TAMKANG JOURNAL OF MATHEMATICS
Volume 55, Number 2, 149-178, 2024
DOI:10.5556/j.tkjm.55.2024.5044

A study of statistical submersions

Aliya Naaz Siddiqui and Kamran Ahmad

Abstract. In the sixties, A. Gray [19] and B. O'Neill [27] come with the notion of Riemannian submersions as a tool to study the geometry of a Riemannian manifold with an additional structure in terms of the fibers and the base space. Riemannian submersions have long been an effective tool to construct Riemannian manifolds with positive or nonnegative sectional curvature in Riemannian geometry and compare certain manifolds within differential geometry. In particular, many examples of Einstein manifolds can be constructed by using such submersions. It is very well known that Riemannian submersions have applications in physics, for example Kaluza-Klein theory, Yang-Mills theory, supergravity and superstring theories.

In [41], Watson popularizes the knowledge of Riemannian submersions between almost Hermitian manifolds under the name of almost Hermitian submersions and many researchers discuss such submersions between various subclasses of almost Hermitian manifolds. Then, Sahin extends Riemannian submersions to many subclasses of almost contact metric manifolds under the title of contact Riemannian submersions in [35]. Afterwards, B. Sahin [34] comes with a self-contained exposition of recent developments in Riemannian submersions and maps. On the other hand, B. Nielsen and Jupp [7] discuss the Riemannian submersion from the viewpoint of statistics. N. Abe and K. Hasegawa [1] introduce the notion of statistical submersions between statistical manifolds by generalizing some basic results of B. O'Neill concerning Riemannian submersions and geodesics. Since then, the study of submersions became an active research subject, and many papers have been published by numerous of geometers (see [8]). The purpose of this article is to provide a comprehensive survey on the study of recent developments in statistical submersions.

Keywords. Statistical manifolds, statistical submersions, Kähler-like statistical submersion, Ssasaki-like statistical submersion, quaternionic Kähler-like statistical submersion, cosymplectic-like statistical submersion, para-Kähler-like statistical submersion, Kenmotsu-like statistical submersion, statistical solitons, Chen-Ricci inequality, Casorati curvatures.

Received date: November 14, 2022; Published online: May 17, 2023.
2010 *Mathematics Subject Classification*. 53A15, 53C05.
Corresponding author: Aliya Naaz Siddiqui.

1 Introduction

A statistical manifold of probability distributions is a Reimannian manifold equipped with a Riemannian metric and a pair of dual (conjugate) torsion-free affine connections. A statistical manifold is a semi-Riemannian manifold equipped with an additional structure given by a pair of dual torsion-free affine connections. Let $\Gamma(TM)$ be the space of all vector fields on a Riemannian or semi-Riemannian manifold (M, g_M).

Definition 1. Let M be a semi-Riemannian manifold and non-degenerate metric g_M, and a torsion-free affine connection by $'\nabla$. The triplet $(M,' \nabla, g_M)$ is called a statistical manifold if $'\nabla$ is compatible to g_M [3, 4].

For a statistical manifold $(M,' \nabla, g_M)$, we describe a second connection $'\nabla^*$ as

$$Gg_M(E,F) = g_M('\nabla_G E, F) + g_M(E, '\nabla^*_G F), \tag{1.1}$$

for any $E, F, G \in \Gamma(TM)$. Here affine connection $'\nabla^*$ is called conjugate (or dual) of the connection $'\nabla$ with respect to the g_M. The affine connection $'\nabla^*$ is torsion-free, $'\nabla^* g_M$ is also symmetric and obeys

$$('\nabla^*)^* =' \nabla.$$

Also,

$$2'\nabla^0 =' \nabla +' \nabla^*, \tag{1.2}$$

where $'\nabla^0$ is the Levi-Civita connection on M.

The Riemann curvature tensor Rim is defined as a map $Rim\colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ by the following:

$$Rim(E,F)G =' \nabla'_E \nabla_F G -' \nabla'_F \nabla_E G -' \nabla_{[E,F]} G,$$

or equivalently we have

$$Rim(E,F) = ['\nabla_E,' \nabla_F] -' \nabla_{[E,F]},$$

where $'\nabla$ denotes an affine connection, $[E, F]$ is the Lie bracket of vector fields and $['\nabla_E,' \nabla_F]$ is a commutator of differential operators.

A space form is a complete Riemannian manifold with constant sectional curvature. A statistical manifold $(M,' \nabla, g_M)$ is said to be of constant curvature $c \in \mathbb{R}$ if

$$Rim(E,F)G = c(g_M(F,G)E - g_M(E,G)F),$$

holds for any $E, F, G \in \Gamma(TM)$.

The theory of statistical manifolds and statistical submanifolds is a recent geometry, which plays a crucial role in several fields of mathematics. Several results have been done by distinguished geometers in this area. Many relevant examples of statistical manifolds are studied such as exponential families, whose points are probability densities of exponential form depending on a finite number of parameters.

The concept of submersion in differential geometry is first exposed by O'Neill [26] and Gray [19]. Most of the research related to the various submersion can be found in [17]. Abe and Hasegawa introduce the concept statistical submersions between statistical manifolds and K. Takano defines the statistical model of the multivariate normal distribution as the Riemannian manifold and constructed a good example of statistical submersion. Then, K. Takano found interesting to study Kähler-like statistical manifold by using a dualistic pair of tensor fields J, J^* of type $(1,1)$ and its statistical submersion [36], Sasaki-like statistical manifold by using a dualistic pair of tensor fields ϕ, ϕ^* of type $(1,1)$ and obtains several geometric properties of statistical submersions which are compatible with almost contact structures in [38].

Motivated by K. Takano's studies, recently A.-D. Vilcu and G.-E. Vilcu [40] put suitable almost quaternionic structures on statistical manifolds and define the new notion called quaternionic Kähler-like statistical manifold. They also investigate new submersion called quaternionic Kähler-like statistical submersion and derive its main properties.

In [18], Furuhata, Hasegawa, Okuyama and Sato define a warped product of two statistical manifolds and then construct a Kenmotsu statistical manifold as the warped product of a holomorphic statistical manifold and a line. Later on, Murathan and Sahin [25] get inspiration from them and used usual product of trivial Euclidean statistical axis and a Kähler-like statistical manifold to construct Kenmotsu-like statistical and cosymplectic-like statistical manifolds. Motivated by above studies, Aytimur and Ozgur [6] come with the concept of cosymplectic-like statistical submersions. For such statistical submersion, they show that the base space is a Kähler-like statistical manifold and each fiber is a cosymplectic-like statistical manifold.

In continuation of study of different classes of statistical submersions, G.-E. Vilcu [39] give new concept of statistical manifolds endowed with almost product structures called para-Kähler-like statistical manifold and its statistical submersion. He derives interesting properties of such statistical submersions [39]. Similar to the Takano's definition for Sasaki-like statistical submersion, Danish, Siddiqui and Aytimur define and discuss nice properties of Kenmotsu-like statistical submersion in [33]. Also, they study Kenmotsu-like statistical submersions with conformal fibers.

2 Preliminaries

Let (M, g_M) and (N, g_N) be two connected semi-Riemannian manifolds of respective index r and s provided $s \leq r$ and $0 \leq r \leq m$, $0 \leq s \leq n$, where $\dim(M) = m$ and $\dim(N) = n$. Then a smooth and onto map $\omega : M \to N$ is said to be a semi-Riemannian submersion, if $\omega_{*|_x} : T_x M \to T_{\omega(x)} N$ is onto for $x \in M$; for each $p \in N$, the fibres $\overline{M} = \omega^{-1}(p)$ are called semi-Riemannian submanifolds of M and the dimension of each fiber is $m - n$; and ω_* preserves the lengths of horizontal vectors. Here, each fiber is denoted by $(\overline{M}, \overline{g} = g|_{\overline{M}})$, the total space is (M, g_M), and the base space is (N, g_N).

If a vector field on M is always tangent to fibers, we call it a vertical vector and a horizontal vector if it is always normal to fibers. In the tangent bundle TM of M, we denote the vertical distribution by $\mathcal{V}(M)$ and the horizontal distribution by $\mathcal{H}(M)$. Then

$$TM = \mathcal{V}(M) \oplus \mathcal{H}(M). \tag{2.1}$$

A horizontal vector field X on M is called basic if X is ω-related to a vector field X_* on N such that $\omega_*(X_x) = X_{*\omega(x)}$. If X and Y are the basic vector fields on M, ω-related to X_*, Y_* on N, we have the following facts:

1. $g_M(X, Y) = g_N(X_*, Y_*) \circ \omega$;
2. $\mathcal{H}[X, Y]$ is the basic vector field and is ω-related to $[X_*, Y_*]$;
3. for any vertical vector field U, $[X, U]$ is vertical.

Let $(M, {}'\nabla, g_M)$ and (N, ∇, g_N) be two statistical manifolds and. Let $\overline{\nabla}$ and $\overline{\nabla}^*$ be the induced affine connections on $\overline{M}$ by ${}'\nabla$ and ${}'\nabla^*$ on M. It is clear that $\overline{\nabla}_U V = \mathcal{V}'\nabla_U V$ and $\overline{\nabla}^*_U V = \mathcal{V}'\nabla^*_U V$, for $U, V \in \Gamma(\mathcal{V})$. It can be simply observed that $\overline{\nabla}$ and $\overline{\nabla}^*$ are torsion-free and conjugate to each other with respect to g_M.

Definition 2. Let a submersion $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ between two statistical manifolds is called a statistical submersion if ω fulfils $\omega_*({}'\nabla_X Y)_x = (\nabla_{\omega_*(X)}\omega_*(Y))_x$ for basic vector fields X, Y on M and $x \in M$.

In [37], K. Takano treats the statistical model of the multivariate normal distribution as the Riemannian manifold and constructed the statistical submersion that the total space is of the multivariate normal distribution.

The geometry of semi-Riemannian submersions is characterized by the author O′Neill's tensors $\mathbf{T}$ and $\mathbf{A}$, defined as follows [26]:

$$\mathbf{T}_E F = \mathcal{V}'\nabla_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}'\nabla_{\mathcal{V}E}\mathcal{V}F, \tag{2.2}$$

and

$$\mathbf{A}_E F = \mathcal{V}'\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}'\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.3}$$

for any $E, F \in \Gamma(TM)$.

It is easy to see that $\mathbf{T}$ and $\mathbf{A}$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions.

We summarize the properties of the tensor fields $\mathbf{T}$ and $\mathbf{A}$. Let V, W be vertical and X, Y be horizontal vector fields on M, then we have

$$\mathbf{T}_V W = \mathbf{T}_W V, \tag{2.4}$$

$$\mathbf{A}_X Y = -\mathbf{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \tag{2.5}$$

By changing ${}'\nabla$ for ${}'\nabla^*$ in (2.2) and (2.3), we define $\mathbf{T}^*$ and $\mathbf{A}^*$ [36]. $\mathbf{A} = 0 = \mathbf{A}^*$ if and only if $\mathcal{H}(M)$ is integrable with respect to ${}'\nabla$ and ${}'\nabla^*$, respectively. For $X, Y \in \mathcal{H}(M)$ and $V, W \in \mathcal{V}(M)$, we turn up

$$g_M(\mathbf{T}_V W, X) = -g_M(W, \mathbf{T}^*_V X) \quad \text{and} \quad g(\mathbf{A}_X Y, V) = -g_M(Y, \mathbf{A}^*_X V). \tag{2.6}$$

Now, we will discuss some useful properties of statistical submersion proposed by Takano [36]. First we have the following lemmas for this study. Therefore, for a statistical submersion $\omega : (M,' \nabla, g_M) \to (N, \nabla, g_N)$, we have [26, 36]

Lemma 2.1. *[36] If X and Y are horizontal vector field, then $\mathbf{A}_X Y = -\mathbf{A}^*_Y X$.*

Lemma 2.2. *[36] For $X, Y \in \mathcal{H}(M)$ and $V, W \in \mathcal{V}(M)$. Then we have*

$$'\nabla_V W = \mathbf{T}_V W + \overline{\nabla}_V W, \qquad '\nabla^*_V W = \mathbf{T}^*_V W + \overline{\nabla}^*_V W, \tag{2.7}$$

$$'\nabla_V X = \mathbf{T}_V X + \mathcal{H}'\nabla_V X, \qquad '\nabla^*_V X = \mathbf{T}^*_V X + \mathcal{H}'\nabla^*{}_V X, \tag{2.8}$$

$$'\nabla_X V = \mathbf{A}_X V + \mathcal{V}'\nabla_X V, \qquad '\nabla^*_X V = \mathbf{A}^*_X V + \mathcal{V}'\nabla^*_X V, \tag{2.9}$$

$$'\nabla_X Y = \mathcal{H}'\nabla_X Y + \mathbf{A}_X Y, \qquad '\nabla^*_X Y = \mathcal{H}'\nabla^*_X Y + \mathbf{A}^*_X Y. \tag{2.10}$$

Furthermore, if X is basic, then $\mathcal{H}'\nabla_V X = \mathbf{A}_X V$ and $\mathcal{H}'\nabla^*_V X = \mathbf{A}^*_X V$.

