $\mathbb{R}^3_1$.

Now, we consider the electric vector field $\mathbf{E}^{(2.Paf)}$ perpendicular to the timelike vector $\mathbf{N}$ according to the type- 2 PAF, that is,

$$\left\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L = 0. \tag{3.18}$$

Theorem 3.5. *The evolution of the electric vector field $\mathbf{E}^{(2.Paf)}$ with respect to the type-2 PAF in $\mathbb{R}^3_1$ is derived by*

$$\begin{aligned} \mathbf{E}_s^{(2.Paf)} &= \rho \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L \mathbf{H} - \rho \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L \mathbf{D}. \\ &\quad + \left(p_3 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L - p_1 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L \right) \mathbf{N}. \end{aligned} \tag{3.19}$$

Proof. The general variation of the electric vector field $\mathbf{E}^{(2.Paf)}$ with respect to the type-2 PAF frame in $\mathbb{R}^3_1$ is given by

$$\mathbf{E}_s^{(2.Paf)} = b_1 \mathbf{H} + b_2 \mathbf{N} + b_3 \mathbf{D}, \tag{3.20}$$

where b_1, b_2 and b_3 are arbitrary smooth functions. Consider no various loss mechanism along with the optical fiber for the type-2 PAF with $\mathbf{E}^{(2.Paf)} \perp \mathbf{N}$. From Eqs.(2.4), (3.4), (3.18) and (3.20), we have

$$b_2 = -p_1 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L + p_3 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L \tag{3.21}$$

with $\left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_M \neq 0$ and $\left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_M \neq 0$. By differentiating Eq.(3.4) with respect to s and using Eq.(3.20) we also obtain

$$b_1 = \rho \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L, \quad b_3 = -\rho \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L, \tag{3.22}$$

where ρ is a parameter. If Eqs. (3.21) and (3.22) are substituted in Eq.(3.20), we obtain Eq.(3.19). □

Theorem 3.6. *In Case II the polarization plane is rotated by an angle p_2 with respect to the type-2 PAF in the Minkowski 3-space.*

Proof. In the second case the Fermi-Walker derivative of the electric vector field $\mathbf{E}^{(2.Paf)}$ with respect to the type-2 PAF in $\mathbb{R}^3_1$ is given by

$$\begin{aligned} {}^{FW}\mathbf{E}_s^{(2.Paf)} &= \mathbf{E}_s^{(2.Paf)} + \left\langle \mathbf{N}, \mathbf{E}^{(2.Paf)} \right\rangle_L \mathbf{N}_s \\ &\quad - \left\langle \mathbf{N}_s, \mathbf{E}^{(2.Paf)} \right\rangle_L \mathbf{N}. \end{aligned} \tag{3.23}$$

If the electric vector field $\mathbf{E}^{(2.Paf)}$ is the Fermi-Walker parallel for the second case, Eqs.(2.4), (3.18) and (3.23) imply

$$\begin{aligned} \mathbf{E}_s^{(2.Paf)} &= \left\langle \mathbf{N}_s, \mathbf{E}^{(2.Paf)} \right\rangle_L \mathbf{N} \\ &= (-p_1 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L + p_3 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L)\mathbf{N}. \end{aligned} \tag{3.24}$$

The electric vector field $\mathbf{E}^{(2.Paf)}$ for the second case is given by:

$$\mathbf{E}^{(2.Paf)} = E^{(2.Paf)H}\mathbf{H} + E^{(2.Paf)D}\mathbf{D}, \tag{3.25}$$

where

$$E^{(2.Paf)H} = \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L, \quad E^{(2.Paf)D} = \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L.$$

Taking the derivative of Eq.(3.25) with respect to s, the change of the electric vector field $\mathbf{E}^{(2.Paf)}$ in the Minkowski 3-space is expressed as

$$\begin{aligned} \mathbf{E}_s^{(2.Paf)} &= (E_s^{(2.Paf)H} - p_2 E^{(2.Paf)D})\mathbf{H} \\ &\quad + (p_3 E^{(2.Paf)D} - p_1 E^{(2.Paf)H})\mathbf{N} \\ &\quad + (p_2 E^{(2.Paf)H} + E_s^{(2.Paf)D})\mathbf{D}. \end{aligned} \tag{3.26}$$

