TAMKANG JOURNAL OF MATHEMATICS
Volume 40, Number 3, 247-256, Autumn 2009

THE SEQUENCE SPACE $F(X_k, f, p, s)$ ON SEMINORMED SPACES

VINOD K. BHARDWAJ AND INDU BALA

Abstract. The object of this paper is to introduce the vector valued sequence space $F(X_k, f, p, s)$ using a modulus function f. Various algebraic and topological properties of this space have been investigated. Our results generalize and unify the corresponding earlier results of Ghosh and Srivastava [4], Maddox [10].

1. Introduction

Ruckle [13] used the idea of a modulus function f (definition given below) to construct a class of FK spaces

$$L(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}.$$

He gave a negative answer to Wilansky's question: Is there a smallest FK space in which the set $\{e_1, e_2, \ldots\}$ of unit vectors is bounded?

The space $L(f)$ is closely related to the space l_1 which is an $L(f)$ space with $f(x) = x$ for all real $x \geq 0$.

The idea of modulus was structured in 1953 by Nakano [12]. Following Ruckle [13] and Maddox [10], we recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

Because of (ii), $|f(x) - f(y)| \leq f(|x - y|)$ so that in view of (iv), f is continuous everywhere on $[0, \infty)$. A modulus may be unbounded (for example, $f(x) = x^p, 0 < p \leq 1$) or bounded (for example, $f(x) = \frac{x}{(1+x)}$).

It is easy to see that $f_1 + f_2$ is a modulus function when f_1 and f_2 are modulus functions, and that the function f^v (v is a positive integer), the composition of a modulus function f with itself v times, is also a modulus function.

Received January 9, 2007.
2000 *Mathematics Subject Classification.* 40A05, 40C05, 46A45.
Key words and phrases. Modulus function, Paranorm, Sequence space, Seminormed algebra.

In this note, we introduce the vector valued sequence space $F(X_k, f, p, s)$ using a modulus function f, which generalizes the work of Ghosh and Srivastava [4], Jakimovski and Russel [5], and Maddox [10]. Some topological results and inclusion relations for $F(X_k, f, p, s)$ have been discussed. We also give some information on multipliers for $F(X_k, f, p, s)$. The composite space $F(X_k, f^v, p, s)$ using composite modulus function f^v has also been studied. Before introducing this sequence space we recall [6, [8(second edition), 15] some terminology and notations.

An algebra X is a linear space together with an internal operation of multiplication of elements of X, such that $xy \in X$, $x(yz) = (xy)z$, $x(y+z) = xy + xz$, $(x+y)z = xz + yz$ and $\lambda(xy) = (\lambda x)y = x(\lambda y)$, for scalar λ.

In some algebras there exists a non-zero element e such that $ex = xe = x$ for all x. If such an e exists it is obviously unique and is called the identity of the algebra. A normed algebra is an algebra which is normed, as a linear space, and in which $\|xy\| \leq \|x\|\|y\|$ for all x, y.

By w we shall denote the space of all scalar sequences and ϕ is the sequence space of finitely nonzero scalar sequences. A sequence algebra is a subspace F of w such that F is closed under the multiplication defined by $xy = (x_k y_k)$; $x = (x_k) \in F$, $y = (y_k) \in F$.

A sequence space F is said to be symmetric if when x is in F, then y is in F when the coordinates of y are those of x, but in a different order.

A sequence space F is said to be balanced if $(a_k x_k)$ is in F whenever (x_k) is in F and $|a_k| \leq 1$ for each k.

A sequence space F is called solid (or normal) if $y = (y_k) \in F$ whenever $|y_k| \leq |x_k|$, $k \geq 1$, for some $x = (x_k) \in F$. If F is both normal and sequence algebra then it is called a normal sequence algebra. ϕ, w, l_1, l_∞ and c_0 are normal sequence algebras whereas c is a sequence algebra but not normal.

A sequence (b_k) of elements of a paranormed space (X, g) is called a Schauder basis for X if and only if , for each $x \in X$, there exists a unique sequence (λ_k) of scalars such that

$$x = \sum_{k=1}^{\infty} \lambda_k b_k$$

i.e., such that

$$g\left(x - \sum_{k=1}^{n} \lambda_k b_k\right) \to 0 \quad (n \to \infty).$$

A norm $\|.\|_F$ on a normal sequence space F is said to be absolutely monotone if $\|x\|_F \leq \|y\|_F$ for $x = (x_k)$, $y = (y_k) \in F$ with $|x_k| \leq |y_k|$ for all $k \in \mathbb{N}$.

Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is stronger than q_2 if there exists a constant L such that $q_2(x) \leq L q_1(x)$ for all $x \in X$. If each is stronger than the other, q_1 and q_2 are said to be equivalent.

The following inequalities (see, e.g., [7; first edition, p. 190]) are needed throughout the paper.

Let $p = (p_k)$ be a bounded sequence of positive real numbers. If $H = \sup_k p_k$, then for any complex a_k and b_k,

(1) $|a_k + b_k|^{p_k} \le C(|a_k|^{p_k} + |b_k|^{p_k})$,
where $C = \max(1, 2^{H-1})$. Also for any complex λ,

(2) $|\lambda|^{p_k} \le \max(1, |\lambda|^H)$.

We now introduce the vector valued sequence space $F(X_k, f, p, s)$ using modulus function f.

Let X_k be a seminormed space over the complex field $\mathbb{C}$ with seminorm q_k for each $k \in \mathbb{N}$, and F be a normal sequence algebra with absolutely monotone norm $\|.\|_F$ and having a Schauder basis (e_k), where $e_k = (0, 0, \ldots, 1, 0, \ldots)$, with 1 in the k-th place. Let $p = (p_k)$ be any sequence of strictly positive real numbers and s be any non-negative real number. By $w(X_k)$, we denote the linear space of all sequences $x = (x_k)$ with $x_k \in X_k$ for each $k \in \mathbb{N}$ under the usual coordinatewise operations:

$$x + y = (x_k + y_k) \ \text{ and } \ \alpha x = (\alpha x_k)$$

for each $\alpha \in \mathbb{C}$. If $x \in w(X_k)$ and $\lambda = (\lambda_k)$ is a scalar sequence then we shall write $\lambda x = (\lambda_k x_k) \in w(X_k)$.

For a modulus function f, we define

$$F(X_k, f, p, s) = \{x = (x_k) \in w(X_k) : (k^{-s}[f(q_k(x_k))]^{p_k}) \in F\}.$$

The norm $\|.\|_F$ and the condition on $\|.\|_F$ are irrelevant as far as the definition of the sets $F(X_k, f, p, s)$ is concerned; they are needed to define a topology on it.

Some well-known spaces are obtained by specializing F, X_k, f, p and s.

(i) If $F = l_1$, $X_k = \mathbb{C}$ for all k, $s = 0$, $f(x) = x$ then $F(X_k, f, p, s) = l(p)$ (Simons [14]).

(ii) If $F = l_1$, $X_k = \mathbb{C}$ for all k, $s = 0$, $f(x) = x$ and $p_k = p$ for all k then $F(X_k, f, p, s) = l_p$.

(iii) If $F = l_1$, $X_k = \mathbb{C}$ for all k, $s = 0$, $f(x) = x$ and $p_k = 1$ for all k then $F(X_k, f, p, s) = l_1$.

(iv) If $F = l_1$, $X_k = \mathbb{C}$ for all k, $s = 0$ and $p_k = 1$ for all k then $F(X_k, f, p, s) = L(f)$ (Ruckle [13]).

(v) If $F = l_1$, $X_k = \mathbb{C}$ for all k and $s = 0$ then $F(X_k, f, p, s) = L(f, p)$ (Bhardwaj [2]).

(vi) If $F = l_1$, $X_k = X$ (a Banach space over $\mathbb{C}$) for all k, $s = 0$, $f(x) = x$ and $p_k = p$ for all k, then $F(X_k, f, p, s) = l_p(X)$ (Leonard [7]).

(vii) If $F = l_1$, $X_k = X$ (a seminormed space over $\mathbb{C}$ with seminorm q)for all k, then $F(X_k, f, p, s) = l(p, f, q, s)$ (Bilgin [3]).

(viii) If $F = l_\infty$, $X_k = X$ (a Banach space over $\mathbb{C}$) for all k, $s = 0$, $f(x) = x$ and $p_k = 1$ for all k, then $F(X_k, f, p, s) = l_\infty(X)$ (Leonard [7], Maddox [9]).