Let $\overline{Rim}$ (resp. $\overline{Rim}^*$) be the curvature tensor with respect to the induced affine connection $\overline{\nabla}$ (resp. $\overline{\nabla}^*$) of each fiber. Moreover, let $\hat{Rim}(X, Y)Z$ (resp. $\hat{Rim}^*(X, Y)Z$) be horizontal vector field such that

$$\omega_*(\hat{Rim}(X, Y)Z) = \hat{Rim}(\omega_* X, \omega_* Y)\omega_* Z$$

at each point $x \in M$, where $\hat{Rim}$ (resp. $\hat{Rim}^*$) be the curvature tensor of N with respect to ∇ (resp. ∇^*). Then we have the following theorem [36]:

Theorem 2.1. *[36, 38] If $\omega : (M,' \nabla, g_M) \to (N, \nabla, g_N)$ is a statistical submersion then for $E, F, G, H \in \mathcal{V}(M)$ and $X, Y, Z, W \in \mathcal{H}(M)$, we have*

$$\begin{aligned} g_M(Rim(E, F)G, H) &= g_M(\overline{Rim}(E, F)G, H) + g_M(\mathbf{T}_E G, \mathbf{T}^*_F H) \\ &\quad - g_M(\mathbf{T}_F G, \mathbf{T}^*_E H), \end{aligned} \tag{2.11}$$

$$\begin{aligned} g_M(Rim(X, Y)Z, W) &= g_M(\hat{Rim}(X, Y)Z, W) + g_M((\mathbf{A}_X + \mathbf{A}^*_X)Y, \mathbf{A}^*_Z W) \\ &\quad - g_M(\mathbf{A}_Y Z, \mathbf{A}^*_X W) + g_M(\mathbf{A}_X Z, \mathbf{A}^*_Y W), \end{aligned} \tag{2.12}$$

$$\begin{aligned} g_M(Rim(X, E)F, Y) &= g_M(('\nabla_X \mathbf{T})_E F, Y) - g_M(('\nabla_E \mathbf{A})_X, F) \\ &\quad + g_M(\mathbf{A}_X E, \mathbf{A}^*_Y F) - g_M(\mathbf{T}_E X, \mathbf{T}^*_F Y), \end{aligned} \tag{2.13}$$

$$\begin{aligned} g_M(Rim(X, E)Y, F) &= g_M(('\nabla_X \mathbf{T})_E Y, F) - g_M(('\nabla_E \mathbf{A})_X Y, F) \\ &\quad - g_M(\mathbf{A}_X E, \mathbf{A}_Y F) - g_M(\mathbf{T}_E X, \mathbf{T}_F Y). \end{aligned} \tag{2.14}$$

Now, we respectively describe the orthonormal basis of $T_x(M)$, $x \in M$, $\mathcal{H}_x(M)$ and $\mathcal{V}_x(M)$ by $\{E_1, E_2, \ldots, E_m\}$, $\{e_1, e_2, \ldots, e_n\}$ and $\{\tilde{e}_1, \tilde{e}_2 \ldots, \tilde{e}_{m-n}\}$ such that $E_i = e_i$, $1 \leq i \leq n$ and

$E_{n+t} = \tilde{e}_t$, $1 \le t \le m-n$. Now, the connection forms in terms of local coordinates with respect to $\{e_1, e_2, \ldots, e_m\}$ of the affine connections $'\nabla$ and $'\nabla^*$ are given by ε_B^A and ε_A^{*B}.

The mean curvature vector field of the fiber with respect to $'\nabla$ and $'\nabla^*$ are given by the horizontal vector fields, accordingly,

$$\mathcal{N} = \frac{1}{m-n}\mathcal{T} = \sum_{t=1}^{m-n} \mathbf{T}_{\tilde{e}_t}\tilde{e}_t \quad \text{and} \quad \mathcal{N}^* = \frac{1}{m-n}\mathcal{T}^* = \sum_{t=1}^{m-n} \mathbf{T}^*_{\tilde{e}_t}\tilde{e}_t.$$

Note that the mean curvature vector fields $\mathcal{N}$ and $\mathcal{N}^*$ vanish identically if and only if the fibers of the statistical submersion ω are minimal.

Definition 3. ω is said to be a statistical submersion with

1. isometric fibers if $\mathbf{T}_U V = 0$,
2. conformal fibers if $\mathbf{T}_U V = \frac{1}{m-n} g_M(U,V)\mathcal{N}$,

for $U, V \in \mathcal{V}(M)$.

The scalar and the normalized scalar curvatures on the vertical space are respectively given by

$$\begin{aligned} R &= \sum_{1 \le t < t' \le m-n} g_M(Rim(\tilde{e}_t, \tilde{e}_{t'})\tilde{e}_{t'}, \tilde{e}_t), \\ \rho &= 2R/(m-n)(m-n-1), \end{aligned}$$

if $m-n \neq 1$.

We put $\mathbf{T}^i_{tt'} = g_M(\mathbf{T}_{\tilde{e}_t}\tilde{e}_{t'}, e_i)$ and $\mathbf{A}^t_{ij} = g_M(\mathbf{A}_{e_i}e_j, \tilde{e}_t)$, then we have

$$\sum_{i=1}^{n}\sum_{t,t'=1}^{m-n} \mathbf{T}^i_{tt'} = \sum_{i=1}^{n}\sum_{t,t'=1}^{m-n} g_M(\mathbf{T}_{\tilde{e}_t}\tilde{e}_{t'}, e_i), \tag{2.15}$$

$$\sum_{t=1}^{m-n}\sum_{i,j=1}^{n} \mathbf{A}^t_{ij} = \sum_{t=1}^{m-n}\sum_{i,j=1}^{n} g_M(\mathbf{A}_{e_i}e_j, \tilde{e}_t). \tag{2.16}$$

On replacing $\mathbf{T}, \mathbf{A}$ by $\mathbf{T}^*, \mathbf{A}^*$, we get equations (2.15) and (2.16) for $\mathbf{T}^{*i}_{tt'}$ and $\mathbf{A}^{*t}_{ij}$, respectively. The squared norms of $\mathbf{T}$ with respect to $'\nabla$ and $\mathbf{T}^*$ with respect to $'\nabla^*$, respectively represented by $\mathcal{C}$ and $\mathcal{C}^*$, are called the vertical Casorati curvatures of the vertical space $\mathcal{V}_x(M)$. Therefore, we have

$$\mathcal{C} = \frac{1}{m-n}\sum_{i=m-n+1}^{m}\sum_{t,t'=1}^{m-n} (\mathbf{T}^i_{tt'})^2, \tag{2.17}$$

$$\mathcal{C}^* = \frac{1}{m-n}\sum_{i=m-n+1}^{m}\sum_{t,t'=1}^{m-n} (\mathbf{T}^{*i}_{tt'})^2. \tag{2.18}$$

Next, we suppose that $\mathcal{L}$ is a s-dimensional vertical subspace of $\mathcal{V}_x(M)$, $s \ge 2$ and let $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_s\}$ be an orthonormal basis of $\mathcal{L}$. Then, the vertical Casorati curvature $\mathcal{C}(\mathcal{L})$ of $\mathcal{L}$ given by

$$\mathcal{C}(\mathcal{L}) = \frac{1}{s}\sum_{i=s+1}^{n}\sum_{t,t'=1}^{s} (\mathbf{T}^i_{tt'})^2. \tag{2.19}$$

The normalized δ-vertical Casorati curvatures $\delta_{\mathcal{C}}(m-n-1)$ and $\hat{\delta}_{\mathcal{C}}(m-n-1)$ of the vertical space are defined by

$$\Big[\delta_{\mathcal{C}}(m-n-1)\Big]_x = \frac{1}{2}\mathcal{C}_x + \Big(\frac{m-n+1}{2(m-n)}\Big)\inf\{\mathcal{C}(L)|L \text{ a hyperplane of } \mathcal{V}_x(M)\}, \tag{2.20}$$

and

$$\Big[\hat{\delta}_{\mathcal{C}}(m-n-1)\Big]_x = 2\mathcal{C}_x - \Big(\frac{2(m-n)-1}{2(m-n)}\Big)\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } \mathcal{V}_x(M)\}. \tag{2.21}$$

From (1.2), we have $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$, $2\delta^0_{\mathcal{C}}(m-n-1) = \delta_{\mathcal{C}}(m-n-1) + \delta^*_{\mathcal{C}}(m-n-1)$ and $2\hat{\delta}^0_{\mathcal{C}}(m-n-1) = \hat{\delta}_{\mathcal{C}}(m-n-1) + \hat{\delta}^*_{\mathcal{C}}(m-n-1)$.

The first author, et al. give a nice example on statistical submersion in [32].

Example 1. [32] Let $\{e_i : 1 \le i \le 6\}$ be an orthonormal frame field on a statistical manifold $(M = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6\}, {}'\nabla, g_M = \sum_{i,j=1}^{6} dx_i \otimes dx_j)$ then an affine connection ${}'\nabla$ is given by

$$'\nabla_{e_1}e_1 = -e_6, \quad '\nabla_{e_2}e_2 = -e_6, \quad '\nabla_{e_3}e_3 = -e_6, \quad '\nabla_{e_4}e_4 = -e_6, \quad '\nabla_{e_5}e_5 = -e_6,$$

$$'\nabla_{e_6}e_6 = 0, \quad '\nabla_{e_6}e_i = 0, \quad '\nabla_{e_i}e_6 = e_i, \quad 1 \le i \le 5,$$

and

$$'\nabla_{e_i}e_j = 0, \quad 1 \le i,j \le 5, \quad i \ne j$$

where

$$\{e_1 = \partial x_1, \quad e_2 = \partial x_2, \quad e_3 = \partial x_3, \quad e_4 = \partial x_4, \quad e_5 = \partial x_5, \quad e_6 = \partial x_6\}$$

is a set of linearly independent vector fields at each point of the manifold $\mathbb{R}^6$ and therefore it forms a basis for the tangent space T_xM, $x \in M$. Thus, $(M, {}'\nabla, g_M)$ is statistical manifold of constant curvature -1 and the scalar curvature is -20. In addition, we can say that $(M, {}'\nabla, g_M)$ is Einstein statistical manifold.

In continuation, it is required to prove that $\omega : (M = \mathbb{R}^6, {}'\nabla, g_M) \to (N = \mathbb{R}^3, \nabla, g_N)$ is a statistical submersion between two statistical manifolds, defined by

$$\omega(x_1, x_2, \dots, x_6) = (y_1, y_2, y_3),$$

where

$$y_1 = \frac{x_1 + x_2}{\sqrt{2}}, \quad y_2 = \frac{x_3 + x_4}{\sqrt{2}} \quad \text{and} \quad y_3 = \frac{x_5 + x_6}{\sqrt{2}}.$$

The Jacobian matrix of ω is given below:

$$\omega_* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Since the rank of matrix above is 3, which is equal to $\dim(N)$. On the other hand, it is easy to see that ω fulfils the conditions of statistical submersion. A straight computations yields

$$\mathcal{V}(M) = span\{V_1 = \frac{1}{\sqrt{2}}(-\partial x_1 + \partial x_2), \quad V_2 = \frac{1}{\sqrt{2}}(-\partial x_3 + \partial x_4),$$

$$V_3 = \frac{1}{\sqrt{2}}(-\partial x_5 + \partial x_6)\},$$

$$\mathcal{H}(M) = span\{H_1 = \frac{1}{\sqrt{2}}(\partial x_1 + \partial x_2), \quad H_2 = \frac{1}{\sqrt{2}}(\partial x_3 + \partial x_4),$$
$$H_3 = \frac{1}{\sqrt{2}}(\partial x_5 + \partial x_6)\},$$

Also, by direct computations, we get

$$\omega_*(H_1) = \partial y_1, \quad \omega_*(H_2) = \partial y_2, \quad \omega_*(H_3) = \partial y_3.$$

Hence, it is easy to see that

$$g_{\mathbb{R}^6}(H_i, H_i) = g_{\mathbb{R}^3}(\omega_*(H_i), \omega_*(H_i)), \quad i = 1, 2, 3.$$

3 Classes of statistical submersions

In this section, we discuss different types of statistical submersions have been studied by known authors till now.

3.1 Kähler-like statistical submersions

Definition 4. [36] If the semi-Riemannian manifold (M, g_M) with the almost complex structure J and another tensor field J^* of type $(1,1)$ satisfying

$$J^2 = -I, \quad g_M(JE, F) + g_M(E, J^*F) = 0, \tag{3.1}$$

for vector fields E and F on M and here I is the identity tensor field of type $(1,1)$ on M. Then (M, g_M, J) is called an almost Hermite-like manifold. Moreover, if the statistical structure on (M, g_M, J) and J is parallel with respect to $'\nabla$, then $(M,' \nabla, g_M, J)$ is known by a Kähler-like statistical manifold.