From Eqs.(2.4) and (3.26), we obtain

$$\begin{pmatrix} E_s^{(2.Paf)H} \\ E_s^{(2.Paf)D} \end{pmatrix} = \begin{pmatrix} 0 & p_2 \\ -p_2 & 0 \end{pmatrix} \begin{pmatrix} E^{(2.Paf)H} \\ E^{(2.Paf)D} \end{pmatrix}, \tag{3.27}$$

it describes a rotation of the polarization plane by an angle p_2 with respect to the type-2 PAF on the fiber. Thus, the proof is completed. □

The magnetic curve produced by the electric vector field $\mathbf{E}^{(2.Paf)}$ along the linearly polarized monochromatic light wave propogating in the optical fiber for the second case is called a Lorentzian type-2 PAF electromagnetic curve for the second case. Therefore, the Lorentzian type-2 PAF electromagnetic curve of the spacelike curve β in the second case is given by (cf. [13])

$$\Phi^N \mathbf{E}^{(2.Paf)} = \mathbf{E}_s^{(2.Paf)} = \mathbf{V}^{(2)} \times_L \mathbf{E}^{(2.Paf)}, \tag{3.28}$$

where $\mathbf{V}^{(2)}$ is any divergence free vector field.

Theorem 3.7. *The Lorentz force equation of the Lorentzian type-2 PAF electromagnetic curve of the spacelike curve β for the second case is given by*

$$\begin{bmatrix} \Phi^N(\mathbf{H}) \\ \Phi^N(\mathbf{N}) \\ \Phi^N(\mathbf{D}) \end{bmatrix} = \begin{bmatrix} 0 & -p_1 & \rho \\ -p_1 & 0 & p_3 \\ -\rho & p_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{H} \\ \mathbf{N} \\ \mathbf{D} \end{bmatrix}. \tag{3.29}$$

Proof. Via Eq.(3.24) and Eq.(3.28), the followings are obtained:

$$\left\langle \Phi^N \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L = -\left\langle \mathbf{E}^{(2.Paf)}, \Phi^N \mathbf{H} \right\rangle_L, \tag{3.30}$$

$$\left\langle \Phi^N \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L = -\left\langle \mathbf{E}^{(2.Paf)}, \Phi^N \mathbf{N} \right\rangle_L, \tag{3.31}$$

$$\left\langle \Phi^N \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L = -\left\langle \mathbf{E}^{(2.Paf)}, \Phi^N \mathbf{D} \right\rangle_L. \tag{3.32}$$

From Eqs.(3.30), (3.31) and (3.32), we obtain Eq.(3.29). □

Theorem 3.8. *The magnetic vector field* $\mathbf{V}^{(2)}$ *with respect to the type-2 PAF in the second case is given by*

$$\mathbf{V}^{(2)} = p_3 \mathbf{H} + \rho \mathbf{N} + p_1 \mathbf{D}. \tag{3.33}$$

Proof. Using Eq.(3.29), we obtain Eq.(3.33). □

Case III. Let us consider an optical fiber $\mathcal{O}$ which can be described by a spacelike curve β with the timelike binormal vector with respect to the type-2 PAF in the Minkowski 3-space $\mathbb{R}^3_1$.