(ix) If $F = w_0$ or w_∞, $X_k = \mathbb{C}$ for all k, $s = 0$ and $p_k = 1$ for all k then the set $F(X_k, f, p, s)$ reduces to $w_0(f)$ and $w_\infty(f)$, respectively (Maddox [10]).

(x) If $F = w_0$ or w_∞, $X_k = X$ (a Banach space over $\mathbb{C}$) for all k, $s = 0$ and $p_k = 1$ for all k, then the set $F(X_k, f, p, s)$ reduces to $w_0(f, X)$ and $w_\infty(f, X)$, respectively (Bhardwaj and Singh [1]).

(xi) If $F = l_\infty$, $X_k = \mathbb{C}$ for all k, $s = 0$, $f(x) = x$ and $p_k = 1$ for all k, then $F(X_k, f, p, s) = l_\infty$.

(xii) If $X_k = E_k$ (Banach spaces over $\mathbb{C}$), $s = 0$ and $p_k = 1$ for all k then $F(X_k, f, p, s) = F(E_k, f)$ (Ghosh and Srivastava [4]).

We denote $F(X_k, f, p, s)$ by $F(X_k, f, s)$ when $p_k = 1$ for all k and by $F(X_k, p, s)$ when $f(x) = x$.

2. Linear Topological Structure of $F(X_k, f, p, s)$ Space and Inclusion Theorems

In this section we examine some algebraic and topological properties of the sequence space defined above and investigate some inclusion relations between these spaces.

Theorem 2.1. *Let $H = \sup p_k < \infty$, then $F(X_k, f, p, s)$ is a linear space over the complex field $\mathbb{C}$.*

Proof. Let $x, y \in F(X_k, f, p, s)$. For $\alpha, \beta \in \mathbb{C}$, there exist positive integers M_α and N_β such that $|\alpha| \le M_\alpha$ and $|\beta| \le N_\beta$. From definition of modulus function (ii) and (iii) and inequalities (1) and (2), we have

$$k^{-s}\Big[f\Big(q_k(\alpha x_k + \beta y_k)\Big)\Big]^{p_k} \le C\bigg(M_\alpha^H k^{-s}\Big[f(q_k(x_k))\Big]^{p_k} + N_\beta^H k^{-s}\Big[f(q_k(y_k))\Big]^{p_k}\bigg),$$

where $C = \max(1, 2^{H-1})$. Since F is a normal sequence algebra, we have $k^{-s}[f(q_k(\alpha x_k + \beta y_k))]^{p_k} \in F$ which shows that $\alpha x + \beta y \in F(X_k, f, p, s)$. Hence $F(X_k, f, p, s)$ is a linear space over $\mathbb{C}$.

Theorem 2.2. *Let f, f_1 and f_2 be modulus functions, s, s_1 and s_2 be non-negative real numbers, then*

(i) $F(X_k, f_1, p, s) \bigcap F(X_k, f_2, p, s) \subseteq F(X_k, f_1 + f_2, p, s)$,

(ii) *If $s_1 \le s_2$, then $F(X_k, f, p, s_1) \subseteq F(X_k, f, p, s_2)$,*

(iii) *If $F_1 \subseteq F_2$, then $F_1(X_k, f, p, s) \subseteq F_2(X_k, f, p, s)$.*

Proof. (i) Let $x = (x_k) \in F(X_k, f_1, p, s) \bigcap F(X_k, f_2, p, s)$.
Then $(k^{-s}[f_1(q_k(x_k))]^{p_k})$ and $(k^{-s}[f_2(q_k(x_k))]^{p_k}) \in F$.
Using (1) we have

$$k^{-s}\Big[(f_1 + f_2)(q_k(x_k))\Big]^{p_k} \le C\{k^{-s}\Big[f_1(q_k(x_k))\Big]^{p_k} + k^{-s}\Big[f_2(q_k(x_k))\Big]^{p_k}\}$$

Since F is normal, $x \in F(X_k, f_1 + f_2, p, s)$.
The proofs of (ii) and (iii) are trivial.