Also, $(M,' \nabla, g_M, J)$ is a Kähler-like statistical manifold if and only if so is $(M,' \nabla^*, g_M, J^*)$ because the following relation holds

$$g_M(('\nabla_G J)E, F) + g_M(E, ('\nabla^*_G J^*)F) = 0,$$

for vector fields E, F and G on M.

Remark 1. [36] It is easy to verify the following:

$$(J^*)^* = J, \quad (J^*)^2 = -I,$$

$$g_M(JE, J^*F) = g_M(E, F).$$

On a Kähler-like statistical manifold $(M,' \nabla, g_M, J)$, the curvature tensor Rim with respect to $'\nabla$ is given by [36]

$$Rim(E,F)G = \frac{c}{4}[g_M(F,G)E - g_M(E,G)F - g_M(F,JG)JE$$

$$+g_M(E,JG)JF + g_M(E,JF)JG$$
$$-g_M(JE,F)JG], \tag{3.2}$$

for vector fields E, F and G on M and here $c \in \mathbb{R}$. On replacing J by J^* in (3.2), one can get the curvature tensor Rim^* with respect to ${}'\nabla^*$.

Definition 5. [36] Let (M, g_M, J) and $(N, g_N, \tilde{J})$ be two almost Hermite-like statistical manifolds. Then a semi-Riemannian submersion $\omega : M \to N$ is said to be an almost Hermite-like statistical submersion, if $\omega_* J = \tilde{J}\omega_*$ hold.

Definition 6. [36] A statistical submersion $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ is said to be a Kähler-like statistical submersion, if $(M, {}'\nabla, g_M, J)$ is a Kähler-like statistical manifold and each fiber is a J-invariant semi-Riemannian submanifold of M.

Example 2. Let us recall example from [36] in which K. Takano shows that $\omega : (M^{2m}_{2s}, {}'\nabla, g_M) \to (N^{2n}_{2r}, \nabla, g_N)$ is a Kähler-like statistical submersion defined by $\omega(x_1, y_1, x_2, y_n, \ldots, x_m, y_m) = (x_1, y_1, x_2, y_n, \ldots, x_n, y_n)$ provided $n \geq m$ and $s \geq r$, where $(M^{2m}_{2s}, {}'\nabla, g_M, J)$ is Kähler-like statistical manifold (see [36]).

K. Takano gives the following interesting results on Kähler-like statistical submersions ω in [36].

Theorem 3.1. *[36] If $\omega : (M, g_M) \to (N, g_N)$ is an almost Hermite-like submersion, then each fiber is an almost Hermite-like manifold.*

Theorem 3.2. *[36] If $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ is a Kähler-like statistical submersion, then $(M, {}'\nabla, g_M, J)$ and $(\overline{M}, \overline{\nabla}, \overline{g}, \overline{J})$ are Kähler-like statistical manifolds, where $\overline{J} = J|_{\overline{M}}$.*

Theorem 3.3. *[36] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ is a Kähler-like statistical submersion. If $rank(\overline{J} + \overline{J}^*) = \dim(\overline{M})$, then we have $\mathbf{A} = 0$.*

Remark 2. K. Takano [36] also stated that the result above is true when $\overline{J} = \overline{J}^*$ holds.

In the same article [36], K. Takano proves the following results on Kähler-like statistical submersion ω with the property that the curvature tensor with respect to the affine connection of M satisfying the certain conditions.

Theorem 3.4. *[36] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ is a Kähler-like statistical submersion. If $rank(\overline{J} + \overline{J}^*) = \dim(\overline{M})$ and the curvature tensor of M fulfils (3.2) with constant c, then the curvature tensor of N fulfils (3.2) with constant c.*

Theorem 3.5. *[36] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ is a Kähler-like statistical submersion. If the curvature tensor of M fulfils (3.2) with constant c, $rank(\overline{J} + \overline{J}^*) = \dim(\overline{M})$ and $\mathcal{H}'\nabla_X \mathcal{N} = 0$, for $X \in \mathcal{H}(M)$. Then we have*

1. *if g_M is positive definite, then $c \leq 0$. Also, if $c = 0$ then each fiber and N are flat and each fiber is a totally geodesic submanifold of M.*
2. *if $X \in \mathcal{H}(M)$ is spacelike (resp. timelike) then $c \geq 0$ and $\mathbf{T}^* X$ is timelike (resp. spacelike) or $c < 0$ and $\mathbf{T}^* X$ is spacelike (resp. timelike).*
3. *if $X \in \mathcal{H}(M)$ is null, then $\mathbf{T}^* X$ is also null.*

Remark 3. K. Takano also mentions that Theorem 3.5 is valid when $\mathcal{N}$ is constant. He also studies the dual case to Theorem 3.5 in [36].

K. Takano [36] proves that a Kähler-like statistical submersion ω has isometric fibers if ω has conformal fibers. The next results on isometric and conformal fibers are also obtained in [36].

Theorem 3.6. *[36] Let $\omega : (M,' \nabla, g_M) \to (N, \nabla, g_N)$ is a Kähler-like statistical submersion with conformal fibers such that the curvature tensor of M fulfils (3.2) with constant c. Then each fiber is a totally geodesic submanifold of M and the curvature tensor of $\overline{M}$ fulfils (3.2) with constant c.*

Theorem 3.7. *[36] Let $\omega : (M,' \nabla, g_M) \to (N, \nabla, g_N)$ is a Kähler-like statistical submersion with conformal fibers such that the curvature tensor of M fulfils (3.2) with constant c. If $\mathrm{rank}(\overline{J}+\overline{J}^*) = \dim(\overline{M})$, then*

1. *the base space N and each fiber $\overline{M}$ are flat.*
2. *the total space M is a locally product space of N and $\overline{M}$.*

3.2 Quaternionic Kähler-like statistical submersions

Let a rank 3-subbundle σ of $End(TM)$ on a differentiable manifold M, then there exists a local basis $\{J_\alpha | \alpha \in \{1, 2, 3\}\}$ on sections of σ satisfying $J_\alpha^2 = -I$, $J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Definition 7. [40] If the semi-Riemannian manifold (M, g_M) endowed with an almost quaternionic structure σ which any canonical local basis $\{J_\alpha | \alpha \in \{1, 2, 3\}\}$ and has three other tensor fields $\{J_\alpha^* | \alpha \in \{1, 2, 3\}\}$ of type $(1, 1)$ satisfying

$$g_M(J_\alpha E, F) + g_M(E, J_\alpha^* F) = 0, \tag{3.3}$$

for vector fields E and F on M. Then (M, σ, g_M) is called an almost Hermite-like quaternionic manifold. Moreover, if the statistical structure is defined on (M, σ, g_M), then $(M,' \nabla, \sigma, g_M)$ is known by an almost Hermite-like quaternionic statistical manifold.

Remark 4. It can be verified that $(J_\alpha^*)^* = J_\alpha$ and $g_M(J_\alpha E, J_\alpha^* F) = g(E, F)$.

Definition 8. [40] An almost Hermite-like quaternionic statistical manifold $(M,' \nabla, \sigma, g_M)$ is said to be a quaternionic Kähler-like statistical manifold if there exist three locally defined 1-forms $\upsilon_1, \upsilon_2, \upsilon_3$ for $\{J_\alpha^* | \alpha = \{1, 2, 3\}\}$ of σ on M such that we have

$$('\nabla_E J_\alpha)F = \upsilon_{\alpha+2}(E) J_{\alpha+1} F - \upsilon_{\alpha+1}(E) J_{\alpha+2} F, \tag{3.4}$$

for vector fields E and F on M, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Theorem 3.8. *[40] $(M,' \nabla, \sigma, g_M)$ is a quaternionic Kähler-like statistical manifold if and only if $(M,' \nabla^*, \sigma^*, g_M)$ is, where σ^* is a subbundle of $End(TM)$ which has canonical local basis $\{J_\alpha^* | \alpha \in \{1, 2, 3\}\}$.*

On quaternionic Kähler-like statistical manifold, the curvature tensor Rim with respect to $'\nabla$ fulfils [40]

$$Rim(E, F)G \quad = \quad \frac{c}{4}\Big\{ g_M(F, G)E - g_M(E, G)F$$

$$+\sum_{\alpha=1}^{3}[g_M(G,J_\alpha F)J_\alpha E - g_M(G,J_\alpha E)J_\alpha F]$$
$$+\sum_{\alpha=1}^{3}[g_M(E,J_\alpha F) - g_M(J_\alpha E,F)]J_\alpha G\Big\}, \quad (3.5)$$

for vector fields E, F, G on M and here c is a real constant. The statistical manifold $(M, '\nabla, \sigma, g_M)$ is said to be of type quaternionic space form. On replacing J by J^* in (3.5), one can get the curvature tensor Rim^* with respect to $'\nabla^*$.

In [40], A.-D. Vilcu and G.-E. Vilcu clearly observed that $(M, '\nabla, \sigma, g_M)$ is said to be a locally hyper-Kähler-like statistical manifold [40] if $\upsilon_\alpha = 0$, $\alpha \in \{1,2,3\}$ in (3.4). Also, if $\{J^*_\alpha | \alpha \in \{1,2,3\}\}$ are globally defined on M, then $(M, '\nabla, J^*_\alpha | \alpha \in \{1,2,3\}, g_M)$ is said to be a hyper-Kähler-like statistical manifold.

The following result is given in [40] based on Theorem 3.8 and above definition.

Corollary 3.9. *[40] $(M, '\nabla, \sigma, g_M)$ is a hyper-Kähler-like statistical manifold if and only if $(M, '\nabla^*, \sigma^*, g_M)$ is.*

Remark 5. A.-D. Vilcu and G.-E. Vilcu [40] noticed that it is easy to prove that $(TM, '\nabla^{TM}, \sigma, g_{TM})$ is a hyper-Kähler-like statistical manifold if and only if $(M, '\nabla, J, g_M)$ is a flat Kähler-like statistical manifold. Here g_{TM} and $'\nabla^{TM}$ denote the Sasaki metric and a torsion free affine connection on TM which is compatible to g_{TM}, respectively.

Now, consider a $(\sigma, \tilde{\sigma})$-holomorphic map as follows [40]: let $f : (M, \sigma, g_M) \to (N, \tilde{\sigma}, g_N)$ be a map between two almost Hermite-like quaternionic manifolds. Then f is a $(\sigma, \tilde{\sigma})$-holomorphic map if and only if there exists a canonical local basis $\{\tilde{J}_\alpha | \alpha = \{1,2,3\}\}$ of $\tilde{\sigma}_{f(x)}$, for any canonical local basis $\{J_\alpha | \alpha = \{1,2,3\}\}$ of σ_x at a point $x \in M$, such that $f_* \circ J_\alpha = \tilde{J}_\alpha \circ f_*$. It is acceptable that a semi-Riemannian submersion $\omega : (M, '\nabla, \sigma, g_M) \to (N, \nabla, \tilde{\sigma}, g_N)$ between two almost Hermite-like quaternionic statistical manifolds is a $(\sigma, \tilde{\sigma})$-holomorphic map if and only if it is a $(\sigma^*, \tilde{\sigma}^*)$-holomorphic map [40].

Definition 9. [40] A statistical submersion $\omega : (M, '\nabla, \sigma, g_M) \to (N, \nabla, \tilde{\sigma}, g_N)$ between two almost Hermite-like quaternionic statistical manifolds is said to be an almost Hermite-like quaternionic statistical submersion if ω is a $(\sigma, \tilde{\sigma})$-holomorphic map.

Definition 10. [40] An almost Hermite-like quaternionic statistical submersion

$$\omega : (M, '\nabla, \sigma, g_M) \to (N, \nabla, \tilde{\sigma}, g_N)$$

is called a quaternionic Kähler-like statistical submersion if $(M, '\nabla, \sigma, g_M)$ is a quaternionic Kähler-like statistical manifold. In particular, if $(M, '\nabla, \sigma, g_M)$ is a (locally) hyper-Kähler-like statistical manifold, then ω is called a (locally) hyper-Kähler-like statistical submersion.

We recall the following interesting and useful results of A.-D. Vilcu and G.-E. Vilcu from [40].

Theorem 3.10. *[40] If $\omega : (M, '\nabla, \sigma, g_M) \to (N, \nabla, \tilde{\sigma}, g_N)$ is an almost Hermite-like quaternionic statistical submersion, then the fibers are almost Hermite-like quaternionic statistical manifolds.*

Theorem 3.11. *[40] If $\omega : (M, {}'\nabla, \sigma, g_M) \to (N, \nabla, \tilde{\sigma}, g_N)$ is a quaternionic Kähler-like statistical submersion, then N is a quaternionic Kähler-like statistical manifold. Moreover, the fibers are also quaternionic Kähler-like statistical manifolds.*

The immediate assertion can be made from Theroem 3.11.