Assume that the electric vector field $\mathbf{E}^{(2.Paf)}$ is perpendicular to the timelike vector $\mathbf{D}$, that is,

$$\left\langle \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L = 0, \tag{3.34}$$

and also consider no various loss mechanism along the optical fiber of the electric vector field $\mathbf{E}^{(2.Paf)}$ in the third case in the Minkowski 3-space. Then,

$$\left\langle \mathbf{E}^{(2.Paf)}, \mathbf{E}^{(2.Paf)} \right\rangle_L = \text{const.} \tag{3.35}$$

Theorem 3.9. *In Case III the evolution of the electric vector field* $\mathbf{E}^{(2.Paf)}$ *with respect to the type-2 PAF in* $\mathbb{R}^3_1$ *is given by*

$$\begin{aligned} \mathbf{E}_s^{(2.Paf)} &= \xi \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L \mathbf{H} - \xi \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L \mathbf{N} \\ &\quad - \left(p_3 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L + p_2 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L \right) \mathbf{D}. \end{aligned} \tag{3.36}$$

Proof. The general evolution of the electric vector field $\mathbf{E}^{(2.Paf)}$ in $\mathbb{R}^3_1$ is given by

$$\mathbf{E}_s^{(2.Paf)} = c_1 \mathbf{H} + c_2 \mathbf{N} + c_3 \mathbf{D}, \tag{3.37}$$

where c_1, c_2 and c_3 are the arbitrary smooth functions. Using Eqs.(2.4), (3.34), (3.35) and (3.37), we get

$$c_1 = \xi \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L, \quad c_2 = -\xi \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L, \tag{3.38}$$

$$c_3 = -p_2 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L - p_3 \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L \tag{3.39}$$

with $\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \rangle_L \neq 0$ and $\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \rangle_L \neq 0$. Here ξ is a parameter. When Eqs.(3.38) and (3.39) are written in Eq.(3.37), we obtain Eq.(3.36). □

Theorem 3.10. *In the third case the polarization plane is rotated by an angle p_1 with respect to the type-2 PAF in $\mathbb{R}^3_1$.*

Proof. The Fermi-Walker derivative of the electric vector field $\mathbf{E}^{(2.Paf)}$ with respect to the type-2 PAF in the third case in $\mathbb{R}^3_1$ is given by

$$\begin{aligned} {}^{FW}\mathbf{E}^{(2.Paf)}_s &= \mathbf{E}^{(2.Paf)}_s + \left\langle \mathbf{D}, \mathbf{E}^{(2.Paf)} \right\rangle_L \mathbf{D}_s \\ &\quad - \left\langle \mathbf{D}_s, \mathbf{E}^{(2.Paf)} \right\rangle_L \mathbf{D}. \end{aligned} \tag{3.40}$$

It follows that if in the third case the electric vector field $\mathbf{E}^{(2.Paf)}$ is Fermi-Walker parallel, then Eqs.(3.34) and (3.40) imply

$$\mathbf{E}^{(2.Paf)}_s = \left\langle \mathbf{D}_s, \mathbf{E}^{(2.Paf)} \right\rangle_L \mathbf{D}. \tag{3.41}$$

Also, in the third case the electric vector field $\mathbf{E}^{(2.Paf)}$ becomes

$$\mathbf{E}^{(2.Paf)} = E^{(2.Paf)H}\mathbf{H} + E^{(2.Paf)N}\mathbf{N}, \tag{3.42}$$

where

$$E^{(2.Paf)H} = \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L, \quad E^{(2.Paf)N} = \left\langle \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L.$$

Via Eq.(2.4) and the derivative of Eq.(3.42) with respect to s, the variation of the electric vector field $\mathbf{E}^{(2.Paf)}$ is obtained as

$$\begin{aligned} \mathbf{E}^{(2.Paf)}_s &= (E^{(2.Paf)H}_s - p_1 E^{(2.Paf)N})\mathbf{H} \\ &\quad + (p_1 E^{(2.Paf)H} + E^{(2.Paf)N}_s)\mathbf{N} \\ &\quad - (p_2 E^{(2.Paf)H} + p_3 E^{(2.Paf)N})\mathbf{D}. \end{aligned} \tag{3.43}$$

From this, Eq. (3.43) implies

$$\begin{pmatrix} E^{(2.Paf)H}_s \\ E^{(2.Paf)N}_s \end{pmatrix} = \begin{pmatrix} 0 & p_1 \\ -p_1 & 0 \end{pmatrix} \begin{pmatrix} E^{(2.Paf)H} \\ E^{(2.Paf)N} \end{pmatrix}. \tag{3.44}$$