Corollary 2.3. *$F(X_k, f, p) \subseteq F(X_k, f, p, s)$ for any modulus function f.*

Theorem 2.4. *$F(X_k, f, p, s)$ is normal and symmetric.*

Proof. Let $x = (x_k) \in F(X_k, f, p, s)$ and $y = (y_k)$ be any sequence in $w(X_k)$ such that $q_k(y_k) \le q_k(x_k)$ for each k. Since f is increasing, we have

$$k^{-s}\Big[f(q_k(y_k))\Big]^{p_k} \le k^{-s}\Big[f(q_k(x_k))\Big]^{p_k}.$$

The normality of F implies that $y = (y_k) \in F(X_k, f, p, s)$ and hence $F(X_k, f, p, s)$ is normal. The proof of the fact that $F(X_k, f, p, s)$ is symmetric is obvious.

Theorem 2.5. *If (X_k, q_k) is seminormed algebra for each $k \in \mathbb{N}$ then $F(X_k, f, p, s)$ is balanced.*

Proof. Let $x = (x_k) \in F(X_k, f, p, s)$ and $a = (a_k)$ be any sequence in $w(X_k)$ such that $q_k(a_k) \le 1$ for each k. Since f is increasing and each X_k is seminormed algebra we have

$$k^{-s}\Big[f(q_k(a_k x_k))\Big]^{p_k} \le k^{-s}\Big[f(q_k(x_k))\Big]^{p_k}$$

The normality of the space F implies that $ax = (a_k x_k) \in F(X_k, f, p, s)$ and hence $F(X_k, f, p, s)$ is balanced.

We conclude this section by considering the metrization of $F(X_k, f, s)$ space.

Theorem 2.6. *For any modulus f, $F(X_k, f, s)$ is a complete topological linear space, paranormed by*

$$g(x) = \Big\|k^{-s}\Big[f(q_k(x_k))\Big]\Big\|_F$$

if X_k is complete under the seminorm q_k for each $k \in \mathbb{N}$.

Proof. Clearly $g(\theta) = 0$ for $\theta = (\theta_1, \theta_2, \ldots)$ the zero element of $F(X_k, f, s)$ (where θ_i is the zero element of X_i for each i). Also $g(-x) = g(x)$. We now show that g is subadditive

$$\begin{aligned} g(x+y) &= \Big\|k^{-s}[f(q_k(x_k + y_k))]\Big\|_F \\ &\le \Big\|k^{-s}[f(q_k(x_k))\Big]\|_F + \Big\|k^{-s}[f(q_k(y_k))]\Big\|_F \end{aligned}$$

$$= g(x) + g(y)$$

as f is increasing and $\|.\|_F$ is absolutely monotone norm.

We now show that the scalar multiplication is continuous. For any complex λ,

$$g(\lambda x) \leq (1 + [|\lambda|])g(x)$$

where $[t]$ denotes the integer part of t, whence $\lambda \to 0$, $x \to \theta$ imply $\lambda x \to \theta$ and also $x \to \theta$, λ fixed imply $\lambda x \to \theta$. Suppose that $\lambda_n \to 0$ and x is fixed in $F(X_k, f, s)$.

Then

$$t = (t_k) = (k^{-s} f(q_k(x_k))) \in F.$$

For arbitrary $\epsilon > 0$, let N be a positive integer such that

$$\left\| t - \sum_{k=1}^{N} t_k e_k \right\|_F = \left\| \sum_{k=N+1}^{\infty} t_k e_k \right\|_F < \frac{\epsilon}{2}$$

since (e_k) is a Schauder basis of F.

Since f is continuous everywhere in $[0, \infty)$,

$$g(u) = \left\| \sum_{k=1}^{N} k^{-s}[f(q_k(ux_k))]e_k \right\|_F,$$

is continuous at 0. So there is $1 > \delta > 0$ such that $g(u) < (\frac{\epsilon}{2})$ for $0 < u < \delta$. Let K be a positive integer such that $|\lambda_n| < \delta$ for $n > K$, then for $n > K$

$$\left\| \sum_{k=1}^{N} k^{-s}[f(q_k(\lambda_n x_k))]e_k \right\|_F < \frac{\epsilon}{2}.$$

Thus

$$\|k^{-s}[f(q_k(\lambda_n x_k))]\|_F < \epsilon \ \text{ for } n > K,$$

so that $g(\lambda x) \to 0$ as $\lambda \to 0$.