Corollary 3.12. *[40] If $\omega : (M, {}'\nabla, \sigma, g_M) \to (N, \nabla, \tilde{\sigma}, g_N)$ is a locally hyper-Kähler-like statistical submersion, then N is a locally hyper-Kähler-like statistical manifold. Moreover, the fibers are also locally hyper-Kähler-like statistical manifolds.*

Theorem 3.13. *[40] If $\omega : (M, {}'\nabla, \sigma, g_M) \to (N, \nabla, \tilde{\sigma}, g_N)$ is a quaternionic Kähler-like statistical submersion, then ω has isometric fibers.*

In particular, Theorem 3.13 implies the following.

Corollary 3.14. *[40] If $\omega : (M, {}'\nabla, \sigma, g_M) \to (N, \nabla, \tilde{\sigma}, g_N)$ is a quaternionic Kähler-like statistical submersion, then $\mathbf{A}^*_X Y = 0$, for horizontal vector fields X, Y on M.*

Corollary 3.15. *[40] If $\omega : (M, {}'\nabla, \sigma, g_M) \to (N, \nabla, \tilde{\sigma}, g_N)$ is a quaternionic Kähler-like statistical submersion, then the horizontal distribution is completely integrable.*

By taking into account of Theorem 3.13, the next result can be proved by using the equations for a statistical submersion given in [36].

Theorem 3.16. *[40] Let $\omega : (M, {}'\nabla, \sigma, g_M) \to (N, \nabla, \tilde{\sigma}, g_N)$ is a quaternionic Kähler-like statistical submersion. If M is a quaternionic space form, then N is also the quaternionic space form and each fiber is a totally geodesic submanifold of M of type quaternionic space form.*

3.3 para-Kähler-like statistical submersions

Definition 11. [39] The triple (M, P, g_M) is said to be an almost para-Hermitian-like manifold if (M, g_M) is a semi-Riemannian manifold endowed with an almost product structure P and another tensor field P^* of type $(1,1)$ satisfying

$$P^2 = I, \quad g_M(PE, F) + g_M(E, P^*F) = 0, \tag{3.6}$$

for vector fields E and F on M. Here, P^* is the negative of the adjoint of P. Moreover, if the statistical structure on (M, P, g_M) and P is parallel with respect to ${}'\nabla$, then $(M, {}'\nabla, P, g_M)$ is said to be a para-Kähler-like statistical manifold.

Remark 6. [39] It is easy to check that the relations $g_M(PE, P^*F) = -g_M(E, F)$ and $(P^*)^* = P$ hold.

On a para-Kähler-like statistical manifold $(M, {}'\nabla, P, g_M)$, the curvature tensor Rim with respect to ${}'\nabla$ fulfils [39]

$$\begin{aligned} Rim(E, F)G &= \frac{c}{4}\Big\{ g_M(F, G)E - g_M(E, G)F \\ &\quad + g_M(PF, G)PE - g_M(PE, G)PF \\ &\quad + [g(E, PF) - g_M(PE, F)]PG \Big\}, \end{aligned} \tag{3.7}$$

for vector fields E, F, G on M, where c is a real constant, then $(M,' \nabla, P, g_M)$ is known by a statistical manifold of type para-Kähler space form. If we replace P by P^*, then we get the expression of the curvature tensor Rim^* with respect to $'\nabla^*$.

In [39], G.-E. Vilcu gives an interesting result that the statistical structure of a para-Kähler-like statistical manifold of constant curvature in the Kurose's sense is a Hessian structure by showing that the curvature tensor field $Rim(E,F)G = c[g_M(F,G)E - g_M(E,G)F]$ with respect to the affine connection $'\nabla$ identically vanishes.

By adopting the similar approach as in [40], G.-E. Vilcu introduces new statistical submersion as follows.

Definition 12. [39] A statistical submersion $\omega : (M,' \nabla, P, g_M) \to (N, \nabla, \tilde{P}, g_N)$ is called an almost para-Hermitian-like statistical submersion, if ω is a para-holomorphic map ($\omega_* \circ P = \tilde{P} \circ \omega_*$). Moreover, ω is called a para-Kähler-like statistical submersion if $(M,' \nabla, P, g_M)$ is a para-Kähler-like statistical manifold.

Example 3. Let us recall example from [39] in which G.-E. Vilcu shows that

$$\omega : (M^{2m},' \nabla, g_M) \to (N^{2n}, \nabla, g_N)$$

is a para-Kähler-like statistical submersion defined by

$$\omega(x_1, y_1, x_2, y_n, \ldots, x_m, y_m) = (x_1, y_1, x_2, y_n, \ldots, x_n, y_n)$$

provided $n \geq m$ and $s \geq r$, where $(M^{2m},' \nabla, g_M, P)$ and $(N^{2m}, \nabla, g_M, \tilde{P})$ are para-Kähler-like statistical manifolds respectively having signature $(s, 2m - s)$ and $(r, 2m - r)$ (see [39]) and .

The following main properties of para-Kähler-like statistical submersion are derived by G.-E. Vilcu in [39].

Theorem 3.17. *[39] If $\omega : (M, P, g_M) \to (N, \tilde{P}, g_N)$ is an almost para-Hermitian -like statistical submersion, then each fiber is an almost para-Hermitian-like statistical manifold.*

Theorem 3.18. *[39] Let $(M,' \nabla, P, g_M)$ be a para-Kähler-like statistical manifold and $(N, \nabla, \tilde{P}, g_N)$ be an almost para-Hermitian-like statistical manifold. If $\omega : (M,' \nabla, P, g_M) \to (N, \nabla, \tilde{P}, g_N)$ is a para-Kähler-like statistical submersion, then $(N, \nabla, \tilde{P}, g_N)$ is a para-Kähler-like statistical manifold. Moreover, the fibres $(\overline{M}, \overline{\nabla}, \overline{P}, \overline{g})$ are also para-Kähler-like statistical manifolds, where $\overline{P} = P|_{\overline{M}}$.*

Theorem 3.19. *[39] Let $(M,' \nabla, P, g_M)$ be a para-Kähler-like statistical manifold and $(N, \nabla, \tilde{P}, g_N)$ be an almost para-Hermitian-like statistical manifold. If $\omega : (M,' \nabla, P, g_M) \to (N, \nabla, \tilde{P}, g_N)$ is a para-Kähler-like statistical submersion and $\mathbf{A}_X Y = \mathbf{A}^*_X Y = 0$, for $X, Y \in \mathcal{H}(M)$, given that $rank(\overline{P} + \overline{P}^*) = \dim(\overline{M})$.*

Immediate corollaries on the result above can be stated as follows.

Corollary 3.20. *[39] If $\omega : (M,' \nabla, P, g_M) \to (N, \nabla, \tilde{P}, g_N)$ is a para-Kähler-like statistical submersion and $\overline{P} = \overline{P}^*$ holds, then $\mathbf{A}_X Y = \mathbf{A}^*_X Y = 0$, for $X, Y \in \mathcal{H}(M)$.*

The tensor fields $\mathbf{A}$ and $\mathbf{A}^*$ of type $(1, 2)$ are equal to zero if and only if the horizontal distribution $\mathcal{H}(M)$ is integrable with respect to $'\nabla$ and $'\nabla^*$, respectively. Thus, we have

Corollary 3.21. *[39] If $\omega : (M, {}'\nabla, P, g_M) \to (N, \nabla, \tilde{P}, g_N)$ is a para-Kähler-like statistical submersion and $\overline{P} = \overline{P}^*$ holds, then $\mathcal{H}(M)$ is completely integrable.*

By following K. Takano's results on Kähler-like statistical submersions, G.-E. Vilcu [39] gives an interesting result for para-Kähler-like statistical submersions as follows.

Theorem 3.22. *[39] If $\omega : (M, {}'\nabla, P, g_M) \to (N, \nabla, \tilde{P}, g_N)$ is a para-Kähler-like statistical submersion such that M is of type para-Kähler space form. Then we have*

1. *If $rank(\overline{P} + \overline{P}^*) = \dim(\overline{M})$, then N is of type para-Kähler space form.*
2. *If ω is a statistical submersion with isometric fibers, then each fiber is a totally geodesic submanifold of M, of type para-Kähler space form.*
3. *If ω is a statistical submersion with isometric fibers such that $rank(\overline{P} + \overline{P}^*) = \dim(\overline{M})$, then the N and each fiber are flat. Moreover, M is a locally product space of N and fiber.*

Remark 7. Several examples of a para-Kähler statistical manifold and its statistical submersion are collected in [39].

3.4 Sasaki-like statistical submersions

Definition 13. [38] If the semi-Riemannian manifold (M, g_M) with the almost contact structure (ϕ, ξ, η) and another tensor field ϕ^* of type $(1, 1)$ satisfying

$$\eta(\xi) = 1, \quad \phi^2 E = -E + \eta(E)\xi, \tag{3.8}$$
$$g_M(\phi E, F) + g_M(E, \phi^* F) = 0, \tag{3.9}$$

for vector fields E and F on M. Then $(M, g_M, \phi, \xi, \eta)$ is known an almost contact metric manifold of certain kind. Moreover, if the statistical structure on $(M, g_M, \phi, \xi, \eta)$ and the following relations hold

$${}'\nabla_E \xi = -\phi E, \qquad ({}'\nabla_E \phi)F = g_M(E, F)\xi - \eta(F)E.$$

Then $(M, {}'\nabla, g_M, \phi, \xi, \eta)$ is called a Sasaki-like statistical manifold.

Remark 8. We can see that $(\phi^*)^* = \phi$, $(\phi^*)^2 E = -E + \eta(E)\xi$, $g_M(\phi E, \phi^* F) = g(E, F) - \eta(E)\eta(F)$, $\phi^* \xi = 0$ and $\eta(\phi^* E) = 0$.

Also, $(M, {}'\nabla, g_M, \phi, \xi, \eta)$ is a Sasaki-like statistical manifold if and only if so is $(M, {}'\nabla^*, g_M, \phi^*, \xi, \eta)$ because the following relations hold

$$g_M(\phi F, {}'\nabla^*_E \xi) = \eta(E)\eta(F) - g_M(E, F), \qquad {}'\nabla^*_E \xi = -\phi^* E,$$

for vector fields E, F and G on M.

On a Sasaki-like statistical manifold, the curvature tensor Rim with respect to ${}'\nabla$ such that

$$\begin{aligned} Rim(E, F)G &= \frac{1}{4}(c+3)[g_M(F, G)E - g_M(E, G)F] \\ &\quad + \frac{1}{4}(c-1)[\eta(E)\eta(G)F - \eta(F)\eta(G)E \end{aligned}$$

$$
\begin{aligned}
&+g_M(E,G)\eta(F)\xi - g_M(F,G)\eta(E)\xi \\
&-g_M(F,\phi G)\phi E + g_M(E,\phi G)\phi F \\
&+g_M(E,\phi F)\phi G - g_M(\phi E,F)\phi G],
\end{aligned}
\tag{3.10}
$$

where c is a constant. On replacing ϕ by ϕ^* in (3.10), we get the curvature tensor Rim^* with respect to $'\nabla^*$.

K. Takano defines Sasaki-like statistical submersion in [38] as follows.

Definition 14. [38] If $\omega : (M, g_M) \to (N, g_N)$ is a semi-Riemannian submersion such that $(M, g_M, \phi, \xi, \eta)$ is an almost contact metric manifold of certain kind, each fiber is a ϕ-invariant semi-Riemannian submanifold of M and tangent to the vector ξ, then ω is said to be an almost contact metric submersion of certain kind.

Definition 15. [38] A statistical submersion $\omega : (M,' \nabla, g_M) \to (N, \nabla, g_N)$ is a called Sasaki-like statistical submersion if $(M,' \nabla, g_M, \phi, \xi, \eta)$ is a Sasaki-like statistical manifold, each fiber is a ϕ-invariant semi-Riemannian submanifold of M and tangent to the vector ξ.

We recall some useful and interesting results on Sasaki-like statistical submersions given by K. Takano in [38] as follows.

Theorem 3.23. *[38] Let $\omega : (M, g_M) \to (N, g_N)$ be an almost contact metric submersion of certain kind. Then the base space is an almost Hermite-like manifold and each fiber is an almost contact metric manifold of certain kind.*

Theorem 3.24. *[38] If $\omega : (M,' \nabla, g_M) \to (N, \nabla, g_N)$ is a Sasaki-like statistical submersion, then the base space $(N, \nabla, g_N, \tilde{J})$ is a Kähler-like statistical manifold and each fiber $(\overline{M}, \overline{\nabla}, \overline{g}, \overline{\phi}, \xi, \eta)$ is a Sasaki-like statistical manifold, where $\overline{\phi} = \phi|_{\overline{M}}$.*

Theorem 3.25. *[38] Let $\omega : (M,' \nabla, g_M) \to (N, \nabla, g_N)$ is a Sasaki-like statistical submersion. If $rank(\overline{\phi} + \overline{\phi}^*) = \dim(\overline{M}) - 1$, or $\overline{\phi} = \overline{\phi}^*$, then we have $\mathbf{A}_X Y = -g_M(X, \phi Y)\xi$, for $X, Y \in \mathcal{H}(M)$.*

Following results are based on curvature tensor of M.