It follows that in the third case Eq.(3.44) gives a rotation of the polarization plane by an angle p_1 with respect to the type-2 PAF in $\mathbb{R}^3_1$. □

The magnetic curve produced by the electric vector field $\mathbf{E}^{(2.Paf)}$ along the linearly polarized monochromatic light wave propogating in the optical fiber is called a Lorentzian type-2 PAF electromagnetic curve for the third case. Therefore, the Lorentzian type-2 PAF electromagnetic curve of the spacelike curve β for the third case in $\mathbb{R}^3_1$ is described by (cf. [13])

$$\Phi^D \mathbf{E}^{(2.Paf)} = \mathbf{E}^{(2.Paf)}_s = \mathbf{V}^{(3)} \times_L \mathbf{E}^{(2.Paf)}, \tag{3.45}$$

where $\mathbf{V}^{(3)}$ is any divergence free vector field.

Theorem 3.11. *The Lorentz force equation of the Lorentzian type-2 PAF electromagnetic curve of the spacelike curve β for the third case is given by*

$$\begin{bmatrix} \Phi^D(\mathbf{H}) \\ \Phi^D(\mathbf{N}) \\ \Phi^D(\mathbf{D}) \end{bmatrix} = \begin{bmatrix} 0 & \xi & -p_2 \\ -\xi & 0 & -p_3 \\ -p_2 & -p_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{H} \\ \mathbf{N} \\ \mathbf{D} \end{bmatrix}. \tag{3.46}$$

Proof. From Eq.(3.24) and Eq.(3.28), one obtains

$$\left\langle \Phi^D \mathbf{E}^{(2.Paf)}, \mathbf{H} \right\rangle_L = -\left\langle \mathbf{E}^{(2.Paf)}, \Phi^D \mathbf{H} \right\rangle_L, \tag{3.47}$$

$$\left\langle \Phi^D \mathbf{E}^{(2.Paf)}, \mathbf{N} \right\rangle_L = -\left\langle \mathbf{E}^{(2.Paf)}, \Phi^D \mathbf{N} \right\rangle_L, \tag{3.48}$$

$$\left\langle \Phi^D \mathbf{E}^{(2.Paf)}, \mathbf{D} \right\rangle_L = -\left\langle \mathbf{E}^{(2.Paf)}, \Phi^D \mathbf{D} \right\rangle_L. \tag{3.49}$$

From Eqs.(3.47), (3.48) and (3.49), we obtain (3.46). □

Theorem 3.12. *In the third case the magnetic vector field* $\mathbf{V}^{(3)}$ *with respect to the type-2 PAF is given by:*

$$\mathbf{V}^{(3)} = \xi \mathbf{D} - p_2 \mathbf{N} + p_3 \mathbf{H}. \tag{3.50}$$

Proof. Using Eq.(3.46), we obtain Eq.(3.50). □

4 The evolution of electric field for the type-3 PAF

In this section, we consider an optical fiber $\mathcal{O}$ described by the type-3 PAF of the timelike curve β in the Minkowski 3-space $\mathbb{R}^3_1$.

Suppose that the electric vector field $\mathbf{E}^{(3.Paf)}$ is perpendicular to the timelike vector $\mathbf{B}$ according to the type- 3 PAF in $\mathbb{R}^3_1$, that is,

$$\left\langle \mathbf{E}^{(3.Paf)}, \mathbf{B} \right\rangle_L = 0. \tag{4.1}$$

Consider no various loss mechanism along with the optical fiber with respect to the type-3 PAF. Then we have

$$\left\langle \mathbf{E}^{(3.Paf)}, \mathbf{E}^{(3.Paf)} \right\rangle_L = \text{const.} \tag{4.2}$$