To show that $F(X_k, f, s)$ is complete, let (x^i) be a Cauchy sequence in $F(X_k, f, s)$. Then $g(x^i - x^j) \to 0$ as $i, j \to \infty$. Hence for each fixed k, $q_k(x_k^i - x_k^j) \to 0$ as $i, j \to \infty$ and so (x_k^i) is a Cauchy sequence in X_k for each fixed k. Since X_k is complete, so there exists a sequence $x = (x_k)$ such that $x_k \in X_k$ for each $k \in \mathbb{N}$ and $q_k(x_k^i - x_k) \to 0$ as $i \to \infty$, for each fixed $k \in \mathbb{N}$. For given $\epsilon > 0$, choose an integer K such that $g(x^i - x^j) < \epsilon$ for $i, j > K$. Since F is normal and (e_k) is a Schauder basis of F,

$$\left\| \sum_{k=1}^{n} k^{-s}[f(q_k(x_k^i - x_k^j))]e_k \right\|_F \leq \left\| k^{-s}[f(q_k(x_k^i - x_k^j))] \right\|_F$$
$$< \epsilon \ \text{ for } \ i, j > K.$$

Since f is continuous, so by taking $j \to \infty$ in the above expression, we get

$$\left\| \sum_{k=1}^{n} k^{-s}[f(q_k(x_k^i - x_k))]e_k \right\|_F < \epsilon \quad \text{for} \quad i > K.$$

Since n is arbitrary, by taking $n \to \infty$, we obtain $g(x^i - x) < \epsilon$ for $i > K$. So (x^i) converges to x in the paranorm of $F(X_k, f, s)$. We now show that $x \in F(X_k, f, s)$. Since $q_k(x_k^i - x_k) \to 0$ as $i \to \infty$, for each fixed k we choose a positive number δ_k^i, $0 < \delta_k^i < 1$, such that

$$f(q_k(x_k^i - x_k)) < \delta_k^i[f(q_k(x_k^i))]$$

Consider

$$\begin{aligned} k^{-s}[f(q_k(x_k))] &\le k^{-s}[f(q_k(x_k^i))] + k^{-s}[f(q_k(x_k - x_k^i))] \\ &< (1 + \delta_k^i)k^{-s}[f(q_k(x_k^i))] \end{aligned}$$

Since F is normal, so $x = (x_k) \in F(X_k, f, s)$. Therefore, $F(X_k, f, s)$ is a complete paranormed space.

Remark 2.7. It can be easily verified that when $F = l_1$, $(X_k, q_k) = (\mathbb{C}, |.|)$ and $s = 0$ the paranorms defined on $F(X_k, f, s)$ and $L(f)$ are the same.

3. The Space of Multipliers of $F(X_k, f, p, s)$

Suppose (X_k, q_k) is seminormed algebra for each $k \in \mathbb{N}$. Define $M[F(X_k, f, p, s)]$, the space of multipliers of $F(X_k, f, p, s)$, as

$$M[F(X_k, f, p, s)] = \Big\{a = (a_k) \in w(X_k) : (k^{-s}[f(q_k(a_k x_k))]^{p_k}) \in F, \text{ for all } x = (x_k) \in F(X_k, f, p, s)\Big\}.$$

Theorem 3.1. *For any modulus f, $l_\infty(X_k) \subseteq M[F(X_k, f, p, s)]$, where*

$$l_\infty(X_k) = \Big\{x = (x_k) \in w(X_k) : \sup_k q_k(x_k) < \infty\Big\}.$$

Proof. $a = (a_k) \in l_\infty(X_k)$ implies $q_k(a_k) < 1+[T]$ for all k, where $T = \sup_k q_k(a_k) < \infty$ and $[T]$ denotes the integer part of T. Let $x = (x_k) \in F(X_k, f, p, s)$. Since X_k is seminormed algebra for each k, by definition of modulus function (ii) and (iii) and inequality (2), we have

$$k^{-s}\Big[f(q_k(a_k x_k))\Big]^{p_k} \le \Big(1 + [T]\Big)^H k^{-s}\Big[f(q_k(x_k))\Big]^{p_k}$$

Since F is normal, $ax \in F(X_k, f, p, s)$, as desired.