Theorem 3.26. *[38] Let $\omega : (M,' \nabla, g_M) \to (N, \nabla, g_N)$ be a Sasaki-like statistical submersion. If $rank(\overline{\phi} + \overline{\phi}^*) = \dim(\overline{M}) - 1$ or $\dim(\overline{M}) = 1$ and the curvature tensor of M fulfils the form (3.10) with c, then the curvature tensor of N fulfils the form (3.2) with $c + 3$.*

The following result is the contact version to the Theorem 3.5. If $\mathcal{H}'\nabla_X \mathcal{N} = 0$, then we have two cases, either $c + 3 = 0$ or $trace(\overline{\phi}) = 0$.

Theorem 3.27. *[38] Let $\omega : (M,' \nabla, g_M) \to (N, \nabla, g_N)$ be a Sasaki-like statistical submersion such that the curvature tensor of the total space fulfils the form (3.10) with c. We assume that $rank(\overline{\phi} + \overline{\phi}^*) = \dim(\overline{M}) - 1$ and $\mathcal{H}'\nabla_X \mathcal{N} = 0$, for $X \in \mathcal{H}(M)$ or $\mathcal{N}$ is a constant vector field. Then the following statements hold.*

1. *if $c + 3 = 0$, then N is flat and each fiber $\overline{M}$ is a totally geodesic submanifold of M such that the curvature tensor fulfils the form (3.10) with -3,*
2. *when $trace(\overline{\phi}) = 0$ and $m - n > 1$,*
   - *(a) if g_M is a positive definite, then $c + 3 \leq 0$,*

(b) *$c+3<0$ and X is spacelike (resp. timelike) or $c+3>0$ and X is timelike (resp. spacelike) if and only if $\mathbf{T}^*X$ is spacelike (resp. timelike),*

(c) *the horizontal vector X is null if and only if $\mathbf{T}^*X$ is null.*

Remark 9. It is quite obvious to prove the dual case of the result above can be seen in [38].

In contact case also, K. Takano proves that if $\omega : (M, '\nabla, g_M) \to (N, \nabla, g_N)$ is a Sasaki-like statistical submersion with conformal fibers, then ω has isometric fibers.

Theorem 3.28. *[38] Let $\omega : (M, '\nabla, g_M) \to (N, \nabla, g_N)$ be a Sasaki-like statistical submersion with conformal fibers such that the curvature tensor of the total space fulfils the form (3.10) with c. Then each fiber $\overline{M}$ is a totally geodesic submanifold of M such that the curvature tensor fulfils the form (3.10) with c.*

Theorem 3.29. *[38] Let $\omega : (M, '\nabla, g_M) \to (N, \nabla, g_N)$ be a Sasaki-like statistical submersion with conformal fibers such that the curvature tensor of the total space fulfils the form (3.10) with c. If $rank(\overline{\phi}+\overline{\phi}^*) = \dim(\overline{M})-1$, then*

1. *the total space M fulfils the form (3.10) with $c=-3$,*
2. *the base space N is flat,*
3. *each fiber $\overline{M}$ fulfils the form (3.10) with $c=-3$.*

The last two results stated above are the contact version of Theorems 3.6 and 3.7, respectively.

3.5 Cosymplectic-like statistical submersions

Definition 16. [6] If the statistical structure on an almost contact metric manifold $(M, g_M, \phi, \xi, \eta)$ of certain kind and the following relations hold

$$'\nabla_E \xi = 0, \qquad '\nabla_E \phi = 0.$$

Then $(M, '\nabla, g_M, \phi, \xi, \eta)$ is called a cosymplectic-like statistical manifold.

On a cosymplectic-like statistical manifold, the curvature tensor Rim with respect to $'\nabla$ is given by

$$\begin{aligned} Rim(E,F)G \quad = \quad & \frac{c}{4}\Big\{ g_M(F,G)E - g_M(E,G)F \\ & + g_M(E,\phi G)\phi F - g_M(F,\phi G)\phi E \\ & + [g_M(E,\phi F) - g_M(\phi E, F)]\phi G \\ & + \eta(E)\eta(G)F - \eta(F)\eta(G)E \\ & + g_M(E,G)\eta(F)\xi - g_M(F,G)\eta(E)\xi \Big\}, \end{aligned} \tag{3.11}$$

where c is a real constant. On changing ϕ by ϕ^* in above relation, we get the expression of the curvature tensor Rim^*.

Remark 10. In [25], Murathan and Sahin construct cosymplectic-like statistical manifold based on the existence of Kähler-like statistical manifold. So, they prove that under the conditions of Proposition 2.2 (see [25]) the product manifold $\mathbb{R} \times M$ is a cosymplectic-like statistical manifold, if $(M, {}'\nabla, g_M, J)$ is Kähler-like statistical manifold and $(\mathbb{R}, \nabla^{\mathbb{R}}, dt)$ is a trivial statistical manifold.

Aytimur and Ozgur [6] get inspiration from the definition of Sasaki-like statistical submersion given by K. Takano in [38] and define cosymplectic-like statistical submersion as follows.

Definition 17. [6] A statistical submersion $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ is called a cosymplectic-like statistical submersion if $(M, {}'\nabla, g_M, \phi, \xi, \eta)$ is a cosymplectic-like statistical manifold, each fiber is a ϕ-invariant Riemannian submanifold of M and tangent to ξ.

Similar to the findings for Sasaki-like statistical submersion, the following important result is also given in [6].

Theorem 3.30. *[6] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ be a cosymplectic-like statistical submersion. Then $(N, \nabla, g_N, \tilde{J})$ is a Kähler-like statistical manifold and $(\overline{M}, \overline{\nabla}, \overline{g}, \overline{\phi}, \xi, \eta)$ is a cosymplectic-like statistical manifold.*

Following K. Takano's results in [38] and [36], some nice results on integrable horizontal distribution for a cosymplectic-like statistical submersion $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ can be found in [6]. It is proved that for a cosymplectic-like statistical submersion ω such that $\dim(\overline{M}) = 1$, the horizontal distribution $\mathcal{H}$ is integrable. Moreover, it is shown that if $rank(\overline{\phi} + \overline{\phi}^*) = \dim(\overline{M}) - 1$ on ω, then $\mathcal{H}$ is integrable in [6]. In the consequence of this result, Aytimur and Ozgur also found a condition $\overline{\phi} = \overline{\phi}^*$ under which $\mathcal{H}(M)$ is again integrable in [6]. Similar to Theorem 3.26, they give following result.

Theorem 3.31. *[6] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ be a cosymplectic-like statistical submersion. If the horizontal distribution is integrable and the curvature tensor of M fulfils the form (3.11) with c, then the curvature tensor of N fulfils the form (3.2) with c.*

Now, we recall the following result similar to Theorem 3.27.

Theorem 3.32. *[6] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ be a cosymplectic-like statistical submersion such that the curvature tensor of M is of the form (3.11). Assume that $\mathcal{H}(M)$ is integrable and $\mathcal{H}'\nabla_X\mathcal{N} = 0$, for $X \in \mathcal{H}(M)$ (or $\mathcal{N}$ is a constant vector field).*

1. *If $c = 0$, then both M and N are flat, each fiber $\overline{M}$ is a totally geodesic submanifold of M.*
2. *If $trace(\overline{\phi}) = 0$ and $c < 0$, then $\dim(\overline{M}) > 1$.*

We note that the dual case of Theorem 3.32 is also discussed in [6]. Next, the result on isometric fibers can also be seen for a cosymplectic-like statistical submersion $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ in [6]. It is proved that if a cosymplectic-like statistical submersion ω with conformal fibers, then ω has isometric fibers. Furthermore, we have

Theorem 3.33. *[6] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ be a cosymplectic-like statistical submersion with isometric fibers such that the curvature tensor of M is of the form (3.11). Then each fiber $\overline{M}$ is a totally geodesic submanifold of M such that the curvature tensor is of the form (3.11).*

Theorem 3.34. *[6] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ be a cosymplectic-like statistical submersion with isometric fibers such that the curvature tensor of M is of the form (3.11). If the horizontal distribution is integrable, then M and N are flat.*

3.6 Kenmotsu-like statistical submersions

Murathan and Sahin [25] set up Kenmotsu-like statistical manifold provided Kähler-like statistical manifold exists. They define β-Kenmotsu-like statistical manifold as follows: an almost contact metric like statistical manifold $(M,'\nabla, g_M, \phi, \xi, \eta)$ is called β-Kenmotsu-like statistical manifold if

$$('\nabla_E\phi)F = \beta\{g_M(E,\phi F)\xi + \eta(F)\phi E\}, \tag{3.12}$$
$$'\nabla_E\xi = \beta\phi^2 E, \tag{3.13}$$

where β is a non-zero smooth function on M. They prove the following theorem [25].

Theorem 3.35. *Let $(M,'\nabla, g_M, J)$ be a Kähler-like statistical manifold and $(\mathbb{R}, \nabla^{\mathbb{R}}, dt)$ be trivial statistical manifold. $\mathbb{R}\times M$. Under the Proposition 2.2 (see [25]), $\mathbb{R}\times_f M$ is a $\beta = \frac{f'}{f}$ Kenmotsu-like statistical manifold.*

On a Kenmotsu-like statistical manifold, the curvature tensor Rim with respect to $'\nabla$ such that

$$\begin{aligned} Rim(E,F)G = {} & \frac{c-3}{4}\Big\{g_M(F,G)E - g_M(E,G)F\Big\} \\ & + \frac{c+1}{4}\Big\{g_M(\phi F,G)\phi E - g_M(\phi E,G)\phi F \\ & -2g_M(\phi E,F)\phi G - g_M(F,\xi)g_M(G,\xi)E \\ & + g_M(E,\xi)g_M(G,\xi)F + g_M(F,\xi)g_M(G,E)\xi \\ & - g_M(E,\xi)g_M(G,F)\xi\Big\}, \end{aligned} \tag{3.14}$$

where $c \in \mathbb{R}$. After, shifting ϕ to ϕ^* in (3.14), we turn up the equation of the curvature tensor Rim^* with respect to the $'\nabla^*$.

Analogous to the Sasaki-like statistical submersion [38], we describe the Kenmotsu-like statistical submersion as follows.

Definition 18. [33] A statistical submersion $\omega : (M,'\nabla, g_M) \to (N, \nabla, g_N)$ is said to be Kenmotsu-like statistical submersion, if $(M,'\nabla, g_M, \phi, \xi, \eta)$ is a Kenmotsu-like statistical manifold, each fiber is a ϕ-invariant semi-Riemannian submanifold of M and tangent to the vector field ξ.

Similar to the Takano's results in [38] and Aytimur and Ozgur's results in [6], Danish, the first author, Mofarreh and Aytimur [33] examine that, for Kenmotsu-like statistical submersion ω, the base space $(N, \nabla, g_N, \tilde{J})$ is a Kähler-like statistical manifold and each fiber $(\overline{M}, \overline{\nabla}, \overline{g}, \overline{\phi}, \xi, \eta)$ is a Kenmotsu-like statistical manifold (see Theorems 3.24 and 3.30). Secondly, they study Kenmotsu-like statistical submersions admit the axioms that the curvature tensor with respect to the affine connection $'\nabla$ of M follows certain conditions (see Theorems 3.26, 3.27). Lastly, they give a glimpse of ω with conformal fibers (see Theorems 3.28 and 3.29).

4 Statistical solitons on statistical submersions

The Ricci flow equation is introduced by Richard S. Hamilton [21]. He put an evolution equation for metrics, called the Ricci fl ow, on a Riemannian manifold (M, g_M)

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)), \quad g(0) = g_M, \tag{4.1}$$

which is used to deform a metric by smoothing out its singularities. Here g and Ric are $(0, 2)$ type symmetric tensor fields. This equation is used to deform a metric by smoothing out its singularities. The differential geometry of Ricci solitons is the natural extensions of Einstein metrics.

The main theorem of his pioneering "Richard S. Hamilton. Three-manifolds with positive Ricci curvature. J. Differential Geom., 17(2):255 306, 198" named paper states that every compact three-dimensional manifold which admits a Riemannian metric with strictly positive Ricci curvature also admits a metric of constant positive curvature.

A Ricci soliton emerges as the limit of the solutions of the Ricci fl ow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g_M, ζ, λ) on a Riemannian manifold (M, g_M) is a generalization of an Einstein metric such that [21]

$$Ric + \frac{1}{2}\mathcal{L}_\zeta g_M + \lambda g_M = 0, \tag{4.2}$$

where Ric is the Ricci tensor, $\mathcal{L}_\zeta$ is the Lie derivative operator along the vector field ζ on M and λ is any real number. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively. During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [23], Meric and Kilic study Riemannian submersions whose total manifolds admit a Ricci soliton.