Theorem 4.1. *The change of the electric vector field* $\mathbf{E}^{(3.Paf)}$ *with respect to the type-3 PAF in* $\mathbb{R}^3_1$ *is given by*

$$\begin{aligned} \mathbf{E}_s^{(3.Paf)} = {} & \lambda \left\langle \mathbf{E}^{(3.Paf)}, \mathbf{F} \right\rangle_L \mathbf{P} - \lambda \left\langle \mathbf{E}^{(3.Paf)}, \mathbf{P} \right\rangle_L \mathbf{F} \\ & - \left(n_3 \left\langle \mathbf{E}^{(3.Paf)}, \mathbf{F} \right\rangle_L + n_2 \left\langle \mathbf{E}^{(3.Paf)}, \mathbf{P} \right\rangle_L \right) \mathbf{B}. \end{aligned} \tag{4.3}$$

Proof. The general variation of the electric vector field $\mathbf{E}^{(3.Paf)}$ with respect to the type-3 PAF in $\mathbb{R}^3_1$ is given by

$$\mathbf{E}_s^{(3.Paf)} = d_1 \mathbf{P} + d_2 \mathbf{F} + d_3 \mathbf{B}, \tag{4.4}$$

where d_1, d_2 and d_3 are arbitrary smooth functions. Via Eqs.(3.27), (4.1), (4.2) and (4.4), it is obtained by

$$d_1 = \lambda \left\langle \mathbf{E}^{(3.Paf)}, \mathbf{F} \right\rangle_L, \quad d_2 = -\lambda \left\langle \mathbf{E}^{(3.Paf)}, \mathbf{P} \right\rangle_L, \tag{4.5}$$

$$d_3 = -n_2 \left\langle \mathbf{E}^{(3.Paf)}, \mathbf{P} \right\rangle_L - n_3 \left\langle \mathbf{E}^{(3.Paf)}, \mathbf{F} \right\rangle_L, \tag{4.6}$$

where $\langle \mathbf{E}^{(3.Paf)}, \mathbf{P} \rangle_L \neq 0$, $\langle \mathbf{E}^{(3.Paf)}, \mathbf{F} \rangle_L \neq 0$ and λ is a parameter. If Eqs.(4.5) and (4.6) are substituted in Eq.(4.4), we obtain Eq.(4.3). □

Theorem 4.2. *The polarization plane for the type-3 PAF in $\mathbb{R}^3_1$ is rotated by an angle n_1.*

Proof. The Fermi-Walker derivative of the electric vector field $\mathbf{E}^{(3.Paf)}$ for the type-3 PAF in $\mathbb{R}^3_1$ is expressed by

$$^{FW}\mathbf{E}_s^{(3.Paf)} = \mathbf{E}_s^{(3.Paf)} - \left\langle \mathbf{B}, \mathbf{E}^{(3.Paf)} \right\rangle_L \mathbf{B}_s + \left\langle \mathbf{B}_s, \mathbf{E}^{(3.Paf)} \right\rangle_L \mathbf{B}. \tag{4.7}$$

If the electric vector field $\mathbf{E}^{(3.Paf)}$ of the type-3 PAF is Fermi-Walker parallel, then Eqs.(4.1) and (4.7) lead to

$$\mathbf{E}_s^{(3.Paf)} = \left\langle \mathbf{B}_s, \mathbf{E}^{(3.Paf)} \right\rangle_L \mathbf{B}. \tag{4.8}$$

Also, the electric vector field $\mathbf{E}^{(3.Paf)}$ is given by

$$\mathbf{E}^{(3.Paf)} = E^{(3.Paf)P}\mathbf{P} + E^{(3.Paf)F}\mathbf{F}, \tag{4.9}$$

where

$$E^{(3.Paf)P} = \left\langle \mathbf{E}^{(3.Paf)}, \mathbf{P} \right\rangle_L, \quad E^{(3.Paf)F} = \left\langle \mathbf{E}^{(3.Paf)}, \mathbf{F} \right\rangle_L.$$