The inclusion seems to be proper but we have not been able to prove it. It is, therefore, an open problem.

Proposition 3.2. *If $e = (e_1, e_2, \ldots) \in F(X_k, f, p, s)$, then $M[F(X_k, f, p, s)] \subseteq F(X_k, f, p, s)$, where e_k is the identity element of X_k for each $k \in \mathbb{N}$.*

Proposition 3.3. *If f is a modulus function such that $f(xy) \le f(x) + f(y)$ for all $x \ge 0$, $y \ge 0$, then $F(X_k, f, p, s) \subseteq M[F(X_k, f, p, s)]$.*

Proof. Let $a = (a_k) \in F(X_k, f, p, s)$ and $x = (x_k) \in F(X_k, f, p, s)$.
Then

$$\Big(k^{-s}[f(q_k(a_k))]^{p_k}\Big) \text{ and } \Big(k^{-s}[f(q_k(x_k))]^{p_k}\Big) \in F.$$

Since X_k is seminormed algebra for each k, using (1) we have,

$$\begin{aligned} k^{-s}\Big[f(q_k(a_k x_k))\Big]^{p_k} &\le k^{-s}\Big[f(q_k(a_k)q_k(x_k))\Big]^{p_k} \\ &\le k^{-s}\Big[f(q_k(a_k)) + f(q_k(x_k))\Big]^{p_k} \\ &\le C\Big(k^{-s}[f(q_k(a_k))]^{p_k} + k^{-s}[f(q_k(x_k))]^{p_k}\Big), \end{aligned}$$

where $C = \max(1, 2^{H-1})$. Since F is a normal sequence algebra, $ax \in F(X_k, f, p, s)$ i.e., $a \in M[F(X_k, f, p, s)]$.

Example 3.4. $f(x) = \log(1 + x)$, is a modulus function which satisfies the condition of Proposition 3.3 i.e., $f(xy) \le f(x) + f(y)$ for $x \ge 0$, $y \ge 0$ (Prop. 2 of Maddox [10]).

4. Composite space $F(X_k, f^v, p, s)$ using composite modulus function f^v

Taking modulus function f^v instead of f in the space $F(X_k, f, p, s)$, we can define the composite space $F(X_k, f^v, p, s)$ as follows:

Definition 4.1. For a fixed natural number v, we define

$$F(X_k, f^v, p, s) = \Big\{x = (x_k) \in w(X_k) : (k^{-s}[f^v(q_k(x_k))]^{p_k}) \in F\Big\}.$$

Theorem 4.2. *Let f be a modulus function and let $v \in \mathbb{N}$. Then*

(i) $F(X_k, f^v, p, s) \subseteq F(X_k, p, s)$ *if* $\lim_{t\to\infty} \frac{f(t)}{t} = \beta > 0$,

(ii) $F(X_k, p, s) \subseteq F(X_k, f^v, p, s)$ *if there exists a positive constant α such that $f(t) \le \alpha t$ for all $t \ge 0$.*

Proof. (i) Let $x = (x_k) \in F(X_k, f^v, p, s)$. Then

$$\Big(k^{-s}[f^v(q_k(x_k))]^{p_k}\Big) \in F$$

Following the proof of Prop. 1 of Maddox [11], we have $\beta = \lim_{t\to\infty} \frac{f(t)}{t} = \inf\{\frac{f(t)}{t} : t > 0\}$, so that $0 \le \beta \le f(1)$. Let $\beta > 0$. By definition of β we have $\beta t \le f(t)$ for all $t \ge 0$. Since f is increasing we have $\beta^2 t \le f^2(t)$. So by induction, we have $\beta^v t \le f^v(t)$. So using inequality (2),

$$\begin{aligned} k^{-s}\Big[q_k(x_k)\Big]^{p_k} &\le k^{-s}\Big[\beta^{-v}(f^v(q_k(x_k)))\Big]^{p_k} \\ &\le \max(1, \beta^{-vH})k^{-s}\Big[f^v(q_k(x_k))\Big]^{p_k} \end{aligned}$$

Since F is normal, $x \in F(X_k, p, s)$ and the proof is complete.