A a statistical manifold $(M, '\nabla, g_M)$ is called Ricci-symmetric if the Ricci operator Q with respect to $'\nabla$ (equivalently, the dual operator Q^* with respect to $'\nabla^*$) is symmetric (see [16, 30]). An extension of Ricci solitons on the statistical manifold M is defined as follows.

Definition 19. A pair (ζ, λ) is called a statistical soliton on a Ricci-symmetric statistical manifold $(M, '\nabla, g_M)$ if the triplet (ζ, λ, g_M) is $'\nabla$-Ricci and $'\nabla^*$-Ricci solitons, that is,

$$'\nabla\zeta + Q + \lambda I = 0, \quad \text{and} \quad '\nabla^*\zeta + Q^* + \lambda I = 0, \tag{4.3}$$

where $g_M(QE, F) = Ric(E, F)$ and $g_M(Q^*E, F) = Ric^*(E, F)$, for all vector fields E and F on M, and Ric and Ric^* indicates the Ricci tensor fields with respect to $'\nabla$ and $'\nabla^*$, respectively.

We recall the following interesting results on statistical submersion $\omega : (M, '\nabla, g_M) \to (N, \nabla, g_N)$ from the statistical solitons onto a statistical manifold given by the first author, et al. in [32].

Lemma 4.1. *[32] Let $\omega : (M, '\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion between statistical manifolds. Then the vertical distribution $\mathcal{V}(M)$ is parallel with respect to the connection $'\nabla$ (resp.*

$'\nabla^*$), if the horizontal parts $\mathbf{T}_V W'$ (resp. $\mathbf{T}^*_V W'$) and $\mathbf{A}_X V$ (resp. $\mathbf{A}^*_X V$) of Lemma 2.2 vanish identically for any $X, Y \in \mathcal{H}(M)$ and $V, W' \in \mathcal{V}(M)$.

Similarly, the horizontal distribution $\mathcal{H}(M)$ is parallel with respect to the connection ∇ (resp. ∇^*), if the vertical parts $\mathbf{T}_V X$ (resp. $\mathbf{T}^*_V X$) and $\mathbf{A}_X Y$ (resp. $\mathbf{A}^*_X Y$) of Lemma 2.2 vanish identically, for any $X, Y \in \mathcal{H}(M)$ and $V, W' \in \mathcal{V}(M)$.

Theorem 4.1. *[32] Let (M, g_M, ζ, λ) be a statistical soliton with vertical potential vector field ζ and $\omega : (M, '\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion. If the vertical distribution $\mathcal{V}(M)$ is parallel, then any fiber of the statistical submersion ω is a statistical soliton which fulfils*

$$g_M('\nabla_U \zeta, V) + g_M(QU, V) + \lambda g_M(U, V) = 0, \tag{4.4}$$

for any $U, V \in \mathcal{V}(M)$.

Theorem 4.2. *[32] Let (M, g_M, ζ, λ) be a statistical soliton with vertical potential vector field ζ and $\omega : (M, '\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion. If the horizontal distribution $\mathcal{H}(M)$ is parallel, then the following are fulfilled.*

1. *If the potential vector field ζ is vertical, then N is an Einstein manifold,*
2. *If the potential vector field ζ is horizontal, then N is a statistical soliton with potential vector field $\zeta' = \omega_* \zeta$.*

Dual case of Theorems 4.1 and 4.2 are stated in [32].

5 Inequalities on statistical submersions

B.-Y. Chen [10, 11] introduces a sharp relationship between Riemannian submersions and minimal immersions. By this result he prominently proves that if a Riemannian manifold M admits a non-trivial Riemannian submersion $\omega : (M, g_M) \to (N, g_N)$ with totally geodesic fibres, then it cannot be isometrically immersed in any Riemannian manifold of non-positively sectional curvature as a minimal submanifold. Then, P. Alegre, B.-Y. Chen and M. I. Munteanu [2] give a sharp relationship between the δ-invariants (see [15]) and Riemannian submersions with totally geodesic fibers. Gulbahar, Meric and Kilic [20] prove some optimal inequalities for Riemannian submersions involving the Ricci curvature. B.-Y. Chen continued this study and formulated a series of fundamental question in the development of inequalities as follows:

Problem. How can we establish simple relationship between the main intrinsic invariants and the main extrinsic invariants of the vertical spaces and horizontal space of a Riemannian manifold admitting a Riemannian submersion?

A solution to the proposed problem is obtained by using the fundamental equations for statistical submersions to establish sharp relationships involving the basic curvature invariants. The purpose of this section to collect all the sharp inequalities are derived for statistical submersions till now.

5.1 Chen-Ricci inequality

In this subsection, we recall Chen–Ricci inequalities for statistical submersions between statistical manifolds derived by the first author, Chen, and Danish in [31]. The next lemmas are required to obtain the main inequalities.

Lemma 5.1. *[31] Let $\omega : (M, '\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion. Then we have*

$$2R = 2\overline{R} - (m-n)^2 g_M(\mathcal{N}, \mathcal{N}^*) + \sum_{t,t'=1}^{m-n} g_M(\mathbf{T}_{\tilde{e}_t}\tilde{e}_{t'}, \mathbf{T}^*_{\tilde{e}_t}\tilde{e}_{t'}), \tag{5.1}$$

where $\overline{R}$ denotes the scalar curvature of $\overline{M}$ with respect to $\overline{\nabla}$.

Lemma 5.2. *[31] Let $\omega : (M, '\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion. Then we have*

$$\begin{aligned} \sum_{i=1}^{n}\sum_{t,t'=1}^{m-n} \left[(\mathbf{T}^i_{tt'})^2 + (\mathbf{T}^{*i}_{tt'})^2\right] &\geq 2\sum_{i=1}^{n}\sum_{2\leq t<t'\leq m-n} \mathbf{T}^i_{tt}\mathbf{T}^{*i}_{t't'} \\ +\sum_{i=1}^{n}\sum_{2\leq t<t'\leq m-n} \left[(\mathbf{T}^i_{tt'})^2 + (\mathbf{T}^{*i}_{tt'})^2\right] &- \sum_{i=1}^{n}\sum_{2\leq t<t'\leq m-n} (\mathbf{T}^i_{tt} + \mathbf{T}^{*i}_{tt}) \\ (\mathbf{T}^i_{t't'} + \mathbf{T}^{*i}_{t't'}) &+ \frac{(m-n)^2}{2}\left(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\right). \end{aligned} \tag{5.2}$$

Lemma 5.3. *[31] Let $\omega : (M, '\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion. Then we have*

$$\begin{aligned} \overline{Ric}(\tilde{e}_1) \geq Ric(\tilde{e}_1) &+ (m-n)^2|\mathcal{N}^0|^2 - \frac{(m-n)^2}{8}\left(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\right) \\ &- \sum_{i=1}^{n}\sum_{t,t'=1}^{m-n}(\mathbf{T}^{0i}_{tt'})^2 - \sum_{i=1}^{n}\sum_{2\leq t<t'\leq m-n}\left[\mathbf{T}^{0i}_{tt}\mathbf{T}^{0i}_{t't'} - (\mathbf{T}^{0i}_{tt'})^2\right]. \end{aligned} \tag{5.3}$$

The Gauss equation for ∇^0 gives

$$\begin{aligned} \sum_{1\leq t<t'\leq m-n} g(Rim^0(\tilde{e}_t, \tilde{e}_{t'})\tilde{e}_{t'}, \tilde{e}_t) &= 2\overline{R}^0 - (m-n)^2|\mathcal{N}^0|^2 \\ &\quad + \sum_{i=1}^{n}\sum_{t,t'=1}^{m-n}(\mathbf{T}^{0i}_{tt'})^2, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \sum_{2\leq t<t'\leq m-n} g(Rim^0(\tilde{e}_t, \tilde{e}_{t'})\tilde{e}_{t'}, \tilde{e}_t) = &\sum_{2\leq t<t'\leq m-n} g(\overline{Rim}^0(\tilde{e}_t, \tilde{e}_{t'})\tilde{e}_{t'}, \tilde{e}_t) \\ &- \sum_{i=1}^{n}\sum_{2\leq t<t'\leq m-n}\left[\mathbf{T}^{0i}_{tt}\mathbf{T}^{0i}_{t't'} - (\mathbf{T}^{0i}_{tt'})^2\right]. \end{aligned} \tag{5.5}$$

Therefore, Chen–Ricci inequality for the vertical distribution $\mathcal{V}$ of a statistical submersion ω is stated below.

Theorem 5.1. *[31] Let $\omega : (M, '\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion between statistical manifolds. Then, for each unit vector $U \in \mathcal{V}_x(M)$, we have*

$$\begin{aligned} 2\overline{Ric}^0(U) + Ric(U) - \overline{Ric}(U) - 2(m-n-1)\max\mathcal{K}^0(U\wedge\cdot) \\ \leq \frac{(m-n)^2}{8}(|\mathcal{N}|^2 + |\mathcal{N}^*|^2), \end{aligned} \tag{5.6}$$

where $\max\mathcal{K}^0(U\wedge\cdot)$ denotes the maximum of the sectional curvature function of M with respect to $'\nabla$ restricted to 2-plane sections of $\mathcal{V}_x(M)$ containing U.

Theorem 5.2. *[31] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion. Then the equality of inequality (5.6) holds identically if and only if the following two conditions are satisfied*

1. $2\mathbf{T}_U U = (m-n)\mathcal{N}(x)$, $\mathbf{T}_U V = 0$ *and*
2. $2\mathbf{T}^*_U U = (m-n)\mathcal{N}^*(x)$, $\mathbf{T}^*_U V = 0$,

for any vector $V \in \mathcal{V}(M)$ orthogonal to U.

Remark 11. It is noticed that if $\overline{Ric} = Ric = 0$, then the Chen–Ricci inequality obtained in [20, Theorem 4.1] (see also [5, Theorem 4.1]) for Rimmannian submersions between Riemannian manifolds can be rediscover from Theorem 5.1.

On putting

$$\begin{aligned}\delta^{\mathcal{V}}(m-n-1)(x) &= \max_{U\in\mathcal{V}(M)}\{2\overline{Ric}^0(U) + Ric(U) - \overline{Ric}(U)\\ &\quad -2(m-n-1)\max\mathcal{K}^0(U\wedge\cdot)\},\end{aligned}$$

the inequality (5.6) turns into a new relation [31]

$$\delta^{\mathcal{V}}(m-n-1) \leq \frac{(m-n)^2}{8}(|\mathcal{N}|^2 + |\mathcal{N}^*|^2). \tag{5.7}$$

The following is an interesting consequence of derived inequality (5.7).

Corollary 5.3. *[31] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion between two statistical manifolds. If*

$$\delta^{\mathcal{V}}(m-n-1)(x) > 0$$

holds at some point $x \in M$, then either $\mathcal{N} \neq 0$ or $\mathcal{N}^* \neq 0$ holds.

Next, for any tensor field Θ, we define [31]

$$\begin{aligned}\widetilde{\delta}\Theta &= -\sum_{i=1}^{n}(\nabla_{e_i}\Theta)_{e_i}, \quad \bar{\delta}\Theta = -\sum_{t=1}^{m-n}(\nabla_{\tilde{e}_t}\Theta)_{\tilde{e}_t},\\ \widetilde{\delta}^*\Theta &= -\sum_{i=1}^{n}(\nabla^*_{e_i}\Theta)_{e_i}, \quad \bar{\delta}^*\Theta = -\sum_{t=1}^{m-n}(\nabla^*_{\tilde{e}_t}\Theta)_{\tilde{e}_t}.\end{aligned}$$

We recall the following lemmas for the second Chen–Ricci inequality for statistical submersions of this subsection.

Lemma 5.4. *[36] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion. Then we have*

$$\begin{aligned}2R = 2\overline{R} + 2\widetilde{R} - 2|\mathbf{A}|^2 + g_M(\mathbf{A}, \mathbf{A}^*) - \widetilde{\delta}\mathcal{T} - \widetilde{\delta}^*\mathcal{T}^*\\ - \bar{\delta}\sigma + \bar{\delta}^*\beta + |\beta|^2 - g_M(\mathbf{T}, \mathbf{T}^*) - (m-n)^2 g_M(\mathcal{N}, \mathcal{N}^*),\end{aligned}$$

where $\widetilde{R}$ denotes the scalar curvature of N with respect to ∇ and β is defined by

$$\beta = \sum_{i=1}^{n}\mathbf{A}_{e_i}e_i. \tag{5.8}$$

Lemma 5.5. *[31] Let $\omega : (M,'\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion. Then we have*

$$\begin{aligned}
2R \geq\ & 2\overline{R} + 2\widetilde{R} - 2|\mathbf{A}|^2 + g_M(\mathbf{A}, \mathbf{A}^*) - \widetilde{\delta}\mathcal{T} - \widetilde{\delta}^*\mathcal{T}^* - \overline{\delta}\beta + \overline{\delta}^*\beta \\
& + |\beta|^2 - 2\sum_{i=1}^{n}\sum_{t,t'=1}^{m-n}(\mathbf{T}^{0i}_{tt'})^2 + \sum_{i=1}^{n}\sum_{2\leq t<t'\leq m-n}\mathbf{T}^{i}_{tt}\mathbf{T}^{*i}_{t't'} \\
& - 2\sum_{i=1}^{n}\sum_{2\leq t<t'\leq m-n}\mathbf{T}^{0i}_{tt}\mathbf{T}^{0i}_{t't'} + \frac{1}{2}\sum_{i=1}^{n}\sum_{2\leq t<t'\leq m-n}[(\mathbf{T}^{i}_{tt'})^2 + (\mathbf{T}^{*i}_{tt'})^2] \\
& + \frac{3(m-n)^2}{4}(|\mathcal{N}|^2 + |\mathcal{N}^*|^2) - 2(m-n)^2|\mathcal{N}^0|^2.
\end{aligned}$$

Consequently, the second Chen–Ricci inequality for statistical submersions proved by the first author, Chen and Danish in [31].