It follows that the evolution of the electric vector field $\mathbf{E}^{(3.Paf)}$ is given by

$$\begin{aligned} \mathbf{E}_s^{(3.Paf)} = {} & (-E_s^{(3.Paf)P} + n_1 E^{(3.Paf)F})\mathbf{P} \\ & +(-n_1 E^{(3.Paf)P} + E_s^{(3.Paf)F})\mathbf{F} \\ & -(-n_2 E^{(3.Paf)P} + n_3 E^{(3.Paf)F})\mathbf{B}. \end{aligned} \tag{4.10}$$

Furthermore, from Eqs.(4.8) and (4.10), one obtains

$$\begin{pmatrix} E_s^{(3.Paf)P} \\ E_s^{(3.Paf)F} \end{pmatrix} = \begin{pmatrix} 0 & n_1 \\ -n_1 & 0 \end{pmatrix} \begin{pmatrix} E^{(3.Paf)P} \\ E^{(3.Paf)F} \end{pmatrix}. \tag{4.11}$$

Therefore, (4.11) gives a rotation of the polarization plane by an angle n_1 for the type-3 PAF in $\mathbb{R}^3_1$. □

The magnetic curve produced by the electric vector field $\mathbf{E}^{(3.Paf)}$ along the linearly polarized monochromatic light wave propogating in the optical fiber for the type-3 PAF is called a Lorentzian type-3 PAF electromagnetic curve. The Lorentzian type-3 PAF electromagnetic curve of the timelike curve β in $\mathbb{R}^3_1$ is defined by (cf. [13])

$$\Phi^B \mathbf{E}^{(3.Paf)} = \mathbf{E}_s^{(3.Paf)} = \mathbf{W} \times_L \mathbf{E}^{(3.Paf)}, \tag{4.12}$$

where $\mathbf{W}$ is any divergence free vector field.

Theorem 4.3. *The Lorentz force equation according to the type 3-PAF of the Lorentzian type-3 PAF electromagnetic curve of the timelike curve β is given by*

$$\begin{bmatrix} \Phi^B(\mathbf{P}) \\ \Phi^B(\mathbf{F}) \\ \Phi^B(\mathbf{B}) \end{bmatrix} = \begin{bmatrix} 0 & \lambda & -n_2 \\ \lambda & 0 & -n_3 \\ -n_2 & -n_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P} \\ \mathbf{F} \\ \mathbf{B} \end{bmatrix}. \tag{4.13}$$

Proof. From Eqs. (4.3) and (4.12), one obtains

$$\left\langle \Phi^B \mathbf{E}^{(3.Paf)}, \mathbf{P} \right\rangle_L = -\left\langle \mathbf{E}^{(3.Paf)}, \Phi^B \mathbf{P} \right\rangle_L, \tag{4.14}$$

$$\left\langle \Phi^B \mathbf{E}^{(3.Paf)}, \mathbf{F} \right\rangle_L = -\left\langle \mathbf{E}^{(3.Paf)}, \Phi^B \mathbf{F} \right\rangle_L, \tag{4.15}$$

$$\left\langle \Phi^B \mathbf{E}^{(3.Paf)}, \mathbf{B} \right\rangle_L = -\left\langle \mathbf{E}^{(3.Paf)}, \Phi^B \mathbf{B} \right\rangle_L. \tag{4.16}$$

Via Eqs. (4.14), (4.15) and (4.16), we obtain Eq. (4.13). □

Theorem 4.4. *The magnetic vector field* $\mathbf{W}$ *with respect to the type-3 PAF is given by*

$$\mathbf{W} = -n_3\mathbf{P} + n_2\mathbf{F} + \lambda\mathbf{B}. \tag{4.17}$$

Proof. Using Eq. (4.13), we obtain Eq. (4.17). □

Conclusion

First of all, we constructed the type-2 and type-3 PAFs in Minkowski 3-space. Later, the evolutions of the electric field were presented with respect to the the type-2 and the type-3 PAFs in the Minkowski 3-space. Also, the type-2 and the type-3 Lorentz equations and the magnetic vector fields with respect to the type-2 and type-3 PAFs in the Minkowski 3-space were found.

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Declaration of Competing Interest

The authors report no declarations of interest.

Data Availability Statement

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