(ii) Let $x = (x_k) \in F(X_k, p, s)$, then $(k^{-s}[q_k(x_k)]^{p_k}) \in F$. Since $f(t) \le \alpha t$ we have $f^v(t) \le \alpha^v t$, so using inequality (2)

$$\begin{aligned} k^{-s}\Big[f^v(q_k(x_k))\Big]^{p_k} &\le k^{-s}\Big[\alpha^v q_k(x_k)\Big]^{p_k} \\ &\le \max\Big(1, \alpha^{vH}\Big)k^{-s}\Big[q_k(x_k)\Big]^{p_k} \end{aligned}$$

Since F is normal, $x \in F(X_k, f^v, p, s)$ and therefore, $F(X_k, p, s) \subseteq F(X_k, f^v, p, s)$.

Example 4.3. $f_1(t) = t + t^{1/2}$ and $f_2(t) = \log(1 + t)$ for all $t \ge 0$ satisfy the conditions given in Theorem 4.2(i), (ii) respectively.

Theorem 4.4. *Let $m, v \in \mathbb{N}$ and $m < v$. If f is a modulus such that $f(t) \le \alpha t$ for all $t \ge 0$, where α is a positive constant, then*

$$F(X_k, p, s) \subseteq F(X_k, f^m, p, s) \subseteq F(X_k, f^v, p, s).$$

Proof. Let $r = v - m$. Since $f(t) \le \alpha t$, we have $f^v(t) < M^r f^m(t) < M^v t$, where $M = 1 + [\alpha]$. Let $x = (x_k) \in F(X_k, p, s)$. By the above inequality, we get

$$\begin{aligned} k^{-s}\Big[f^v(q_k(x_k))\Big]^{p_k} &< M^{rH}k^{-s}\Big[f^m(q_k(x_k))\Big]^{p_k} \\ &< M^{vH}k^{-s}\Big[q_k(x_k)\Big]^{p_k} \end{aligned}$$

Since F is normal, the required inclusion follows.

References

[1] V. K. Bhardwaj and N. Singh, *Banach space valued sequence spaces defined by a modulus*, Indian J. Pure Appl. Math., **32**(2001), 1869–1882.

[2] V. K. Bhardwaj, *A generalization of a sequence space of Ruckle*, Bull. Cal. Math. Soc., **95**(2003), 411-420.

[3] T. Bilgin, *The sequence space $l(p, f, q, s)$ on seminormed spaces*, Bull. Cal. Math. Soc., **86**(1994), 295-304.

[4] D. Ghosh and P. D. Srivastava, *On some vector valued sequence spaces defined using a modulus function*, Indian J. Pure Appl. Math., **30**(1999), 819–826.

[5] A. Jakimovski and D. C. Russel, *Representation of continous linear functionals on a subspace of a countable cartesian product of Banach spaces*, Studia Math., **72**(1982), 273–284.

[6] P. K. Kamthan and M. Gupta, Sequence Spaces and Series, Marcel Dekker Inc., New York, 1981.

[7] I. E. Leonard, *Banach sequence spaces*, J. Math. Anal. Appl., **54**(1976), 245–265.

[8] I. J. Maddox, Elements of Functional Analysis, Cambridge Univ. Press, 1970(first edition), 1988(second edition).

[9] I. J. Maddox, *Infinite matrices of operators*, Lecture Notes in Mathematics 786, Springer-Verlag Berlin, Heidelberg, New York, 1980.

[10] I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Camb. Philos. Soc., **100**(1986), 161–166.

[11] I. J. Maddox, *Inclusion between FK spaces and Kuttner's Theorem*, Math. Proc. Camb. Philos. Soc., **101**(1987), 523–527.

[12] H. Nakano, *Concave modulars*, J. Math. Soc. Japan, **5**(1953), 29–49.

[13] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math., **25**(1973), 973–978.

[14] S. Simons, *The sequence spaces $l(p_v)$ and $m(p_v)$*, Proc. London Math. Soc., **15**(1965), 422–436.

[15] A. Wilansky, Functional Analysis, Blaisdell Publishing Company, New York, 1964.

Department of Mathematics, Kurukshetra University, Kurukshetra -136 119, INDIA.

E-mail: vinodk_bhj@rediffmail.com

Department of Mathematics, Rajiv Gandhi Government College, Saha (Ambala)-133 104, INDIA.

E-mail: bansal_indu@rediffmail.com