Theorem 5.4. *[31] Let $\omega : (M,'\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion between statistical manifolds. Then, for each unit vector $U \in \mathcal{V}_x(M)$, we have*

$$\begin{aligned}
2\{\overline{Ric}(U) - Ric(U) - \widetilde{R}\} \geq\ & 4\big[\overline{R} - R\big] + 6\big[\overline{R}^0 - R^0\big] \\
& - 2\big[\overline{Ric}^0(U) - Ric^0(U)\big] - 2|\mathbf{A}|^2 + g_M(\mathbf{A}, \mathbf{A}^*) - \widetilde{\delta}\mathcal{T} - \widetilde{\delta}^*\mathcal{T}^* \\
& - \overline{\delta}\beta + \overline{\delta}^*\beta + |\beta|^2 - 4(m-n)^2|\mathcal{N}^0|^2 + \frac{3(m-n)^2}{4}\big(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\big).
\end{aligned} \tag{5.9}$$

Theorem 5.5. *[31] Let $\omega : (M,'\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion between statistical manifolds. Then the equality of inequality (5.9) holds identically if and only if*

1. $2\mathbf{T}_U U = (m-n)\mathcal{N}(x), \quad \mathbf{T}_U V = 0,$
2. $2\mathbf{T}^*_U U = (m-n)\mathcal{N}^*(x), \quad \mathbf{T}^*_U V = 0,$

for any $V \in \mathcal{V}(M)$ orthogonal to U.

Remark 12. If the horizontal distribution $\mathcal{H}$ of ω is integrable, then (5.9) reduces to

$$\begin{aligned}
\overline{Ric}(U) - Ric(U) - \widetilde{R} \geq\ & 2\big[\overline{R} - R\big] + 3\big[\overline{R}^0 - R^0\big] - \overline{Ric}^0(U) + Ric^0(U) \\
& - \frac{1}{2}(\widetilde{\delta}\mathcal{T} - \widetilde{\delta}^*\mathcal{T}^*) - 2(m-n)^2|\mathcal{N}^0|^2 + \frac{3(m-n)^2}{8}(|\mathcal{N}|^2 + |\mathcal{N}^*|^2).
\end{aligned}$$

5.2 $\delta(2,2)$ Chen-type inequality

B.-Y. Chen introduces a new type of curvature invariant, called $\delta(2)$-invariant or known today as the first Chen invariant in [12]. For an n-dimensional Riemannian manifold M, it is defined by $\delta(2) = R - \inf \mathcal{K}$, where R denotes the scalar curvature and $\mathcal{K}$ the sectional curvature function of M. Then, he derives the Chen first inequality for any submanifold in a Riemannian space form. Later on, he introduces a sequence of δ-invariants on M, denoted by $\delta(n_1, \ldots, n_k)$, for integers $n_1, \ldots, n_k \geq 2$ such that $n_1 < n$ and $n_1 + \cdots + n_k \leq n$ (see [13, 14]). In particular, the $\delta(n-1)$-invariant and $\delta(2,2)$-invariant for a Riemannian n-manifold M are given respectively by

$$\begin{aligned}
\delta(n-1)(x) &= \max Ric(x), \\
\delta(2,2)(x) &= R(x) - \inf(\mathcal{K}(L_1) + \mathcal{K}(L_2)),
\end{aligned}$$

where $\max Ric$ is the maximum of the Ricci curvature, L_1, L_2 are mutually orthogonal 2-plane sections at $x \in M$, and $\mathcal{K}(L_i)$ denotes the sectional curvature of L_i for $i = 1, 2$. The corresponding inequality for $\delta(n-1)$ is known today as the Chen-Ricci inequality.

We need the following algebraic lemma from [24] to prove the $\delta(2,2)$ Chen-type inequality for statistical submersions in [31].

Lemma 5.6. *[24] Let $x_1, x_2, x_3, \ldots, x_m$ be $m \geq 4$ real numbers. Then we have*

$$\sum_{1\leq\alpha<\beta\leq m} x_\alpha x_\beta - x_1x_2 - x_3x_4 \leq \frac{m-3}{2(m-2)}\left(\sum_{\alpha=1}^{m} x_\alpha\right)^2.$$

Equality holds if and only if $x_1 + x_2 = x_3 + x_4 = x_5 = \cdots = x_m$.

Theorem 5.6. *[31] Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion between statistical manifolds. Then, for any orthonormal vectors $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4 \in \mathcal{V}_x(M)$, we have*

$$\begin{aligned}
&[\overline{R}^0 - \overline{\mathcal{K}}^0(\tilde{e}_1\wedge\tilde{e}_2) - \overline{\mathcal{K}}^0(\tilde{e}_3\wedge\tilde{e}_4)] - [\overline{R} - \overline{\mathcal{K}}(\tilde{e}_1\wedge\tilde{e}_2) - \overline{\mathcal{K}}(\tilde{e}_3\wedge\tilde{e}_4)]\\
&\quad - 2[R^0 - \mathcal{K}^0(\tilde{e}_1\wedge\tilde{e}_2) - \mathcal{K}^0(\tilde{e}_3\wedge\tilde{e}_4)]\\
&\qquad + [R^{\nabla,\nabla^*} - \mathcal{K}^{'\nabla,'\nabla^*}(\tilde{e}_1\wedge\tilde{e}_2) - \mathcal{K}^{'\nabla,'\nabla^*}(\tilde{e}_3\wedge\tilde{e}_4)]\\
&\qquad\quad \leq \frac{(m-n)^2(m-n-3)}{4(m-n-2)}\big[|\mathcal{N}|^2 + |\mathcal{N}^*|^2\big],
\end{aligned} \tag{5.10}$$

where $\mathcal{K}^{'\nabla,'\nabla^*}$ denotes the sectional curvature of M with respect to the dual affine connections given by [28, 29]

$$\begin{aligned}
2\mathcal{K}^{'\nabla,'\nabla^*}(\tilde{e}_1\wedge\tilde{e}_2) &= g_M(Rim(\tilde{e}_1,\tilde{e}_2)\tilde{e}_2,\tilde{e}_1)\\
&\quad + g_M(Rim^*(\tilde{e}_1,\tilde{e}_2)\tilde{e}_2,\tilde{e}_1).
\end{aligned} \tag{5.11}$$

Theorem 5.7. *Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion. Then the equality sign of (5.10) holds identically if and only if we have*

1. $\mathbf{T}^i_{11} + \mathbf{T}^i_{22} = \mathbf{T}^i_{33} + \mathbf{T}^i_{44} = \mathbf{T}^i_{55} = \cdots = \mathbf{T}^i_{m-nm-n}$
2. $\mathbf{T}^{*i}_{11} + \mathbf{T}^{*i}_{22} = \mathbf{T}^{*i}_{33} + \mathbf{T}^{*i}_{44} = \mathbf{T}^{*i}_{55} = \cdots = \mathbf{T}^{*i}_{m-nm-n}$
3. $\mathbf{T}^i_{tt'} = \mathbf{T}^{*i}_{tt'} = 0$

for $1 \leq t < t' \leq m-n$, $t \neq t'$, $(t,t') \neq (1,2), (2,1), (3,4), (4,3)$, $i \in \{1, 2, \ldots, n\}$.

The following corollary is an immediate consequence of Theorem 5.6.

Corollary 5.8. *Let $\omega : (M, {}'\nabla, g_M) \to (N, \nabla, g_N)$ be a statistical submersion. If*

$$\begin{aligned}
&[\overline{R}^0 - \overline{\mathcal{K}}^0(\tilde{e}_1\wedge\tilde{e}_2) - \overline{\mathcal{K}}^0(\tilde{e}_3\wedge\tilde{e}_4)]\\
&\quad - [\overline{R} - \overline{\mathcal{K}}(\tilde{e}_1\wedge\tilde{e}_2) - \overline{\mathcal{K}}(\tilde{e}_3\wedge\tilde{e}_4)]\\
&\qquad - 2[R^0 - \mathcal{K}^0(\tilde{e}_1\wedge\tilde{e}_2) - \mathcal{K}^0(\tilde{e}_3\wedge\tilde{e}_4)]\\
&\qquad\quad + [R^{'\nabla,'\nabla^*} - \mathcal{K}^{'\nabla,'\nabla^*}(\tilde{e}_1\wedge\tilde{e}_2) - \mathcal{K}^{'\nabla,'\nabla^*}(\tilde{e}_3\wedge\tilde{e}_4)] > 0
\end{aligned}$$

holds at some point, then either $\mathcal{N} \neq 0$ or $\mathcal{N}^* \neq 0$.

5.3 Inequality involving vertical Casorati curvatures

A new concept of curvature for regular surfaces in Euclidean space of dimension three introduced by Casorati as the normalized sum of the squared principal curvatures of the surface in [9]. Nowadays, it is called a Casorati curvature. For a submanifold B in a Riemannian manifold M, the Casorati curvature (an extrinsic invariant) of B is defined as the normalized square of the length of the second fundamental form.

From isometric immersions, many research problems for optimal inequalities on submanifolds in different kinds ambient spaces involving δ-Casorati curvatures are driven. Recently, in [22], Lee, et al. establish optimal inequalities for a Riemannian submersion between a space form and a Riemannian manifold involving the Casorati curvature of the vertical space. In this subsection, we recall the lower bounds for the normalized scalar curvature on statistical submersion with the normalized δ-vertical Casorati curvatures obtained by the first author in [32].

Theorem 5.9. *[32] Let $\omega : (M,'\nabla, g_M) \to (N,\nabla,g_N)$ be a statistical submersion between statistical manifolds. Then, the normalized δ-vertical Casorati curvatures $\delta^0_{\mathcal{C}}(m-n-1)$ and $\hat{\delta}^0_{\mathcal{C}}(m-n-1)$ satisfy*

$$\bar{\rho} - \rho \leq \delta^0_{\mathcal{C}}(m-n-1) + \frac{1}{(m-n-1)}\mathcal{C}^0 - \frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\Big) \tag{5.12}$$

and

$$\bar{\rho} - \rho \leq \hat{\delta}^0_{\mathcal{C}}(m-n-1) + \frac{1}{(m-n-1)}\mathcal{C}^0 - \frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\Big) \tag{5.13}$$

Theorem 5.10. *[32] Let $\omega : (M,'\nabla, g_M) \to (N,\nabla,g_N)$ be a statistical submersion. Then the equality sign of (5.12) and (5.13) hold identically if and only if with respect to suitable orthonormal basis $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{m-n}\}$ on the vertical space and $\{e_{m-n+1}, e_{m-n+2}, \ldots, e_n\}$ on the horizontal space, the components of $\mathbf{T}$ satisfy*

1. $\mathbf{T}^{0i}_{11} = \mathbf{T}^{0i}_{22} = \cdots = \mathbf{T}^{0i}_{m-n-1m-n-1} = \frac{1}{2}\mathbf{T}^{0i}_{m-nm-n}$
2. $\mathbf{T}^{i}_{tt'} = -\mathbf{T}^{*i}_{tt'}, \quad t,t' \in \{1,2,\ldots,m-n\}$, *provided* $t \neq t'$

for $i \in \{m-n+1, m-n+2, \ldots, n\}$.

Some immediate inequalities for submersions from different kinds of statistical manifold.

Theorem 5.11. *[32] Let $\omega : (M,'\nabla, g_M) \to (N,\nabla,g_N)$ be a statistical submersion from a statistical manifold of constant curvature c to a statistical manifolds. Then, the normalized δ-vertical Casorati curvatures $\delta^0_{\mathcal{C}}(m-n-1)$ and $\hat{\delta}^0_{\mathcal{C}}(m-n-1)$ satisfy*

$$\bar{\rho} \leq \delta^0_{\mathcal{C}}(m-n-1) + c + \frac{1}{(m-n-1)}\mathcal{C}^0 - \frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\Big), \tag{5.14}$$

and

$$\bar{\rho} \leq \hat{\delta}^0_{\mathcal{C}}(m-n-1) + c + \frac{1}{(m-n-1)}\mathcal{C}^0 - \frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2 + |\mathcal{N}|^2\Big), \tag{5.15}$$

(1) Kähler-like statistical manifold

$$\bar{\rho} \quad \leq \quad \delta^0_{\mathcal{C}}(m-n-1) + \frac{1}{(m-n-1)}\mathcal{C}^0 + \frac{c}{4}$$

$$
\begin{aligned}
&+\frac{c}{4(m-n)(m-n-1)}\{2|\mathcal{P}|^2 - trace^2(\mathcal{P}) - trace(\mathcal{P})^2\} \\
&-\frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\rho} \quad \leq \quad & \hat{\delta}^0_{\mathcal{C}}(m-n-1) + \frac{1}{(m-n-1)}\mathcal{C}^0 + \frac{c}{4} \\
&+\frac{c}{4(m-n)(m-n-1)}\{2|\mathcal{P}|^2 - trace^2(\mathcal{P}) - trace(\mathcal{P})^2\} \\
&-\frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\Big).
\end{aligned}
$$

(2) Sasaki-like statistical manifold

$$
\begin{aligned}
\overline{\rho} \quad \leq \quad & \delta^0_{\mathcal{C}}(m-n-1) + \frac{1}{(m-n-1)}\mathcal{C}^0 + \frac{c+3}{4} \\
&+\frac{c-1}{4(m-n)(m-n-1)}\{(2-m+n) - trace^2(\mathcal{F}) \\
&-trace(\mathcal{F})^2 + 2|\mathcal{F}|^2\} - \frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\rho} \quad \leq \quad & \hat{\delta}^0_{\mathcal{C}}(m-n-1) + \frac{1}{(m-n-1)}\mathcal{C}^0 + \frac{c+3}{4} \\
&+\frac{c-1}{4(m-n)(m-n-1)}\{(2-m+n) - trace^2(\mathcal{F}) \\
&-trace(\mathcal{F})^2 + 2|\mathcal{F}|^2\} - \frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\Big).
\end{aligned}
$$

(3) Quaternionic Kähler-like statistical manifold

$$
\begin{aligned}
\overline{\rho} \quad \leq \quad & \delta^0_{\mathcal{C}}(m-n-1) + \frac{1}{(m-n-1)}\mathcal{C}^0 + \frac{c}{4} \\
&+\frac{c}{4(m-n)(m-n-1)}\sum_{\alpha=1}^{3}\{2|\mathcal{P}_\alpha|^2 - trace^2(\mathcal{P}_\alpha) \\
&-trace(\mathcal{P}_\alpha)^2\} - \frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\rho} \quad \leq \quad & \hat{\delta}^0_{\mathcal{C}}(m-n-1) + \frac{1}{(m-n-1)}\mathcal{C}^0 + \frac{c}{4} \\
&+\frac{c}{4(m-n)(m-n-1)}\sum_{\alpha=1}^{3}\{2|\mathcal{P}_\alpha|^2 - trace^2(\mathcal{P}_\alpha) \\
&-trace(\mathcal{P}_\alpha)^2\} - \frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2 + |\mathcal{N}^*|^2\Big).
\end{aligned}
$$

(4) Cosymplectic-like statistical manifold

$$\begin{aligned}\overline{\rho} \;\leq\;& \delta^0_{\mathcal{C}}(m-n-1)+\frac{1}{(m-n-1)}\mathcal{C}^0+\frac{c}{4}\\&+\frac{c}{4(m-n)(m-n-1)}\{(2-m+n)-trace^2(\mathcal{F})\\&-trace(\mathcal{F})^2+2|\mathcal{F}|^2\}-\frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2+|\mathcal{N}^*|^2\Big),\end{aligned}$$

and

$$\begin{aligned}\overline{\rho} \;\leq\;& \hat{\delta}^0_{\mathcal{C}}(m-n-1)+\frac{1}{(m-n-1)}\mathcal{C}^0+\frac{c}{4}\\&+\frac{c}{4(m-n)(m-n-1)}\{(2-m+n)-trace^2(\mathcal{F})\\&-trace(\mathcal{F})^2+2|\mathcal{F}|^2\}-\frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2+|\mathcal{N}^*|^2\Big).\end{aligned}$$

(5) Kenmotsu-like statistical manifold

$$\begin{aligned}\overline{\rho} \;\leq\;& \delta^0_{\mathcal{C}}(m-n-1)+\frac{1}{(m-n-1)}\mathcal{C}^0+\frac{c-3}{4}\\&+\frac{c+1}{4(m-n)(m-n-1)}\{(2-m+n)-trace^2(\mathcal{F})\\&-trace(\mathcal{F})^2+2|\mathcal{F}|^2\}-\frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2+|\mathcal{N}^*|^2\Big),\end{aligned}$$

and

$$\begin{aligned}\overline{\rho} \;\leq\;& \hat{\delta}^0_{\mathcal{C}}(m-n-1)+\frac{1}{(m-n-1)}\mathcal{C}^0+\frac{c-3}{4}\\&+\frac{c+1}{4(m-n)(m-n-1)}\{(2-m+n)-trace^2(\mathcal{F})\\&-trace(\mathcal{F})^2+2|\mathcal{F}|^2\}-\frac{m-n}{2(m-n-1)}\Big(|\mathcal{N}|^2+|\mathcal{N}^*|^2\Big).\end{aligned}$$

Here, $\mathcal{P}$ denotes the tangential component of J. Likewise, $\mathcal{F}$ and $\mathcal{P}_\alpha$ are the tangential components of ϕ and J_α, respectively.

References

[1] N. Abe and K. Hasegawa, An affine submerion with horizontal distribution and its application, Differential Geom. Appl., **14** (2001), 235-250.

[2] P. Alegre, B.-Y. Chen, M.I. Munteanu, Riemannian submersions, δ–invariants, and optimal inequality, Ann. Global Anal. Geom., **42(3)** (2012), 317-331.

[3] S. Amari, Differential-Geometrical Methods in Statistics, Lecture Notes in Statistics, Springer-Verlag: New York, USA, **28** (1985).

[4] S. Amari and H. Nagaoka, Methods of Information Geometry, AMS & Oxford University Press, 2000.

[5] H. Aytimur and C. Ozgur, Sharp inequalities for anti-invariant Riemannian submersions from Sasakian space forms, *arXiv:1901.04172v1* [math.DG] January 14, 2019.

[6] H. Aytimur and C. Ozgur, On Cosymplectic-Like Statistical Submersions, Mediterr. J. Math., **16(70)** (2019).

[7] O.E. Barndroff-Nielson and P.E. Jupp, Differential geometry, profile likelihood, L-sufficient and composite transformation models, Ann. Statist., **16** (1988), 1009-1043.

[8] A. Beri, İ. Küpeli Erken and C. Murathan, Anti-invariant Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds, Turk. J. Math., **40**(3) (2016), 540-552.

[9] F. Casorati, Nuova defnizione della curvatura delle superfcie e suo confronto con quella di Gauss. (New defnition of the curvature of the surface and its comparison with that of Gauss). Rend. Inst. Matem. Accad. Lomb. Ser. II, **22(8)** (1889) 335-346.

[10] B.-Y. Chen, Riemannian submersions, minimal immersions and cohomology class, Proc. Japan Acad. Ser. A Math. Sci., **81(10)** (2005), 162-167.

[11] B.-Y. Chen, Examples and classification of Riemannian submersions satisfying a basic equality, Bull. Austral. Math. Soc., **72(3)** (2005), 391-402

[12] B.-Y. Chen, Some pinching and classification theorems for minimal submanifolds, Arch. Math., **60** (1993), 568–578.

[13] B.-Y. Chen, Strings of Riemannian invariants, inequalities, ideal immersions and their applications. in: The Third Pacific Rim Geometry Conference (Seoul, 1996) (International Press, Cambridge, MA, 1998) pp. 7–60.

[14] B.-Y. Chen, Some new obstructions to minimal and Lagrangian isometric immersions, Japan. J. Math., **26** (2000), 105–127.

[15] B.-Y. Chen, Pseudo-Riemannian Geometry, δ-invariants and Applications, World Scientific Publishing (Hackensack, NJ, 2011).

[16] M. Crasmareanu, A New Approach to Gradient Ricci Solitons and Generalizations, Filomat **32(9)** (2018).

[17] M. Falcitelli, S., Ianus, and A.M. Pastore, Riemannian submersions and related topics, World Scientific, River Edge, NJ, 2004.

[18] H. Furuhata, I. Hasegawa, Y., Okuyama and K. Sato, Kenmotsu statistical manifolds and warped product, J. Geom., DOI 10.1007/s00022-017-0403-1 (2017).

[19] A. Gray, Pseudo-Riemannian almost product manifolds and submersion, J. Math. Mech., **16** (1967) 715-737.

[20] M. Gulbahar, Ş.E. Meriç and E. Kılıç, Sharp inequalities involving the Ricci curvature for Riemannian submersions, Kragujevac J. Math., **41(2)** (2017), 279–293.

[21] R.S. Hamilton, The Ricci flow on surfaces, Mathematics and general relativity, (Santa Cruz. CA, 1986), Contemp. Math. 71, Amer. Math. Soc., (1988) 237-262.

[22] C.W. Lee, J.W. Lee, B. Şahin and G.E. Viclu, Optimal inequalities for Riemannian maps and Riemannian submersions involving Casorati curvatures, Annali di Matematica Pura ed Applicata, https://doi.org/10.1007/s10231-020-01037-7.

[23] Ş.E. Meriç and E. Kılıç, Riemannian submersions whose total manifolds admit a Ricci soliton, Intern. J. Geom. Meth. Mod. Phys., **16(12)** (2019) 1950196 .

[24] A. Mihai and I. Mihai, The $\delta(2,2)$-invariant on statistical submanifolds in Hessian manifolds of constant Hessian curvature, Entropy, **22**(2) (2020), Art. 164, 8 pp.

[25] C. Murathan and B. Şahin, A study of Wintgen like inequality for submanifolds in statistical warped product manifolds, J. Geom., **109**(30) (2018).

[26] B. O'Neill, The fundamental equations of a submersion, Mich. Math. J. **13** (1966) 458-469.

[27] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983

[28] B. Opozda, Bochner's technique for statistical structures, Ann. Glob. Anal. Geom., **48(4)** (2015), 357-395.

[29] B. Opozda, A sectional curvature for statistical structures, Linear Algebra Appl., **497** (2016), 134–161.

[30] A.N. Siddiqui, B.-Y. Chen and O. Bahadir, Statistical solitons and inequalities for statistical warped product submanifolds, Mathematics **7**(9), 797 (2019).

[31] A.N. Siddiqui, B.-Y. Chen and M.D. Siddiqi, Chen inequalities for statistical submersions between statistical manifolds, Intern. J. Geom. Meth. Mod. Phys., **18(04)**, 2150049 (2021).

[32] A.N. Siddiqui, M.D. Siddiqi, A.H. Alkhaldi and A. Ali, Lower bounds on statistical submersions with vertical Casorati curvatures, Intern. J. Geom. Meth. Mod. Phys., **19(03)**, 2250044 (2022) https://doi.org/10.1142/S021988782250044X.

[33] M.D. Siddiqi, A.N. Siddiqui, F. Mofarreh and H. Aytimur, A study of Kenmotsu-like statistical submersions, Symmetry **14(8)** (2022), 1681.

[34] B. Şahin, Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications. Elsiever (2017).

[35] B. Şahin, Anti-invariant Riemannian submersions from almost Hermitian manifolds, Cent. Eur. J. Math., **8** (3) (2010), 437-447.

[36] K. Takano, Statistical manifolds with almost complex structures and its statistical submerions, Tensor, N. S. **65**(2004) 123-137.

[37] K. Takano, Examples of the statistical submerions on the statistical model, Tensor, N. S. **65**(2004) 170-178.

[38] K. Takano, Statistical manifolds with almost contact structures and its statistical submersions, J. Geom., **85(12)**(2006), 171-187.

[39] G.E. Vilcu, Almost Product Structures on Statistical Manifolds and para-Kähler-Like Statistical Submersions, Bull. Sci. Math., **171(4)** 103018 (2021).

[40] A.D. Vilcu and G.E. Vilcu, Statistical manifolds with almost quaternionic structures and quaternionic Kähler-like statistical submersions, Entropy **17** (2015) 6213-6228.

[41] B. Watson, $\{G, G'\}$-Riemannian submersions and nonlinear gauge field equations of general relativity, In: Rassias, T. (ed.) Global Analysis - Analysis on manifolds, dedicated M. Morse. Teubner-Texte Math., 57 (1983), 324-349, Teubner, Leipzig.

Aliya Naaz Siddiqui Division of Mathematics, School of Basic & Applied Sciences, Galgotias University, Greater Noida, Uttar Pradesh 203201, India

E-mail: aliyanaazsiddiqui9@gmail.com

Kamran Ahmad Division of Mathematics, School of Basic & Applied Sciences, Galgotias University, Greater Noida, Uttar Pradesh 203201, India

E-mail: qskamrankhan@gmail.com